

A Nondegenerate Vuong Test and A Post Selection Confidence Interval for Semi/Nonparametric Models *

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Abstract

This paper proposes a new model selection test for the statistical comparison of semi/non-parametric models based on a general quasi-likelihood ratio criterion. An important feature of the new test is its uniformly exact asymptotic size in the overlapping nonnested case, as well as in the easier nested and strictly nonnested cases. The uniform size control is achieved without using pre-testing, sample-splitting, or simulated critical values. We also show that the test has nontrivial power against all \sqrt{n} -local alternatives and against some local alternatives that converge to the null faster than \sqrt{n} . Finally, we provide a framework for conducting uniformly valid post model selection inference for model parameters. The finite sample performance of the nondegenerate test and that of the post model selection inference procedure are illustrated in a mean-regression example by Monte Carlo.

JEL Classification: C14, C31, C32

Keywords: Asymptotic Size, Model Selection/Comparison Test, Post Model Selection Inference, Semi/Nonparametric Models

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1 Introduction

Model selection is an important issue in many empirical work. For example, in economic studies, there are often competing theories for one phenomenon. Even when there is only one theory, it can rarely pin down an empirical model to take to the data. Model selection tests are tools to determine the best model out of multiple competing models with a pre-specified statistical confidence level. One such test was proposed in Vuong (1989) to select from two parametric likelihood models according to their Kullback-Leibler information criterion (KLIC). The test determines the statistical significance of KLIC difference and, when the difference is significant, draws the directional conclusion that one model is closer to the truth than the other. This test has been widely used in empirical work due to its straightforward interpretation and implementation,¹ and it has been extended to many settings besides the likelihood one.

The studentized quasi-likelihood ratio (QLR) test statistic used in Vuong (1989) may have different asymptotic distributions under the null hypothesis, depending on whether the asymptotic variance of the QLR is degenerate. The degeneracy is unknown when the models compared are overlapping nonnested. In this case, a test based on such a test statistic and a standard critical value may not be uniformly valid and adding a pretest of the degeneracy does not provide a satisfactory solution, as shown in Shi (2015b). What is especially troubling is that the QLR-based test has a bias term that favors complex models. As a result, a user could manipulate the model selection result by unnecessarily increasing or decreasing the complexity of certain models. Shi (2015b) develops a solution in the context of parametric models, but Shi's test does not apply to semi/nonparametric models where the problem is in fact exacerbated.

The first contribution of this paper is to extend the conceptual idea of Shi (2015b) to semi/nonparametric models. Like Shi's test, our test corrects for bias caused by difference in model complexity and achieves uniform asymptotic validity regardless of model relationship. Unlike Shi's test, our revised QLR statistic is uniformly asymptotically normal, leading to a very simple testing procedure. The nonparametric component in one or both of the models, while making the asymptotic theory much more complicated, remarkably simplifies the testing procedure relative to Shi (2015b). We use linear sieve approximation for the nonparametric components (ref, e.g., Chen (2007)). As such, the asymptotic theory also provides a good approximation for parametric models with a large number of parameters.

The second contribution of this paper is a valid inference for the model parameters after the model selection test. Post model selection inference on one hand is unavoidable in most

¹See, e.g., Cameron and Heckman (1998), Coate and Conlin (2004), Paulson et al. (2006), Gowrisankaran and Rysman (2012), Moines and Pouget (2013), Barseghyan et al. (2013), Karaivanov and Townsend (2014), Kendall et al. (2015), Gandhi and Serrano-Padial (2015), to name only a few.

applications, and on the other hand is difficult to do correctly. For example, if post-model selection confidence intervals are constructed as if no model selection had been conducted, Leeb and Pötscher (2005) show that the resulting confidence intervals may have coverage probabilities very different from the nominal level. In this paper, we provide two types of uniformly asymptotically valid confidence intervals for parameters post model selection.

The rest of the introduction is devoted to the discussion of related literature.

The literature on the QLR model selection test. Although the QLR test proposed in Vuong (1989) has been widely used in the empirical studies and extended to many non-likelihood settings,² its property on the size control draws researchers' attention only recently. As mentioned above, the model selection part of this paper extends the conceptual idea of Shi (2015b) to semi/non-parametric models and propose a test with uniform size control for semi/non-parametric models. A few other papers in the literature of the Vuong test also achieve uniform asymptotic size control. These include Li (2009), Schennach and Wilhelm (2017), Hsu and Shi (2017) and Shi (2015a). These papers do not deal with semi/non-parametric models and each achieves uniform size control by a different technique. Li (2009) achieves uniformity thanks to the simulation noise brought about by numerical integration. Schennach and Wilhelm (2017) employ a sophisticated split-sample technique. Hsu and Shi (2017) introduce artificial noise to their test statistic. Shi (2015a) uses a pretest with a diverging threshold.

The consistent model specification testing literature. Although the main advantage of our revised QLR test is in the overlapping nonnested cases, it can be applied to and has uniform asymptotic similarity in the nested cases as well. In such cases, our test is a model specification test of the nested model against the alternative of the nesting model. As such, it is related to Hong and White (1995), Fan and Li (1996), Lavergne and Vuong (2000), and Aït-Sahalia et al. (2001) among others (see e.g. Aït-Sahalia et al. (2001) for a comprehensive literature review). Our test reduces to the heteroskedasticity-robust version of Hong and White (1995) based on series regression when a parametric conditional mean model is compared to a nonparametric one, and reduces to a series regression-based version of Aït-Sahalia et al.'s (2001) test when two nested nonparametric regressions are compared based on a weighted mean-squared error criterion. Our test applies to the testing problems in Fan and Li (1996) and Lavergne and Vuong (2000) but differs from the tests therein.

Post model selection inference. Our post model selection (PMS) inference has two parts. The first part regards conditional inference on model-specific parameters. This part is inspired by Tibshirani et al. (2016), who provide valid p-values and confidence intervals for post Lasso inference in a linear regression context with Gaussian noise. Their result is extended in Tibshirani et al. (2015) and Tian and Taylor (2015) to other linear regressions settings. We generalize Tibshirani

²Extensions include Lavergne and Vuong (1996), Rivers and Vuong (2002), Kitamura (2000), among others.

et al. (2016) to post model test inference for general semi-nonparametric models, and provide asymptotically exact confidence intervals without imposing special structures on the models or requiring knowledge of a variance-covariance matrix. The second part of our PMS inference analysis regards inference on common parameters of the two models. This part shares the objective of the methods surveyed in Belloni et al. (2014). However, this type of post selection inference is highly context specific, and the surveyed methods do not apply to post selection inference in general models.

The nonnested hypotheses literature. Since Vuong’s (1989) test is most commonly used to select between nonnested models, it is often linked to the literature of nonnested hypotheses featuring Cox (1961, 1962), Atkinson (1970), Pesaran (1974), Pesaran and Deaton (1978), Mizon and Richard (1986), Gourieroux and Monfort (1995), Ramalho and Smith (2002), and Bontemps et al. (2008) among others. This literature does not share the objective of Vuong’s test. Rather than focusing on the relative fit of the models, earlier part of this literature focuses on testing the correct specification of one model with power directed toward the other model. Later part of this literature focuses on the ability of one model to encompass empirical features of the other model. To our knowledge, the uniform validity of these tests when the models under consideration are overlapping nonnested has not been studied, and may be an interesting topic for future research.³

The rest of the paper is organized as follows. Section 2 sets up our testing framework and gives three examples. Section 3 describes our test in detail. Section 4 establishes the asymptotic size and the local power of our test. Section 5 illustrates the construction of our test in the mean-regression context. Section 6 provides the uniformly valid post model selection inference procedures. Section 7 shows Monte Carlo results of a mean-regression example. Section 8 applies the proposed nondegenerate test and conditional confidence interval to a schooling choice example, and Section 9 concludes. Proofs and other supplemental materials are included in the Supplemental Appendix.

Notation. Let C , C_1 and C_2 be generic positive constants whose values do not change with the sample size. For any column vector a , let a' denote its transpose and $\|a\|$ its ℓ_2 -norm. For any square matrix A , $A(i, j)$ denotes the element in the i th row and j th column of A , $\|A\|$ denotes the operator norm, and A^+ denotes its Moore-Penrose inverse. Let $\rho_{\min}(A)$ and $\rho_{\max}(A)$ be the smallest and largest eigenvalues of A in terms of absolute value, respectively. Let $tr(A)$ denote the trace of matrix A . For any square matrices A_1 and A_2 , $diag(A_1, A_2)$ denotes the block diagonal matrix with A_1 being the leading submatrix. Let $N(\mu, \Sigma)$ stand for a normal random vector with mean μ and variance-covariance matrix Σ . For any (possibly random) positive sequences $\{a_n\}_{n=1}^{\infty}$

³The lack of uniform size control of the Cox test when the DGP space is not restricted is illustrated in Loh (1985). However, uniform size control under reasonable restrictions on the DGP space for the Cox test and other nonnested hypotheses tests is still an interesting problem yet to be explored.

and $\{b_n\}_{n=1}^\infty$, $a_n = O_P(b_n)$ means that $\lim_{c \rightarrow \infty} \limsup_n \Pr(a_n/b_n > c) = 0$; and $a_n = o_P(b_n)$ means that for all $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \Pr(a_n/b_n > \varepsilon) = 0$. For any $p \in (0, 1)$, let z_p denote the p quantile of the standard normal distribution.

2 General Setup

2.1 Setup

Let $Z \in \mathcal{Z} \subseteq R^{d_z}$ be an observable random vector with distribution F_0 . Let \mathcal{M}_1 and \mathcal{M}_2 be two models about F_0 ; that is, \mathcal{M}_1 and \mathcal{M}_2 are two sets of probability distributions on R^{d_z} defined by modeling assumptions. We are interested in testing the null hypothesis of equal fit:

$$H_0 : f(\mathcal{M}_1, F_0) = f(\mathcal{M}_2, F_0), \quad (2.1)$$

where $f(\cdot, \cdot)$ is a generic measure of fit. The alternative hypothesis can be either

$$H_1^{2\text{-sided}} : f(\mathcal{M}_1, F_0) \neq f(\mathcal{M}_2, F_0) \text{ or } H_1^{1\text{-sided}} : f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0). \quad (2.2)$$

The two-sided test indicates that the two models have (statistically) significantly different fit for the observed data when it rejects H_0 , and the one-sided test indicates that model \mathcal{M}_1 fits the observed data significantly better when it rejects H_0 . It is the goal of this paper to develop a simple test of equal fitting with uniform asymptotic validity and good power properties.

The fit measure $f(\cdot, \cdot)$ is context-specific and should be chosen to best suit the empirical model comparison need. We focus on a given fit measure of the following form:

$$f(\mathcal{M}_j, F_0) = \max_{\alpha_j \in \mathcal{A}_j} E_{F_0} [m_j(Z; \alpha_j)] = E_{F_0} [m_j(Z; \alpha_{F_0,j}^*)], \text{ for } j = 1, 2, \quad (2.3)$$

where $E_{F_0}[\cdot]$ denotes the expectation taken under F_0 , $m_j(\cdot; \cdot)$ is a user-chosen link function that is the central component of the fit measure, α_j is the parameter in model \mathcal{M}_j , \mathcal{A}_j is the possibly infinite-dimensional parameter space, and $\alpha_{F_0,j}^*$ is the pseudo-true parameter value of model j defined as $\alpha_{F_0,j}^* = \arg \max_{\alpha_j \in \mathcal{A}_j} E_{F_0} [m_j(Z; \alpha_j)]$.⁴

To fix ideas, consider the most common examples of \mathcal{M}_j and $f(\mathcal{M}_j, F_0)$, $j = 1, 2$:

⁴Following the literature (see, e.g., Stone (1985) and Ai and Chen (2007)), we assume that the pseudo true parameter $\alpha_{F_0,j}^*$ exists, is unique, and lies in the interior of \mathcal{A}_j for $j = 1, 2$ throughout the paper. The sufficient conditions to ensure the existence of the pseudo true parameter $\alpha_{F_0}^*$ in general semi/nonparametric models are: (i) the population function $Q_{F_0}(\alpha) = E_{F_0} [m(Z, \alpha)]$ is continuous at any $\alpha \in \mathcal{A}$ under certain metric d (e.g., the L_2 -metric or the uniform metric); and (ii) the parameter space \mathcal{A} is compact with respect to d . Low level sufficient conditions for the existence and uniqueness of $\alpha_{F_0,j}^*$ in specific models can be found in Stone (1985) and Ai and Chen (2007). See Section 5 for more discussion in the regression models.

Example 1 (Likelihood Ratio) Consider $Z = (W', X)'$. Many structural models used in empirical economics can be written as a conditional likelihood model of Z given X , i.e. (ignoring the model index j)

$$\mathcal{M} = \{F : dF_{Z|X}(z|x)/d\mu_z = \phi(z|x; \alpha), \forall z, \text{ for some } \alpha \in \mathcal{A}\}, \quad (2.4)$$

where $F_{Z|X}$ is the conditional distribution of Z given X implied by F , $dF_{Z|X}(z|x)/d\mu_z$ is the Radon-Nykodym density of $F_{Z|X}$ with respect to a basic measure (μ_z) on the space of Z , ϕ is a known function, α is a possibly infinite-dimensional unknown parameter, and \mathcal{A} is its parameter space. For such a model, a natural fit measure is the population conditional log-likelihood, which is the $f(\mathcal{M}, F_0)$ defined in equation (2.3) with

$$m(Z; \alpha) = \log \phi(Z|X; \alpha). \quad (2.5)$$

Note that with $f(\mathcal{M}, F_0)$ defined this way, $\{f(\mathcal{M}, F_0) - f(\{F_0\}, F_0)\}$ is the Kullback-Leibler pseudo-distance from model \mathcal{M} to the true distribution F_0 . Vuong's (1989) original test is designed for such a likelihood context with α for both models being finite-dimensional, although Shi (2015b) shows that it may have size distortion. Shi (2015b) proposes a uniformly valid procedure for the parametric likelihood case.

Example 2 (Squared Error) Consider $Z = (Y, X)'$, where Y is a dependent variable, X is a vector of regressors. A mean-regression model may be written as

$$\mathcal{M} = \{F : E_F[Y|X = x] = g(x; \alpha), \forall x, \text{ for some } \alpha \in \mathcal{A}\}, \quad (2.6)$$

where $g(\cdot; \cdot)$ is a known regression function, α is a possibly infinite-dimensional unknown parameter and \mathcal{A} is its parameter space.⁵ For such a model, a commonly used fit measure is the population regression mean-squared error, which is $f(\mathcal{M}, F_0)$ defined in equation (2.3) with

$$m(Z; \alpha) = -|Y - g(X; \alpha)|^2 / 2. \quad (2.7)$$

Example 3 (Check Function) Consider $Z = (Y, X)'$, where Y is a dependent variable, X is a vector of regressors. A quantile-regression model may be written as

$$\mathcal{M} = \{P : Q_{\tau, F}(Y|X = x) = g(x; \alpha), \forall x, \text{ for some } \alpha \in \mathcal{A}\}, \quad (2.8)$$

⁵Sometimes, regression models are used without explicitly or implicitly assuming the best fitting regression function to be $E(Y|X = x)$. Nonetheless, the regression mean-squared error criterion often still is used to compare the models. In those cases, the test developed in this paper still applies.

where $Q_{\tau,F}(Y|X)$ is the conditional τ -th quantile of Y given X under F with $\tau \in (0, 1)$, $g(\cdot; \cdot)$ is a known regression function, α is a possibly infinite-dimensional unknown parameter, and \mathcal{A} is its parameter space. Similar to the example above, a reasonable fit measure is the expected check function of Y from the best conditional τ -th quantile function in the model, which is $f(\mathcal{M}, F_0)$ defined in equation (2.3) with

$$m(Z; \alpha) = (I \{Y \leq g(X; \alpha)\} - \tau) [Y - g(X; \alpha)] \quad (2.9)$$

where $I \{\cdot\}$ denotes the indicator function.

2.2 Model Relationships

The following terms for model relationships are mentioned in the introduction, and will be used in later sections when we discuss the uniform validity of our test in detail.

Definition 1 (Strictly Nonnested) Models \mathcal{M}_1 and \mathcal{M}_2 are strictly nonnested if there does not exist a pair $(\alpha_1, \alpha_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that $m_1(z; \alpha_1) = m_2(z; \alpha_2) \forall z \in \mathcal{Z}$.

Definition 2 (Overlapping) Models \mathcal{M}_1 and \mathcal{M}_2 are overlapping if they are not strictly nonnested.

Definition 3 (Nested) Model \mathcal{M}_1 nests model \mathcal{M}_2 if, for each $\alpha_2 \in \mathcal{A}_2$, there exists an $\alpha_1 \in \mathcal{A}_1$ such that $m_1(z; \alpha_1) = m_2(z; \alpha_2)$ for any $z \in \mathcal{Z}$.

Clearly, the overlapping case include the nested case. If the models are overlapping but not nested, we say that the models are *overlapping nonnested*. If the models are mutually nested (i.e. \mathcal{M}_1 nests \mathcal{M}_2 , and \mathcal{M}_2 nests \mathcal{M}_1), then the models are *observationally equivalent*.⁶ We exclude the case where the models are observationally equivalent from our discussion, since in this trivial case, H_0 always holds regardless of the true data distribution and no statistical method can distinguish the two. The model relationship determines whether the random variable $m_1(Z; \alpha_{F_0,1}^*) - m_2(Z; \alpha_{F_0,2}^*)$ is always, never, or sometimes degenerate (i.e., almost surely zero) under H_0 .⁷ ⁸ Since whether $m_1(Z; \alpha_{F_0,1}^*) - m_2(Z; \alpha_{F_0,2}^*)$ is degenerate or not affects the asymptotic distribution of standard quasi-likelihood ratio statistic, uniformity issue arises when its status is unknown.

⁶This definition of model equivalence is consistent with that in Pesaran and Ulloa (2008).

⁷This variable is clearly not almost surely zero under H_1 , because its mean is different from zero.

⁸Some readers may confuse the degeneracy of $m_1(Z; \alpha_1^*) - m_2(Z; \alpha_2^*)$ under H_0 with the observational equivalence of the models \mathcal{M}_1 and \mathcal{M}_2 . The former does not imply the latter, as one can easily see in the following simplistic example. Let \mathcal{M}_1 be a mean-regression model $E[Y|X] = \alpha_1(X)$ with the space \mathcal{A}_1 of α_1 including the zero function, and let \mathcal{M}_2 be another mean-regression model $E[Y|X] = 0$. Then our H_0 is the same as the hypothesis that $E[Y|X] = 0$ a.s.. Under H_0 , the difference in squared residuals is degenerate to zero. But the models \mathcal{M}_1 and \mathcal{M}_2 are clearly not observationally equivalent.

As we will see, the test statistic that we construct is asymptotically standard normal under H_0 regardless of whether $m_1(Z; \alpha_{F_0,1}^*) - m_2(Z; \alpha_{F_0,2}^*)$ is degenerate. This leads to a test that is uniformly asymptotically valid across all cases and all types of model relationship. Such uniformity is of practice importance for a number of reasons. First, in many nonnested model selection scenarios, the competing models are not completely incompatible to each other, in which case they are overlapping. Second, establishing strict nonnestedness is difficult for structural models used in empirical analysis. Using our test obviates the need for doing this. Third, even when the models are strictly nonnested, tests ignoring the uniformity issue may still have severe size distortion (over-rejection) in finite samples when both models can closely describe the data distribution, while our test does not suffer from this kind of distortion.

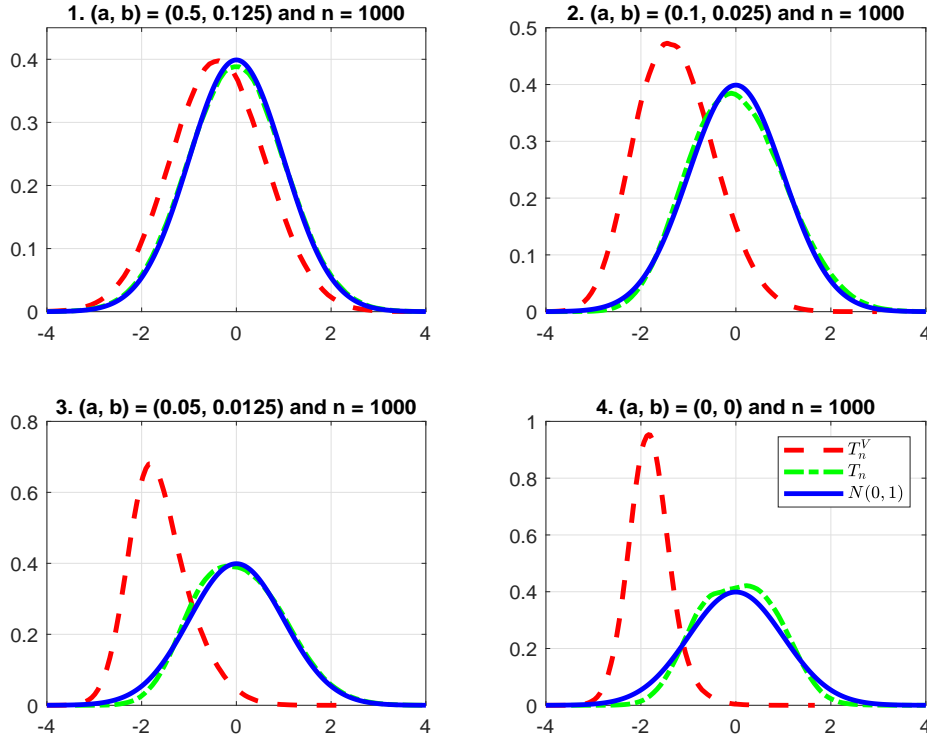
2.3 Illustration of the Uniformity Issue

To further illustrate the uniformity issue, we presents a simple simulation study in Figure 1 compares two parametric linear regression models based on their mean-squared error. We show both the distribution of the standardized QLR statistic (as used in Vuong (1989), T_n^V) and our test statistic (T_n) in the figure. Here, model 1 has two regressors and model 2 has 17 regressors.

The red dashed line represents the finite sample density of T_n^V defined in (3.5) below. In the pointwise asymptotic framework, under H_0 , T_n^V has asymptotic standard normal distribution when the latent parameters $(a, b) \neq 0$ and asymptotic weighted chi-square distribution when $(a, b) = 0$. Suppose that one conducts model selection test using the critical value from the standard normal distribution. Although such a test is justified by the asymptotic distribution of T_n^V when (a, b) are not zero, we see that it over-rejects under the null even in this case, as illustrated in the first three scenarios considered in Figure 1. When the latent parameters (a, b) are close to zero, this test is severely over-sized and strongly in favor of the large model, i.e., model 2. As the figure also shows, the standard normal distribution is a poor approximation to the finite sample density of T_n^V when (a, b) are not far enough away from zero, this also suggests that it is tricky to use pre-testing of the latent model structure construct a valid model selection test.

The green dash-dotted line represents the finite sample density of our revised QLR statistic T_n defined in (3.16) below. It is clear that the distribution of T_n is robust against small values of (a, b) , and its finite sample density is very close to the standard normal. Thus, the test using T_n and critical value from the standard normal has better size control than the test based on T_n^V and it is also not biased by the relative complexities of the two models.

Figure 1: Finite Sample Densities of T_n^V and T_n under the Null Hypothesis



Notes: (i). The simulated data is generated from the equation $Y_i = 0.5X_{1,i} + aX_{2,i} + b \sum_{k=1}^{16} X_{2+k,i} + u_i$, where (a, b) is set to different values in the four subgraphs and the values guarantee equal fitting of the candidate models, and $(X_{1,i}, \dots, X_{18,i}, u_i)'$ is a standard normal random vector; (ii) model 1: $Y_i = X_{1,i}\theta_{1,1} + aX_{2,i}\theta_{1,2} + u_{1,i}$ is compared with model 2: $Y_i = X_{2,i}\theta_{2,2} + b \sum_{k=1}^{16} X_{2+k,i}\theta_{2,2+k} + u_{2,i}$ in their expected squared errors; (iii) the finite sample densities of the existing QLR statistic T_n^V and our statistic T_n are approximated using 1,000,000 simulated samples.

3 Description of Our Model Selection Test

Suppose that there is an i.i.d. sample $\{Z_i\}_{i=1}^n$ of Z . In this section we describe our test for (2.1) based on this sample. The construction of the test is grounded on the asymptotic expansion established in the next section. We focus on the steps of the construction in this section for easy reference for potential users of the test.

We use linear sieve approximation for the unknown functions, and use sieve M-estimator for estimation.⁹ The specific procedure is explained now. For $j = 1, 2$, let \mathcal{A}_{j,k_j} denote a finite

⁹Many properties of the sieve M-estimator, including consistency, rate of convergence and asymptotic normality are established in the literature. See, e.g., Chen (2007) for a recent survey on this topic.

dimensional approximation of the parameter space \mathcal{A}_j , which satisfies

$$\mathcal{A}_{j,k_j} = \{\alpha_{j,k_j}(\cdot) : \alpha_{j,k_j}(\cdot) = \alpha_j(\beta_{j,k_j}) \equiv P_{j,k_j}(\cdot)' \beta_{j,k_j} : \beta_{j,k_j} \in B_{j,k_j} \subseteq R^{k_j}\}, \quad (3.1)$$

where $P_{j,k_j}(\cdot) = [p_{j,1}(\cdot), \dots, p_{j,k_j}(\cdot)]'$ is a k_j -dimensional vector of user-chosen approximating functions such as polynomials and splines, k_j is a positive integer which may diverge with the sample size n . In the rest of the paper, we write $\alpha_{k_j}(\cdot) = \alpha_{j,k_j}(\cdot)$, $P_{k_j}(\cdot) = P_{j,k_j}(\cdot)$ and $\beta_{k_j} = \beta_{j,k_j}$ for $j = 1, 2$ for ease of notation.

To construct the test, we first estimate the fit of each model with the sample analogue estimator. For $j = 1, 2$, define

$$\widehat{f}(\mathcal{M}_j, F_0) = n^{-1} \sum_{i=1}^n m_j(Z_i; \widehat{\alpha}_{k_j}) \quad (3.2)$$

where $\widehat{\alpha}_{k_j} = \alpha_j(\widehat{\beta}_{k_j})$ is an M-estimator defined with

$$\widehat{\beta}_{k_j} = \arg \max_{\beta_{k_j} \in B_{j,k_j}} n^{-1} \sum_{i=1}^n m_j[Z_i; \alpha_j(\beta_{k_j})]. \quad (3.3)$$

For notation simplicity, we define the pseudo-density ratio:

$$\ell(Z; \alpha) = m_1(Z; \alpha_1) - m_2(Z; \alpha_2) \quad (3.4)$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathcal{A}_1 \times \mathcal{A}_2$. We also define $\alpha_{F_0}^* = (\alpha_{F_0,1}^*, \alpha_{F_0,2}^*)$, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, $\mathbf{k} = (k_1, k_2)$, $\beta_{\mathbf{k}} = (\beta'_{k_1}, \beta'_{k_2})'$, $\mathcal{A}_{\mathbf{k}} = \mathcal{A}_{1,k_1} \times \mathcal{A}_{2,k_2}$, $\alpha_{\mathbf{k}} = \alpha(\beta_{\mathbf{k}}) = (\alpha_1(\beta_{k_1}), \alpha_2(\beta_{k_2}))$, and $\widehat{\alpha}_{\mathbf{k}} = (\widehat{\alpha}_{k_1}, \widehat{\alpha}_{k_2})$.

Since the null hypothesis H_0 is equivalent to $E_{F_0}[\ell(Z; \alpha_{F_0}^*)] = 0$, one may be tempted to suggest treating $E_{F_0}[\ell(Z; \alpha_{F_0}^*)]$ as a parameter and constructing a Student t-like test for this hypothesis. In other words, the suggestion would be to construct the test statistic

$$T_n^V \equiv \bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}})(n^{-1/2}\widehat{\omega}_n(\widehat{\alpha}_{\mathbf{k}}))^{-1}, \quad (3.5)$$

where $\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}})$ is the sample analogue estimator of $E_{F_0}[\ell(Z; \alpha_{F_0}^*)]$ and $n^{-1/2}\widehat{\omega}_n(\widehat{\alpha}_{\mathbf{k}})$ is the sample analogue of its standard deviation:

$$\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) = n^{-1} \sum_{i=1}^n \ell(Z_i; \widehat{\alpha}_{\mathbf{k}}) \text{ and } \widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}}) = n^{-1} \sum_{i=1}^n [\ell(Z_i; \widehat{\alpha}_{\mathbf{k}}) - \bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}})]^2. \quad (3.6)$$

Then one would construct tests of the form: $\varphi_n^{V,2\text{-sided}}(p) = 1\{|T_n^V| > z_{1-p/2}\}$ or $\varphi_n^{V,1\text{-sided}}(p) = 1\{T_n^V > z_{1-p}\}$. In fact, such tests are analogous extensions of Vuong's (1989) (one-step) test to the semi/non-parametric context. Thus, we refer to them as the "naive extension" tests hereafter.

The rationale behind the naive extension test is that $n^{1/2}\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) = n^{1/2}\bar{\ell}_n(\alpha_{F_0}^*) + o_p(1) \rightarrow_d N(0, \omega_{F_0, *}^2)$ and $\hat{\omega}_n^2 = \omega_{F_0, *}^2 + o_p(1)$, where $\omega_{F_0, *}^2 = Var_{F_0}(\ell(Z; \alpha_{F_0}^*))$. However, this asymptotic approximation can be very poor when $\omega_{F_0, *}^2$ is close to or equal to zero. When the models are overlapping nonnested, both small positive values and the zero value are possible for $\omega_{F_0, *}^2$ under H_0 , depending on the *unknown* data distribution F_0 . Thus, the naive extension test often fails to have the correct level in a finite sample.¹⁰

The intuition of the failure of the naive extension test can be seen from the following heuristic second order expansion of the QLR statistic.¹¹ Let

$$\beta_{k_j}^* = \arg \max_{\beta_{k_j} \in B_{j, k_j}} E_{F_0} [m_j(Z; \alpha_j(\beta_{k_j}))], \quad (3.7)$$

where we suppress the dependence of $\beta_{k_j}^*$ on F_0 for notational convenience. We assume that the sieve coefficients $\beta_{k_j}^*$ are in the interior of their spaces B_{j, k_j} for any k_j . Let $\alpha_{k_j}^*(\cdot) = P_{k_j}(\cdot)' \beta_{k_j}^*$. Then $\alpha_{k_j}^*$ is the sieve approximator of the pseudo true parameter $\alpha_{F_0, j}^*$ on the finite dimensional space \mathcal{A}_{j, k_j} . Let $\ell_{\alpha, \mathbf{k}}(Z; \alpha)$ be the “score” function of $\ell(Z; \alpha)$ evaluated at $\alpha \in \mathcal{A}_{\mathbf{k}}$. When $\ell(Z; \alpha(\beta_{\mathbf{k}}))$ is differentiable in $\beta_{\mathbf{k}}$, we can let

$$\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}) = \partial \ell(Z; \alpha_{\mathbf{k}}) / \partial \beta_{\mathbf{k}} \text{ and } \bar{\ell}_{\alpha, \mathbf{k}, n}(\alpha_{\mathbf{k}}^*) = n^{-1} \sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \quad (3.8)$$

where $\alpha_{\mathbf{k}}^* = (\alpha_{k_1}^*, \alpha_{k_2}^*)$. Then a second order Taylor expansion of $\bar{\ell}_n(\alpha_{\mathbf{k}}^*)$ around $\hat{\alpha}_{\mathbf{k}}$ gives:

$$\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - E_{F_0}[\ell(Z; \alpha_{F_0}^*)] \approx \bar{\ell}_n(\alpha_{\mathbf{k}}^*) - E_{F_0}[\ell(Z; \alpha_{F_0}^*)] - 2^{-1} \bar{\ell}_{\alpha, \mathbf{k}, n}(\alpha_{\mathbf{k}}^*)' H_{F_0, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, \mathbf{k}, n}(\alpha_{\mathbf{k}}^*), \quad (3.9)$$

where

$$H_{F_0, \mathbf{k}} = \text{diag} \left(\frac{\partial^2 E_{F_0}[m_1(Z; \alpha_{k_1})]}{\partial \beta_{k_1} \partial \beta_{k_1}'}, -\frac{\partial^2 E_{F_0}[m_2(Z; \alpha_{k_2})]}{\partial \beta_{k_2} \partial \beta_{k_2}'} \right) = \text{diag}(H_{F_0, k_1}, -H_{F_0, k_2}). \quad (3.10)$$

Appropriate conditions and the central limit theorem imply that $n^{1/2} \{ \bar{\ell}_n(\alpha_{\mathbf{k}}^*) - E_{F_0}[\ell(Z; \alpha_{F_0}^*)] \} \rightarrow_d N(0, \omega_{F_0, *}^2)$ and $n^{1/2} \bar{\ell}_{\alpha, \mathbf{k}, n}(\alpha_{F_0}^*) \rightarrow_d N(0, D_{F_0, \mathbf{k}})$, where

$$D_{F_0, \mathbf{k}} = E_{F_0}[\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)']. \quad (3.11)$$

¹⁰A pretest for whether $\ell(\cdot; \alpha_{F_0}^*) = 0$ could be performed. But the two-step procedure may (a) not be uniformly asymptotically valid if the pretest does not use a conservative critical value, and (b) not be powerful because the pretest makes rejection difficult.

¹¹The use of higher order expansion to develop more robust asymptotic theory is not new. It has been used in many contexts including, for example, Jun and Pinkse (2012).

The latter implies that $n\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'H_{F_0,\mathbf{k}}^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)$ is approximately $\sum_{j=1}^{|\mathbf{k}|}\lambda_j\chi_j^2(1)$, where $|\mathbf{k}| = k_1 + k_2$, $\{\chi_j^2(1)\}_{j=1}^{|\mathbf{k}|}$ are independent chi-squares with one degree of freedom and $\{\lambda_j\}_{j=1}^{|\mathbf{k}|}$ are the eigenvalues of $D_{F_0,\mathbf{k}}H_{F_0,\mathbf{k}}^{-1}$. Thus,

$$n\{\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - E_{F_0}[\ell(Z; \alpha_{F_0}^*)]\} \approx n^{1/2}N(0, \omega_{F_0,*}^2) - 2^{-1}\sum_{j=1}^{|\mathbf{k}|}\lambda_j\chi_j^2(1). \quad (3.12)$$

Note that since $E[\chi_j^2(1)] = 1$, we have $E[\sum_{j=1}^{|\mathbf{k}|}\lambda_j\chi_j^2(1)] = \sum_{j=1}^{|\mathbf{k}|}\lambda_j$, which is typically nonzero and can be of comparable scale as $n^{1/2}\omega_{F_0,*}$, the standard deviation of $n\bar{\ell}_n(\alpha_{F_0}^*)$. This means that, even when the null hypothesis H_0 holds ($E_{F_0}[\ell(Z; \alpha_{F_0}^*)] = 0$), the numerator of the statistic T_n^V may not be centered around zero, causing the naive extension test to be biased. A similar expansion of the denominator unveils that $n\hat{\omega}_n(\hat{\alpha}_{\mathbf{k}})^2$ is a biased estimator of $\omega_{F_0,*}^2$ as well, and the dominating term of the bias is coincidentally $2^{-1}\sum_{j=1}^{|\mathbf{k}|}\lambda_j^2$. Thus, the naive extension test not only has a numerator bias that leads it to favor one model over the other when both have equal fit, but also has a denominator bias that tends to make it conservative. The two biases could cancel each other in certain context, but in general do not, and can exacerbate each other when the power against one-sided alternatives is considered.

Our nondegenerate test corrects the two biases by estimating and removing them. Specifically, we construct estimators $\hat{\lambda}_j : j = 1, \dots, |\mathbf{k}|$ and propose the bias removed statistics:

$$\tilde{\ell}_n = \bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) + (2n)^{-1}\sum_{j=1}^{|\mathbf{k}|}\hat{\lambda}_j \text{ and } \tilde{\omega}_n^2 = \hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) - (2n)^{-1}\sum_{j=1}^{|\mathbf{k}|}\hat{\lambda}_j^2. \quad (3.13)$$

Then the approximation in (3.12) implies that under H_0 ,

$$n\tilde{\ell}_n \approx n^{1/2}N(0, \omega_{F_0,*}^2) - 2^{-1}\sum_{j=1}^{|\mathbf{k}|}\lambda_j(\chi_j^2 - 1). \quad (3.14)$$

Recall that $|\mathbf{k}| \rightarrow \infty$ as $n \rightarrow \infty$ in semi/non-parametric models, and apply the central limit theorem on the sum of independent mean-zero variables $\chi_j^2 - 1 : j = 1, \dots, |\mathbf{k}|$ to find that the second term is approximately normal as well. We also show that the two terms are asymptotically independent, suggesting that $n\tilde{\ell}_n$ is asymptotically mean-zero normal under H_0 . Moreover, $n\tilde{\omega}_n^2$ also consistently estimate the variance of this mean-zero normal limit. As a result, we have

$$T_n^0 = \frac{n\tilde{\ell}_n}{n^{1/2}\tilde{\omega}_n} \rightarrow_d N(0, 1), \text{ as } n \rightarrow \infty. \quad (3.15)$$

There is a minor issue with using T_n^0 as our test statistic because $\tilde{\omega}_n^2$ is defined as the difference of two non-negative terms. In finite sample, this difference can be zero or negative even though the probability of that happening approaches zero as $n \rightarrow \infty$. To avoid this finite sample irregularity, we recommend a slightly regularized version:

$$T_n = \frac{n\tilde{\ell}_n}{n^{1/2}\hat{\sigma}_n}, \text{ where } \hat{\sigma}_n^2 = \max \left\{ \tilde{\omega}_n^2, (2n)^{-1} \sum_{i=1}^{|\mathbf{k}|} \hat{\lambda}_j^2 \right\}. \quad (3.16)$$

The regularization has no effect on the asymptotic distribution as we show that $(2n)^{-1} \sum_{i=1}^{|\mathbf{k}|} \hat{\lambda}_j^2$ is less than or equal to $\tilde{\omega}_n^2$ asymptotically. Thus, we still have $T_n \rightarrow_d N(0, 1)$ as $n \rightarrow \infty$.

Estimating $\lambda_j : j = 1, \dots, |\mathbf{k}|$ is straightforward as they are eigenvalues of $D_{F_0, \mathbf{k}} H_{F_0, \mathbf{k}}^{-1}$. It is in fact unnecessary to estimate these eigenvalues individually since it is clear from the discussion above that all we need are the two sums: $\sum_{j=1}^{\mathbf{k}} \lambda_j$ and $\sum_{j=1}^{\mathbf{k}} \lambda_j^2$, which are equal to $\text{tr}(D_{F_0, \mathbf{k}} H_{F_0, \mathbf{k}}^{-1})$ and $\text{tr}((D_{F_0, \mathbf{k}} H_{F_0, \mathbf{k}}^{-1})^2)$, respectively, by matrix algebra identities. These can be constructed in a plug-in manner once we have estimates \hat{D}_n and \hat{H}_n for $D_{F_0, \mathbf{k}}$ and $H_{F_0, \mathbf{k}}$. When $\ell(Z; \cdot)$ is differentiable, we let

$$\hat{D}_n = n^{-1} \sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \hat{\alpha}_{\mathbf{k}}) \ell_{\alpha, \mathbf{k}}(Z_i; \hat{\alpha}_{\mathbf{k}})' \text{ and } \hat{H}_n = n^{-1} \sum_{i=1}^n \frac{\partial^2 \ell(Z_i; \hat{\alpha}_{\mathbf{k}})}{\partial \beta_{\mathbf{k}} \partial \beta_{\mathbf{k}}'}. \quad (3.17)$$

The score functions $\ell_{\alpha, \mathbf{k}}(Z_i; \hat{\alpha}_{\mathbf{k}})$ and estimators of the Hessian matrix are available case by case in the literature when differentiability does not hold. For example, suitable choices for the nonparametric quantile regression example are given in Belloni et al. (2011).

The two-sided test and the one-sided test of H_0 in (2.1) of nominal size $p \in (0, 1)$ are, therefore,

$$\varphi_n^{2\text{-sided}}(p) = 1\{|T_n| > z_{1-p/2}\} \text{ and } \varphi_n^{1\text{-sided}}(p) = 1\{T_n > z_{1-p}\} \quad (3.18)$$

respectively. The test does not select a better fitting model when it does not reject the null hypothesis. Such indeterminacy reflects the data fact that the fit of the two models are not statistically significantly different. In practice, if a model must be selected, one needs to analyze other, perhaps nonstatistical, features of the models. Often times the researcher has a preferred model based on features such as dimensionality and interpretability, and can set that one as the benchmark model. The benchmark model is selected when the null of equal fit is not rejected.

We show the uniform asymptotic validity of the above tests in the next section. Specifically, we show that:

$$\lim_{n \rightarrow \infty} \inf_{F_0 \in \mathcal{F}_0} E_{F_0}[\varphi_n(p)] = \lim_{n \rightarrow \infty} \sup_{F_0 \in \mathcal{F}_0} E_{F_0}[\varphi_n(p)] = p, \quad (3.19)$$

where $\varphi_n = \varphi_n^{2\text{-sided}}$ or $\varphi_n = \varphi_n^{1\text{-sided}}$, and \mathcal{F}_0 is the set of data generating processes (DGPs) that

the null hypothesis and the assumptions (given below) allow, which shows that the tests that we propose are asymptotically size-exact and similar.

4 Uniform Asymptotic Validity

In this section, we establish the uniform asymptotic validity and the local power of our test under high-level assumptions. These assumptions are verified in a nonparametric mean-regression example and in a quantile-regression example in Supplemental Appendices C and D respectively.

We begin by stating the regularity conditions on the DGP space \mathcal{F} and null DGP space \mathcal{F}_0 . In the assumptions below, $\{\xi_{\mathbf{k}}\}_{\mathbf{k}}$ is an array of non-decreasing positive numbers which may diverge with $|\mathbf{k}| = k_1 + k_2$, and may not depend on F_0 .

Assumption 4.1 *The set \mathcal{F} is the set of F_0 's such that*

- (a) $\{Z_i\}_{i \geq 1}$ are i.i.d. draws from F_0 ;
- (b) for every \mathbf{k} , $E_{F_0}[\ell(Z; \alpha(\beta_{\mathbf{k}}))]$ is twice-differentiable in $\beta_{\mathbf{k}}$;
- (c) the sieve approximator $\alpha_{\mathbf{k}}^*$ satisfies $E_{F_0}[\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] = \mathbf{0}_{|\mathbf{k}|}$ for every \mathbf{k} ;
- (d) $E_{F_0}[\ell(Z; \alpha_{F_0}^*)^2] < C$, and for every \mathbf{k} , $E_{F_0}[\|\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)\|^4] \leq C\xi_{\mathbf{k}}|\mathbf{k}|$;
- (e) $E_{F_0}\left[\left| \frac{\ell(Z; \alpha_{F_0}^*) - E_{F_0}(\ell(Z; \alpha_{F_0}^*))}{\omega_{F_0, *}} \right|^4\right] < C$ whenever $\omega_{F_0, *}^2 \equiv \text{Var}_{F_0}[\ell(Z; \alpha_{F_0}^*)] > 0$;
- (f) for $j = 1, 2$, $-C \leq \rho_{\min}(H_{F_0, k_j}) \leq \rho_{\max}(H_{F_0, k_j}) \leq -C^{-1}$ and $\rho_{\max}(D_{F_0, \mathbf{k}}) \leq C$ for all \mathbf{k} .

Assumption 4.2 $\mathcal{F}_0 = \{F_0 \in \mathcal{F} : E_{F_0}[\ell(Z; \alpha_{F_0}^*)] = 0\}$.

Assumption 4.1(b) ensures that the matrix $H_{F_0, \mathbf{k}}$ in (3.10) is well defined. Assumption 4.1(c) generally follows from the first order optimality condition of $\alpha_{\mathbf{k}}^*$. Let $\lambda_{F_0, 1}, \dots, \lambda_{F_0, |\mathbf{k}|}$ denote the $|\mathbf{k}|$ eigenvalues of $D_{F_0, \mathbf{k}}^{1/2} H_{F_0, \mathbf{k}}^{-1} D_{F_0, \mathbf{k}}^{1/2}$, and let

$$\sigma_{F_0, n}^2 \equiv \omega_{F_0, *}^2 + (2n^2)^{-1}(n-1)\omega_{F_0, U, \mathbf{k}}^2 \quad (4.1)$$

where $\omega_{F_0, U, \mathbf{k}}^2 \equiv \sum_{j=1}^{|\mathbf{k}|} \lambda_{F_0, j}^2 \equiv \text{tr}((D_{F_0, \mathbf{k}} H_{F_0, \mathbf{k}}^{-1})^2)$. Assumptions 4.1(d) and (f) together ensure that $\omega_{F_0, *}^2$, $D_{F_0, \mathbf{k}}$, $\omega_{F_0, U, \mathbf{k}}^2$, and $\sigma_{F_0, n}^2$ are well defined. The array $\xi_{\mathbf{k}}$ depends on the approximating function used. For example, it is the order of k_j^2 on the j th direction if power series is used for model j , and it is the order of k_j if Fourier or spline series is used. Assumption 4.1(e) implies the Linderberg condition on the pseudo-density ratio.

The definition of the supremum (infimum) operator implies that, to show the uniformity results (3.19), it is sufficient to consider all sequences of DGPs $\{F_n\}_{n \geq 1}$ in \mathcal{F}_0 . Moreover, to study the local power properties, we need to consider sequences of DGPs $\{F_n\}_{n \geq 1}$ in $\mathcal{F} \setminus \mathcal{F}_0$. In general, we consider sequences $\{F_n\}_{n \geq 1}$ in \mathcal{F} . For any $F_n \in \mathcal{F}$, we let $\alpha_{j, n}^*$ abbreviate $\alpha_{F_n, j}^*$, and let α_n^* abbreviate $(\alpha_{1, n}^*, \alpha_{2, n}^*)$. Let $\bar{\ell}_{\alpha, n}(\alpha) = n^{-1} \sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha)$ for any $\alpha \in \mathcal{A}$.

Assumption 4.3 Under any sequence of DGP's $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have

- (a) $\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) = \bar{\ell}_n(\alpha_n^*) - 2^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*) + o_p(n^{-1/2} \sigma_{F_n, n})$;
- (b) $(n \sigma_{F_n, n}^2)^{-1} = o(1)$ and $|\mathbf{k}| \xi_{\mathbf{k}} (n^2 \sigma_{F_n, n}^2)^{-1} = o(1)$.

Assumption 4.3(a) is a second order expansion of $\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}})$ around α_n^* . We verify this assumption in the nonparametric mean regression example (Supplemental Appendix C) and the nonparametric quantile regression example (Supplemental Appendix D). With the formula of this expansion, we can add more details to the heuristic discussion in Section 3. The variance of the leading term, $n^{-1} \omega_{F_n, * }^2$, in the expansion comes from estimating the expectation, and the variance of the second term, approximately $2^{-1} n^{-2} \omega_{F_n, U, \mathbf{k}}^2$, comes from estimating α_n^* . The quantity $\omega_{F_n, * }^2$ can be either zero or positive in the overlapping nonnested case. Indeed, it can converge to zero at any rate in that case. On the other hand, the quantity $\omega_{F_n, U, \mathbf{k}}^2$ typically is nonzero.¹² The relative magnitude of the two terms is proportional to $\frac{n \omega_{F_n, * }^2}{\omega_{F_n, U, \mathbf{k}}^2}$, which can be zero or positive. It is such ambiguity of the relative asymptotic order of the two expansion terms that makes a uniformly valid test difficult to construct.¹³

Assumption 4.3(b) is an important condition for the uniform asymptotic validity of our test. The first part of it ensures that the approximation residual in Assumption 4.3 (a) diminishes at a fast enough rate as the sample size grows. The second part of the assumption allows us to apply a U-statistic central limit theorem to the quadratic term $2^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*)$. To understand this assumption, note that $\sigma_{F_n, n}^2 = \omega_{F_n, * }^2 + (2n^2)^{-1} (n-1) \omega_{F_n, U, \mathbf{k}}^2$. If $\omega_{F_n, * }^2$ is bounded below by a positive constant (as is typical for strictly nonnested models), Assumption 4.3(b) is satisfied as long as $|\mathbf{k}| \xi_{\mathbf{k}} n^{-2} = o(1)$ as $n \rightarrow \infty$, which simply requires that the number of sieve terms not to grow too fast. Otherwise, Assumption 4.3(b) imposes restriction on the U-statistic variance $\omega_{F_n, U, \mathbf{k}}^2 \equiv \text{tr} \left((H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}})^2 \right)$. Specifically, it requires, as $n \rightarrow \infty$, that

$$\omega_{F_n, U, \mathbf{k}}^2 \rightarrow \infty \text{ and } |\mathbf{k}| \xi_{\mathbf{k}} (n \omega_{F_n, U, \mathbf{k}}^2)^{-1} = o(1). \quad (4.2)$$

This is satisfied if $|\mathbf{k}|$ grows with n and there are not too many near zero eigenvalues for the matrix $H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}}$. Both can be assessed in practice because \mathbf{k} is user-chosen and $H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}}$ can be consistently estimated. Moreover, the requirement that $|\mathbf{k}|$ grows with n is natural and

¹²For example, consider $\mathcal{M}_1: Y = X_1' \beta_1 + X_2' \beta_2 + u$ and $\mathcal{M}_2: Y = X_1' \beta_1 + u$. Suppose that $X = (X_1', X_2')'$ is uncorrelated with u and $E_{F_0}[XX'] = I_{|\mathbf{k}|}$ for simplicity. The null hypothesis H_0 is equivalent to $\beta_2 = 0$ and there is $\ell(Z; \alpha_n^*) = 0$ under H_0 as a result. Yet, $2^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha,n}(\alpha_n^*) = 2^{-1} n^{-2} \sum_{i=1}^n \sum_{j=1}^n u_i u_j X_{2,i}' X_{2,j}$ which is clearly not degenerate. See Hong and White (1995) for more sophisticated examples.

¹³Ambiguity of this type also arises in the analysis of weak instruments and weak identification, where the common techniques include pretesting with conservative critical value, Anderson-Rubin type robust procedures, and conditional likelihood inference. The first two in general do not yield asymptotically similar tests, indicating power loss under some data generating processes, while the last one is not a general technique that can be applied here.

necessary in the literature of series estimation of semi/nonparametric models.¹⁴

Under the above assumptions, the following intermediate result holds.

Theorem 4.1 *Suppose that Assumptions 4.1 and 4.3 hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

$$\frac{n(\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - E_{F_n}[\ell(Z; \alpha_n^*)]) + (1/2)\text{tr}(\hat{D}_n(\alpha_{\mathbf{k}}^*)H_{F_n, \mathbf{k}}^{-1})}{n^{1/2}\sigma_{F_n, n}} \rightarrow_d N(0, 1), \quad (4.3)$$

where $\hat{D}_n(\alpha_{\mathbf{k}}^*) = n^{-1} \sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'$.

Remark 1 *Note that Theorem 4.1 applies whether or not $F_n \in \mathcal{F}_0$. In the case that $F_n \in \mathcal{F}_0$ for all n , it again covers two special sub-cases: (i) The statistic $n^{1/2}\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}})$ is non-degenerate ($F_n = F$ for some F and for all n , and $\omega_{F, *}^2 > 0$); (ii) the statistic $n^{1/2}\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}})$ is degenerate ($F_n = F$ for some F and for all n , and $\omega_{F, *}^2 = 0$). More importantly, it allows $\omega_{F_n, *}^2$ to converge to zero at all rates, and thus covers all types of DGP sequences in the overlapping nonnested case.*

When $\omega_{F_n, *}^2$ converges to zero at an equal or faster rate than n^{-1} or is exactly zero, the asymptotic normality in (4.3) is achieved by the central limit theorem of U-statistic which requires that $|\mathbf{k}|$ grows with n . The normal approximation of the U-statistic is widely used in the literature of model specification test. See e.g., Hall (1984), Hong and White (1995), Horowitz and Härdle (1994), Fan and Li (1996), Aït-Sahalia et al. (2001) and Donald et al. (2003). Theorem 4.1 shares similar features with the results in these papers, in that they also require the number of approximating functions to diverge with n or the bandwidth of kernel functions to go to zero with n .

In order to use the intermediate result in Theorem 4.1, we need to construct consistent estimators of $\hat{D}_n(\alpha_{\mathbf{k}}^*)$, $H_{F_n, \mathbf{k}}$, and $\sigma_{F_n, n}^2$. The estimators that we consider are respectively the \hat{D}_n , the \hat{H}_n , and the $\hat{\sigma}_n^2$ defined in the previous section. Assumption 4.4 below ensures their consistency. In this assumption, $\delta_n = \min\{n^{1/2}\sigma_{F_n, n}|\mathbf{k}|^{-1}, 1\}$, and $\ell_F(\alpha) = E_F[\ell(Z; \alpha)]$ for all $F \in \mathcal{F}$ and $\alpha \in \mathcal{A}$.

Assumption 4.4 *Under any sequence of DGP's $\{F_n\}_{n \geq 1}$ with $F_n \in \mathcal{F}$ for all n , we have:*

- (a) $\|\hat{H}_n - H_{F_n, \mathbf{k}}\| = o_p(\delta_n)$, $\|\hat{D}_n - \hat{D}_n(\alpha_{\mathbf{k}}^*)\| = o_p(\delta_n)$ and $\|\hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}\| = o_p(\delta_n)$;
- (b) $n^{-1} \sum_{i=1}^n |\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_n^*)|^2 = \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)'(H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}) \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) + o_p(\sigma_{F_n, n}^2)$;
- (c) $n^{-1} \sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) [\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_n^*)] = o_p(\sigma_{F_n, n}^2)$;
- (d) $|\mathbf{k}| n^{-1} = o(1)$.

¹⁴The asymptotic theory established in this paper also provides a good approximation for the comparison of parametric models with fixed but large $|\mathbf{k}|$. Simulation results in Supplemental Appendix F show that our test works well even when $|\mathbf{k}|$ is only 7.

Conditions in Assumption 4.4 are verified in the nonparametric mean-regression example in Supplemental Appendix C. Under this assumption, we can easily show that the large sample bias of $n\bar{\ell}_n(\hat{\alpha}_{\mathbf{k},n})$ can be estimated up to the appropriate rate:

Lemma 4.1 *Suppose that Assumptions 4.1(c) and (e)-(g), and 4.4(a) hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

$$\text{tr}(\widehat{D}_n \widehat{H}_n^{-1}) - \text{tr}(\widehat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1}) = o_p(n^{1/2} \sigma_{F_n, n}).$$

Next, we derive the convergence of $\widehat{\sigma}_n^2$. First, we show the convergence of $\widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}})$ in the following lemma.

Lemma 4.2 *Suppose that Assumptions 4.1, 4.3 and 4.4 hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

$$|\widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}}) - (\omega_{F_n, *}^2 + n^{-1} \omega_{F_n, U, \mathbf{k}}^2)| = o_p(\sigma_{F_n, n}^2).$$

Remark 2 *Note that $\widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}})$ may be viewed as a sample-analogue estimator of $\omega_{F_n, *}^2$. Lemma 4.2 shows that, in general, $\widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}})$ over-estimates $\omega_{F_n, *}^2$. In fact, it even over-estimates the overall asymptotic variance of the size-corrected quasi-likelihood ratio statistic: $\sigma_{F_n, n}^2$, by $(2n^2)^{-1}(n+1)\omega_{F_n, U, \mathbf{k}}^2$. The upward bias is due to the estimation error in $\widehat{\alpha}_{\mathbf{k}}$.*

Lemma 4.2 suggests that $\sigma_{F_n, n}^2$ can be consistently estimated by estimating and then removing the large-sample bias $(2n^2)^{-1}(n+1)\omega_{F_n, U, \mathbf{k}}^2$ from $\widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}})$. This motivates the estimator $\widehat{\sigma}_n^2$ defined in the previous section. In the definition of $\widehat{\sigma}_n^2$, $\text{tr}((\widehat{D}_n \widehat{H}_n^{-1})^2)$ is used to estimate $\omega_{F_n, U, \mathbf{k}}^2$. The lemma below shows that this estimator of $\omega_{F_n, U, \mathbf{k}}^2$ is consistent in an appropriate sense, and so is the resulting bias-removed estimator of $\sigma_{F_n, n}^2$.

Lemma 4.3 *Suppose that Assumptions 4.1, 4.3 and 4.4 hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

- (a) $\text{tr}((\widehat{D}_n \widehat{H}_n^{-1})^2) - \omega_{F_n, U, \mathbf{k}}^2 = o_p(n \sigma_{F_n, n}^2)$, and
- (b) $\widehat{\omega}_n^2 - \sigma_{F_n, n}^2 = o_p(\sigma_{F_n, n}^2)$, where $\widehat{\omega}_n^2 = \widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}}) - (2n)^{-1} \text{tr}((\widehat{D}_n \widehat{H}_n^{-1})^2)$ as defined in (3.13).

Lemma 4.3 is used to show the consistency of $\widehat{\sigma}_n^2$: $\widehat{\sigma}_n^2 - \sigma_{F_n, n}^2 = o_p(\sigma_{F_n, n}^2)$. This along with Theorem 4.1 and Lemmas 4.1–4.2 immediately leads to the uniform asymptotic size control and the asymptotic similarity results in (3.19). These results also immediately lead to a local power formula because the assumptions used for them do not require $F_n \in \mathcal{F}_0$. These are summarized in the theorem below.

Theorem 4.2 *Suppose that Assumptions 4.1-4.4 hold. Then:*

(a) (3.19) holds for $\varphi_n = \varphi_n^{2\text{-sided}}$ and $\varphi_n = \varphi_n^{1\text{-sided}}$.

(b) Under any sequence $F_n \in \mathcal{F}$ such that $F_n \rightarrow F_0$ for some $F_0 \in \mathcal{F}_0$ in the Kolmogorov-Smirnov distance, and that $n^{1/2} E_{F_n} [\ell(Z; \alpha_n^*)] / \sigma_{F_n, n} \rightarrow c$ for some constant $c \in R$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} E_{F_n} [\varphi_n^{2\text{-sided}}(p)] &= 2 - \Phi(z_{1-p/2} - c) - \Phi(z_{1-p/2} + c), \text{ and} \\ \lim_{n \rightarrow \infty} E_{F_n} [\varphi_n^{1\text{-sided}}(p)] &= 1 - \Phi(z_{1-p} - c), \end{aligned}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

Remark 3 *Note that $\sigma_{F_n, n} = O(1)$, and it can be $o(1)$ when $\omega_{F_n, *}^2 \rightarrow 0$. Thus, part (b) of the theorem implies that the test has nontrivial power against all local alternatives with $E_{F_n} [\ell(Z; \alpha_n^*)]$ converging to 0 at the rate $n^{1/2}$, and against alternatives with $E_{F_n} [\ell(Z; \alpha_n^*)]$ converging to 0 at a rate faster than $n^{1/2}$ if $\omega_{F_n, *}^2 \rightarrow 0$. Such power property is not shared by a pre-test based model selection test like that in Shi (2015a), or a model selection test that uses added noise to augment the variance either through sample splitting or other means.*

Remark 4 *As we have discussed, Shi (2015b) proposes a nondegenerate test for the parametric case. Her test statistic, if directly applied to the sieve approximation of the semi/nonparametric models, takes the following form*

$$T_n^{\text{para}}(c) = \frac{n\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) + 2^{-1}\text{tr}(\hat{D}_n \hat{H}_n^{-1})}{n^{1/2} \left(\hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) + cn^{-1}\text{tr}((\hat{D}_n \hat{H}_n^{-1})^2) \right)^{1/2}}, \quad (4.4)$$

where $c \geq 0$ is a tuning parameter. Compared with $T_n^{\text{para}}(c)$, our test statistic T_n has the same numerator but a different denominator. By Lemma 4.3(b), $\frac{\hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}})}{\sigma_{F_n, n}^2} - \frac{(2n)^{-1}\text{tr}((\hat{D}_n \hat{H}_n^{-1})^2)}{\sigma_{F_n, n}^2} \rightarrow_p 1$, which implies that $\hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) > (2n)^{-1}\text{tr}((\hat{D}_n \hat{H}_n^{-1})^2)$ with probability approaching one. This and the definition of $\hat{\sigma}_n^2$ together imply that $\hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) \geq \hat{\sigma}_n^2$ with probability approaching one, which in turn implies that $|T_n^{\text{para}}(c)| \leq |T_n|$ with probability approaching one for any $c \geq 0$. On the other hand, the critical value of the test proposed in Shi (2015b) by construction is not smaller than the critical value of our test. Therefore the asymptotic theory established in this section automatically justifies the test proposed in Shi (2015b) in terms of asymptotic size control when applied to the semi/nonparametric models. However, when $|\mathbf{k}|$ is large, there are a large number of nuisance parameters (which are not consistently estimable) for Shi's (2015b) approach to consider, which makes it difficult to use. In contrast, our test is much easier to use, also has asymptotic size control, and has better power in the semi/nonparametric setting, where the better power is implied by its bigger test statistic and smaller critical value. Moreover, the asymptotic standard normal

distribution of our test statistic T_n also makes the post model selection inference easy in practice as we discuss in later sections.

5 Example: Semi/Nonparametric Mean-Regression

In this section we illustrate the construction of our test using the nonparametric mean-regression example. We verify the high-level assumptions in this example in Supplemental Appendix C. Another illustrating example—quantile-regression—is given in Supplemental Appendix D, where we also verify the high level assumptions.

For $j = 1, 2$, model j is to maximize $E_{F_0}[-2^{-1}|Y - \alpha_j(X_j)|^2]$ over $\alpha_j \in \mathcal{A}_j$, where $\alpha_j(x)$ is a possibly infinite dimensional parameter, \mathcal{A}_j is its parameter space, and F_0 denotes the joint distribution of $Z \equiv (Y, X_1, X_2)$. The regressors X_1 and X_2 of the two models may be nested, overlapping, or strictly non-nested sets of variables. Even when the regressors are strictly nonnested sets of variables (i.e., there are no common regressors across the two regressions), the two regression models are still overlapping according to the definitions in Section 2.2 because it is possible that $\alpha_1(X_1) = \alpha_2(X_2) = \text{Constant}$.

The model studied in this section covers a richer class of models than it looks. Depending on what one sets \mathcal{A}_j to be, it can represent a fully nonparametric mean-regression model, a partial linear model, a separable model, or a parametric linear model. See below for an example. We do not require that there exists an $\alpha_j \in \mathcal{A}_j$ such that $\alpha_j(X_j) = E_{F_0}[Y|X_j]$ a.s.

The sieve approximating functions for this case have to do with the structure of \mathcal{A}_j . For example, suppose that we have a partial linear model $\alpha_j(X_j) = \beta_1 X_{j,1} + g(X_{j,2})$. Then, we should let $P_{k_j}(X_j) = [p_{j,1}(X_j), \dots, p_{j,k_j}(X_j)]'$ such that $p_{j,1}(X_j) = X_{j,1}$ and the rest of the sequence of $p_{j,\ell}(X_j)$'s be an appropriate sieve approximation of $g(X_{j,2})$, such as splines or polynomials on $X_{j,2}$.

The sieve M-estimator is simply the sieve least squares estimator:

$$\widehat{\alpha}_{k_j}(\cdot) = P_{k_j}(\cdot)' \widehat{\beta}_{k_j} \quad \text{with} \quad \widehat{\beta}_{k_j} = (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-1} \mathbf{P}'_{k_j,n} \mathbf{Y}_n, \quad (5.1)$$

where $\mathbf{P}_{k_j,n} = [P_{k_j}(X_{j,1}), \dots, P_{k_j}(X_{j,n})]'$ for $j = 1, 2$, and $\mathbf{Y}_n = (Y_1, \dots, Y_n)'$. The link function is

$$\ell(Z; \alpha) = 2^{-1}|Y - \alpha_2(X_2)|^2 - 2^{-1}|Y - \alpha_1(X_1)|^2. \quad (5.2)$$

Using the above two displays, the pseudo-likelihood ratio and the standard error statistics can be constructed easily following (3.6).

The pseudo true parameter $\alpha_j^*(\cdot)$ is defined as $\alpha_j^* = \arg \max_{\alpha_j \in \mathcal{A}_j} E_{F_0}[-2^{-1}|Y - \alpha_j(X_j)|^2]$, which depends on the functional form restrictions imposed on the parameter space \mathcal{A}_j . If there is no functional form restriction, then $\alpha_j^*(X_j) = E_{F_0}[Y|X_j]$. If an additive form is imposed,

i.e., $\alpha_j(X_j) = g(X_{j,1}) + \dots + g(X_{j,q})$ for some finite q , the pseudo true parameter exists and is unique under general conditions (see Condition 1 and Lemma 1 in Stone (1985)). When a partially linear form is imposed, i.e., $\alpha_j(X_j) = X'_{j,1}\beta_1 + g(X_{j,2})$, then the pseudo true parameter $\alpha_j^*(X_j) = X'_{j,1}\beta_1^* + g^*(X_{j,2})$ where

$$\beta_1^* = (E_{F_0} [X_{j,1}^* X_{j,1}^{*'}])^{-1} E_{F_0} [X_{j,1}^* Y^*] \text{ and } g^*(X_{j,2}) = E_{F_0} [Y - X'_{j,1}\beta_1^* | X_{j,2}] \quad (5.3)$$

where $X_{j,1}^* = X_{j,1} - E_{F_0} [X_{j,1} | X_{j,2}]$ and $Y^* = Y - E_{F_0} [Y | X_{j,2}]$.

Let $u_{k_j} = Y - \alpha_{k_j}^*(X_j)$, where $\alpha_{k_j}^*(\cdot) = P_{k_j}(\cdot)' \beta_{k_j, F_0}^*$ and

$$\beta_{k_j, F_0}^* = \arg \min_{\beta_{k_j} \in R^{k_j}} E_{F_0} \left[|Y - P_{k_j}(X_j)' \beta_{k_j}|^2 \right]. \quad (5.4)$$

By the first order optimality condition for $u_{k_j} = Y - P_{k_j}(X_j)' \beta_{k_j, F_0}^*$, we have $E_{F_0} [u_{k_j} P_{k_j}(X_j)] = \mathbf{0}_{k_j \times 1}$. With the sieve approximation in (3.1), $\ell(Z; \alpha(\beta_{\mathbf{k}}))$ is differentiable in $\beta_{\mathbf{k}}$. Thus, the score function can be obtained by the chain rule:

$$\ell_{\alpha, \mathbf{k}}(Z; \alpha) = ((Y - \alpha_1(X_1))P_{k_1}(X_1)', -(Y - \alpha_2(X_2))P_{k_2}(X_2)')'. \quad (5.5)$$

Then, the expectation of the outer product of the score function evaluated at $\alpha_{\mathbf{k}}^*$ is

$$D_{F_0, \mathbf{k}} = \begin{pmatrix} E_{F_0} [u_{k_1}^2 P_{k_1}(X_1) P_{k_1}(X_1)'] & -E_{F_0} [u_{k_1} u_{k_2} P_{k_1}(X_1) P_{k_2}(X_2)'] \\ -E_{F_0} [u_{k_1} u_{k_2} P_{k_2}(X_2) P_{k_1}(X_1)'] & E_{F_0} [u_{k_2}^2 P_{k_2}(X_2) P_{k_2}(X_2)'] \end{pmatrix}, \quad (5.6)$$

and the population Hessian matrix is:

$$H_{F_0, \mathbf{k}} = \text{diag}(-E_{F_0} [P_{k_1}(X_1) P_{k_1}(X_1)'], E_{F_0} [P_{k_2}(X_2) P_{k_2}(X_2)']). \quad (5.7)$$

It is natural to use the plug-in estimators of $D_{F_0, \mathbf{k}}$ and $H_{F_0, \mathbf{k}}$:

$$\widehat{D}_{n, \mathbf{k}} = \begin{pmatrix} n^{-1} \sum_{i=1}^n \widehat{u}_{1,i}^2 P_{k_1}(X_{1,i}) P_{k_1}(X_{1,i})' & -n^{-1} \sum_{i=1}^n \widehat{u}_{1,i} \widehat{u}_{2,i} P_{k_1}(X_{1,i}) P_{k_2}(X_{2,i})' \\ -n^{-1} \sum_{i=1}^n \widehat{u}_{1,i} \widehat{u}_{2,i} P_{k_2}(X_{2,i}) P_{k_1}(X_{1,i})' & n^{-1} \sum_{i=1}^n \widehat{u}_{2,i}^2 P_{k_2}(X_{2,i}) P_{k_2}(X_{2,i})' \end{pmatrix}, \quad (5.8)$$

where the residual $\widehat{u}_{j,i} = Y_i - \widehat{\alpha}_{k_j}(X_{j,i})$; and

$$\widehat{H}_{n, \mathbf{k}} = \text{diag} \left(-n^{-1} \sum_{i=1}^n P_{k_1}(X_{1,i}) P_{k_1}(X_{1,i})', n^{-1} \sum_{i=1}^n P_{k_2}(X_{2,i}) P_{k_2}(X_{2,i})' \right). \quad (5.9)$$

Finally, the test statistic may be constructed easily using the above quantities following (3.16).

6 Uniformly Valid Post Selection Test Inference

Up to this point, we have focused on how to properly conduct model selection that takes into account sample noise. Sometimes, model selection is the sole purpose of a research project (e.g., Coate and Conlin (2004) and Gandhi and Serrano-Padial (2015)). But, sometimes, one is also interested in the model parameters that are estimated using the same data set on which the model selection test is conducted. Leeb and Pötscher (2005) show the size-distortion of naive post-model-selection (PMS) inference that does not account for the randomness of model selection. Uniformly valid post model selection test inference procedures for possibly misspecified semi/nonparametric models have not been developed in the literature.

The QLR model selection test framework treats the parameters in the two models as separate parameters in the sense that there is no across-model restrictions. In practice, while some parameters of a model may only have meaningful interpretation in its own model environment, it is also possible that a parameter from one model and a parameter from the other model represent the same economic parameter of interest. Thus, we treat these two different scenarios separately when considering post model selection test inference.

In the first scenario, the parameter of interest is only well-defined in model \mathcal{M}_j ($j = 1$ or 2), and the researcher is interested in it only when \mathcal{M}_j is selected by the model selection test. In this scenario, we would like to make the inference conditional on the event that \mathcal{M}_1 is selected. Leeb and Pötscher (2006) pointed out that in general it is impossible to approximate the conditional distribution of the parameter estimator given that the model is selected. Instead of studying the conditional distribution, we take a different route, and construct confidence interval for the parameter using a conditionally asymptotically pivotal statistic. We devote subsection 6.2 to this approach.

In the second scenario, the parameter of interest, θ , is well-defined in both models: it equals $\psi_1(\alpha_1)$ in model \mathcal{M}_1 and equals $\psi_2(\alpha_2)$ in model \mathcal{M}_2 for two known functionals $\psi_1 : \mathcal{A}_1 \rightarrow R$ and $\psi_2 : \mathcal{A}_2 \rightarrow R$. Its (pseudo)-true value is determined by the better fitting model:

$$\theta^* = \psi_1(\alpha_1^*)1(f(\mathcal{M}_1, F_0) \geq f(\mathcal{M}_2, F_0)) + \psi_2(\alpha_2^*)1(f(\mathcal{M}_1, F_0) < f(\mathcal{M}_2, F_0)). \quad (6.1)$$

For example, if the competing models are two regression models, θ^* could be the expected point prediction from the better fitting model. We devote subsection 6.3 below to this problem.

To prepare for subsections 6.2 and 6.3, we let $\psi_1(\alpha_1^*)$ and $\psi_2(\alpha_2^*)$ be estimated by the plug-in estimators $\psi_1(\widehat{\alpha}_{k_1})$ and $\psi_2(\widehat{\alpha}_{k_2})$ respectively. Both subsections 6.2 and 6.3 rely on the joint normal limiting distribution of $(\psi_1(\widehat{\alpha}_{k_1}), \psi_2(\widehat{\alpha}_{k_2}), \bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}))'$ (after proper re-centering and rescaling), which we derive in the next subsection.

6.1 Preliminaries

We first introduce some notation. Let $\ell_{\alpha,k_1}(Z; \alpha_1)$ denote the sub-vector of the first k_1 coordinates of $\ell_{\alpha,\mathbf{k}}(Z; \alpha)$, and let $\ell_{\alpha,k_2}(Z; \alpha_2)$ denote minus the sub-vector of the last k_2 coordinates of $\ell_{\alpha,\mathbf{k}}(Z; \alpha)$. Let $D_{F_0,k_j} = E_{F_0}[\ell_{\alpha,k_j}(Z; \alpha_{F_0,j}^*)\ell_{\alpha,k_j}(Z; \alpha_{F_0,j}^*)']$ for $j = 1, 2$. Also define

$$\psi_{\alpha,k_j}(\alpha_j) = \frac{\partial \psi_j(\alpha_j(\beta_{k_j}))}{\partial \beta_{k_j}} \text{ and } v_{\psi,k_j}^* = \left(\psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_0,k_j}^{-1} D_{F_0,k_j} H_{F_0,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right)^{1/2}, \quad (6.2)$$

where v_{ψ,k_j}^* is the well-established formula for the asymptotic standard deviation of functionals of sieve-M estimator.

Let \widehat{v}_{ψ,k_j}^* denote the estimator of v_{ψ,k_j}^* which is defined as

$$\widehat{v}_{\psi,k_j}^{*2} = \psi_{\alpha,k_j}(\widehat{\alpha}_{k_j})' \widehat{H}_{k_j,n}^{-1} \widehat{D}_{k_j,n} \widehat{H}_{k_j,n}^{-1} \psi_{\alpha,k_j}(\widehat{\alpha}_{k_j})$$

where $\widehat{H}_{k_j,n}$ and $\widehat{D}_{k_j,n}$ are the leading $k_j \times k_j$ submatrices of \widehat{H}_n and \widehat{D}_n respectively for $j = 1$, and the last $k_j \times k_j$ submatrices of $-\widehat{H}_n$ and \widehat{D}_n respectively for $j = 2$.

We shall derive the asymptotic distribution of

$$\widehat{G}_{n,F_n} \equiv \begin{pmatrix} \frac{n[\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) - E_{F_n}(\ell(Z; \alpha_n^*))] + (1/2)\text{tr}(\widehat{D}_n \widehat{H}_n^{-1})}{n^{1/2} \widehat{\sigma}_n} \\ n^{1/2} [\psi_1(\widehat{\alpha}_{k_1}) - \psi_1(\alpha_{1,n}^*)] (\widehat{v}_{\psi,k_1}^*)^{-1} \\ n^{1/2} [\psi_2(\widehat{\alpha}_{k_2}) - \psi_2(\alpha_{2,n}^*)] (\widehat{v}_{\psi,k_2}^*)^{-1} \end{pmatrix}. \quad (6.3)$$

For this purpose, define the correlation coefficients

$$\begin{aligned} \rho_{0j,F_0} &= \psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_0,k_j}^{-1} E_{F_0} \left[\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell(Z; \alpha_n^*) \right] (v_{\psi,k_j}^* \sigma_{F_0,n})^{-1} \text{ for } j = 1, 2, \\ \rho_{12,F_0} &= \psi_{\alpha,k_1}(\alpha_{k_1}^*)' H_{F_0,k_1}^{-1} D_{F_0,k_1,k_2} H_{F_n,k_2}^{-1} \psi_{\alpha,k_2}(\alpha_{k_2}^*) (v_{\psi,k_1}^* v_{\psi,k_2}^*)^{-1}, \end{aligned} \quad (6.4)$$

where $D_{F_0,k_1,k_2} = E_{F_0} [\ell_{\alpha,k_1}(Z; \alpha_{k_1}^*) \ell_{\alpha,k_2}(Z; \alpha_{k_2}^*)']$.

For any sequence $\{F_n\}_{n \geq 1}$, we write $\rho_{0j,n} = \rho_{0j,F_n}$ and $\rho_{12,n} = \rho_{12,F_n}$ for ease of notation. The following lemma gives the limiting distribution of \widehat{G}_{n,F_n} under an arbitrary sequence $F_n \in \mathcal{F}$, which extends the asymptotic distribution result in Section 4 to joint convergence.

Lemma 6.1 *Suppose that Assumptions 4.1, 4.3, and B.1-B.2 in Supplemental Appendix B hold. Then under any sequence $\{F_n\}_{n \geq 1}$ and any subsequence $\{u_n\}$ of $\{n\}$ such that with $F_n \in \mathcal{F}$ for all n , $\rho_{0j,u_n} \rightarrow \rho_{0j}$ and $\rho_{12,u_n} \rightarrow \rho_{12}$ for some ρ_{0j} and $\rho_{12} \in [-1, 1]$, we have*

$$\widehat{G}_{n,F_n} \rightarrow_d N(\mathbf{0}_3, \Sigma_G)$$

where Σ_G is symmetric with $\Sigma_G(i, i) = 1$ for $i = 1, 2, 3$, $\Sigma_G(i, i + 1) = \rho_{ij}$ for $i = 0, 1$ and $\Sigma_G(1, 3) = \rho_{02}$.

Lemma 6.1 follows immediately from Lemmas 4.2 and 4.3 in Section 4, and Lemmas B.1, and B.2 in Supplemental Appendix B and hence is omitted.

6.2 Conditional Inference for Model-Specific Parameters

In this subsection, we consider the conditional inference of a functional – denoted $\psi_1(\alpha_1^*)$ – of the parameter in model \mathcal{M}_1 given that \mathcal{M}_1 is selected.¹⁵ Specifically, we construct a level $1 - p$ conditional confidence interval, $CI_{\psi_1}(1 - p)$ such that

$$\liminf_{n \rightarrow \infty} \inf_{F_0 \in \mathcal{F}_n} \Pr_{F_0}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1 - p) | T_n \geq t) = 1 - p, \quad (6.5)$$

where \mathcal{F}_n is a sequence of subsets of \mathcal{F} defined below. Note that we allow c to be an arbitrary number, which the user can choose according to her interpretation of the event that \mathcal{M}_1 is selected.

To describe our conditional confidence interval, first define a function $\Psi : R \times (-\infty, \infty] \times [-1, 1] \rightarrow R$:

$$\Psi(c, h, \rho) = \begin{cases} [\Phi(c) - \Phi(c - h/\rho)] / [1 - \Phi(c - h/\rho)] & \text{if } \rho > 0 \text{ and } h \in R \\ \Phi(c) & \text{if } \rho = 0 \text{ or } h = \infty \\ \Phi(c)/\Phi(c - h/\rho) & \text{if } \rho < 0 \text{ and } h \in R. \end{cases} \quad (6.6)$$

For any $t \in R$ and $p \in (0, 1)$, let $c_{1,p}$ be the solution to the equation:

$$\Psi(c_{1,p}, T_n - t, \hat{\rho}_{01,n}) = p, \quad (6.7)$$

where $\hat{\rho}_{0j,n} = \psi_{\alpha, k_j}(\hat{\alpha}_{k_j})' \hat{H}_{k_j, n}^{-1} (\hat{v}_{\psi, k_j}^* \hat{\sigma}_n)^{-1} n^{-1} \sum_{i=1}^n \ell_{\alpha, k_j}(Z_i; \hat{\alpha}_{k_j}) \ell(Z_i; \hat{\alpha}_{\mathbf{k}})$, for $j = 1, 2$. This equation only needs to be solved when $T_n \geq t$ because the confidence interval is only needed then. The equation always has a unique solution when $T_n \geq t$ because $\Psi(c, h, \rho)$ is a strictly increasing function in θ with range $(0, 1)$, for any $h \geq 0$ and any $\rho \in [-1, 1]$. Our conditional confidence interval is of the form:

$$CI_{\psi_1}(1 - p) = [\psi_1(\hat{\alpha}_{k_1}) - n^{-1/2} c_{1, 1-p/2} \hat{v}_{\psi, k_1}^*, \psi_1(\hat{\alpha}_{k_1}) + n^{-1/2} c_{1, p/2} \hat{v}_{\psi, k_1}^*]. \quad (6.8)$$

These critical values depend on T_n and hence are not approximations of the conditional quantiles of $\sqrt{n}(\psi_1(\hat{\alpha}_{k_1}) - \psi_1(\alpha_1^*)) / \hat{v}_{\psi, k_1}^*$ given $T_n > t$. Therefore, the validity of our construction is

¹⁵Conditional inference for a functional of the parameter in model \mathcal{M}_2 given that \mathcal{M}_2 is selected is analogous and thus omitted.

not contradictory to the impossibility results in Leeb and Pötscher (2006). The construction of the critical values is inspired by the construction in Tibshirani et al. (2016) of valid p-values and confidence intervals for post Lasso inference in a linear regression context with known Gaussian noise.¹⁶ We generalize Tibshirani et al. (2016) to post model selection test inference for general semi-nonparametric models, and provide asymptotically exact confidence intervals without imposing special structure on the models compared or requiring knowledge about the variance-covariance Σ_G of the statistics \widehat{G}_{n,F_n} .

The formal justification of the above construction requires us to rule out the case where $n^{1/2}E_{F_n}[\ell(Z; \alpha_n^*)]/\widehat{\sigma}_n \rightarrow -\infty$ because in that case the conditioning event occurs with diminishing probability, and the conditional distribution of our test statistic becomes difficult to characterize. We rule out this troublesome case by considering

$$\mathcal{F}_n = \{F_0 \in \mathcal{F} : n^{1/2}E_{F_0}[\ell(Z; \alpha_n^*)]\sigma_n^{-1} - t \geq -C\}, \quad (6.9)$$

for some large $C > 0$. The formal validity result is stated as Theorem 6.1 below. The proof of this theorem is given in Appendix B.

Theorem 6.1 *Suppose that Assumptions 4.1, 4.3 and B.1–B.2 in Supplemental Appendix B hold. Then equation (6.5) holds with \mathcal{F}_n defined in (6.9).*

6.3 Inference for Common Parameters

In this subsection, we consider the inference for the parameter θ that equals $\psi_1(\alpha_1)$ in model \mathcal{M}_1 and $\psi_2(\alpha_2)$ in model \mathcal{M}_2 . Let $\ell_0 = f(\mathcal{M}_1, F_0) - f(\mathcal{M}_2, F_0)$. Then the pseudo-true value of θ is

$$\theta^* = \psi_1(\alpha_1^*)1(\ell_0 \geq 0) + \psi_2(\alpha_2^*)1(\ell_0 < 0). \quad (6.10)$$

Note that θ^* is a function of $(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0)$. Because this function is discontinuous, we cannot obtain uniformly asymptotically valid inference via the Delta method even though the vector $(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0)$ has an asymptotically jointly normal estimator by Lemma 6.1. Instead, we construct a confidence interval for θ^* by projecting a joint confidence set for $(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0^*)$.

We let the joint confidence set of $(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0)$ of confidence level $1-p$ to be all (x_1, x_2, x_0) such that

$$\widehat{G}_n(x_1, x_2, x_0)' \widehat{\Sigma}_G^{-1} \widehat{G}_n(x_1, x_2, x_0) \leq \chi_{1-p}^2(3), \quad (6.11)$$

¹⁶Asymptotically conservative one-sided inference is also available in Tibshirani et al. (2016) when the variance of the noise is unknown.

where $\chi_{1-p}^2(3)$ is the $1 - p$ quantile of the chi-squared distribution with three degrees of freedom,

$$\widehat{\Sigma}_G = \begin{pmatrix} 1 & \widehat{\rho}_{01,n} & \widehat{\rho}_{02,n} \\ \widehat{\rho}_{01,n} & 1 & \widehat{\rho}_{12,n} \\ \widehat{\rho}_{02,n} & \widehat{\rho}_{12,n} & 1 \end{pmatrix}, \text{ and } \widehat{G}_n(x_1, x_2, x_0) = \begin{pmatrix} T_n - n^{1/2}x_0/\widehat{\sigma}_n \\ n^{1/2}(\psi_1(\widehat{\alpha}_{1,n}) - x_1)/\widehat{v}_{\psi,k_1}^* \\ n^{1/2}(\psi_2(\widehat{\alpha}_{2,n}) - x_2)/\widehat{v}_{\psi,k_2}^* \end{pmatrix}$$

where $\widehat{\rho}_{0j,n}$ is defined in the previous subsection for $j = 1, 2$ and

$$\widehat{\rho}_{12,n} = \psi_{\alpha,k_1}(\widehat{\alpha}_{k_1})' \widehat{H}_{k_1,n}^{-1} \widehat{D}_{k_1,k_2,n} \widehat{H}_{k_2,n}^{-1} \psi_{\alpha,k_2}(\widehat{\alpha}_{k_2}) (\widehat{v}_{\psi,k_1}^* \widehat{v}_{\psi,k_2}^*)^{-1}$$

where $\widehat{D}_{k_1,k_2,n} = n^{-1} \sum_{i=1}^n \ell_{\alpha,k_1}(Z_i; \widehat{\alpha}_{k_1}) \ell_{\alpha,k_2}(Z_i; \widehat{\alpha}_{k_2})'$. Then the projected confidence set of confidence level $1 - p$ for θ^* is

$$CI_{\theta}(1-p) = \{\theta = x_1 1(x_0 \geq 0) + x_2 1(x_0 < 0) : \widehat{G}_n(x_1, x_2, x_0)' \widehat{\Sigma}_G^{-1} \widehat{G}_n(x_1, x_2, x_0) \leq \chi_{1-p}^2(3)\}. \quad (6.12)$$

Theorem 6.2 below shows the uniform asymptotic validity of this confidence interval. The proof of this theorem is given in Appendix B.

Theorem 6.2 *Suppose that Assumptions 4.1, 4.3 and B.1–B.2 in Supplemental Appendix B hold. In addition, suppose that there is a constant $C > 0$ such that under all $F_0 \in \mathcal{F}$, we have $\rho_{\min}(\Sigma_G) > C^{-1}$. Then $\liminf_{n \rightarrow \infty} \Pr_{F_0 \in \mathcal{F}} \Pr_{F_0}(\theta^* \in CI_{\theta}(1-p)) \geq 1 - p$.*

7 Simulation Studies

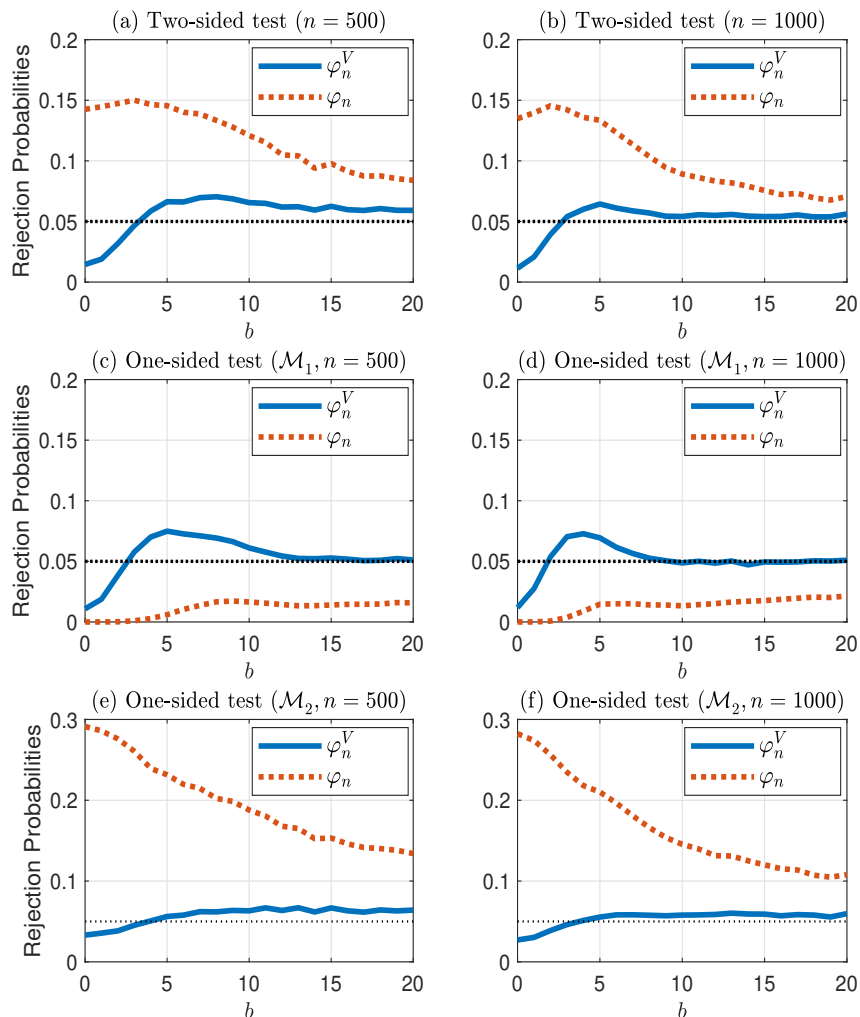
In this section, we report Monte Carlo simulation results on the finite sample performance of the nondegenerate test and the conditional confidence interval $CI_{\psi}(1-p)$.

Consider the following two models,

$$\mathcal{M}_1 : E[Y|X_1] = \beta_{10} + X_1\beta_{11} \text{ and } \mathcal{M}_2 : E[Y|X_2, X_3] = X_2\beta_{21} + g(X_3), \quad (7.1)$$

where $(\beta_{10}, \beta_{11})' \in R^2$, $\beta_{21} \in R$ and $g(\cdot) \in C^{\infty}([0, 1])$. This example readily fits into the framework of regression model studied in Section 5 with $\alpha_1(x_1) = \beta_{10} + \beta_{11}x_1$, $\mathcal{A}_1 = \{b_0 + x_1b_1 : (b_0, b_1)' \in R^2\}$, $\alpha_2(x_2, x_3) = x_2\beta_{21} + g(x_3)$, and $\mathcal{A}_2 = \{x_2b_2 + g(x_3) : b_2 \in R, g(\cdot) \in C^{\infty}([0, 1])\}$.

Figure 2: Null Rejection Rates of the Tests



To generate the data, let X_1, X_2 be independent standard normal random variables, and let X_3 be a uniform random variable independent of X_1 and X_2 . Let ε be standard normal and independent of X_1, X_2 and X_3 . Let

$$Y = 1 + X_1 a + X_2 b + c\sqrt{2} \sin(10\pi X_3) + \varepsilon. \quad (7.2)$$

7.1 Uniform Model Selection Test

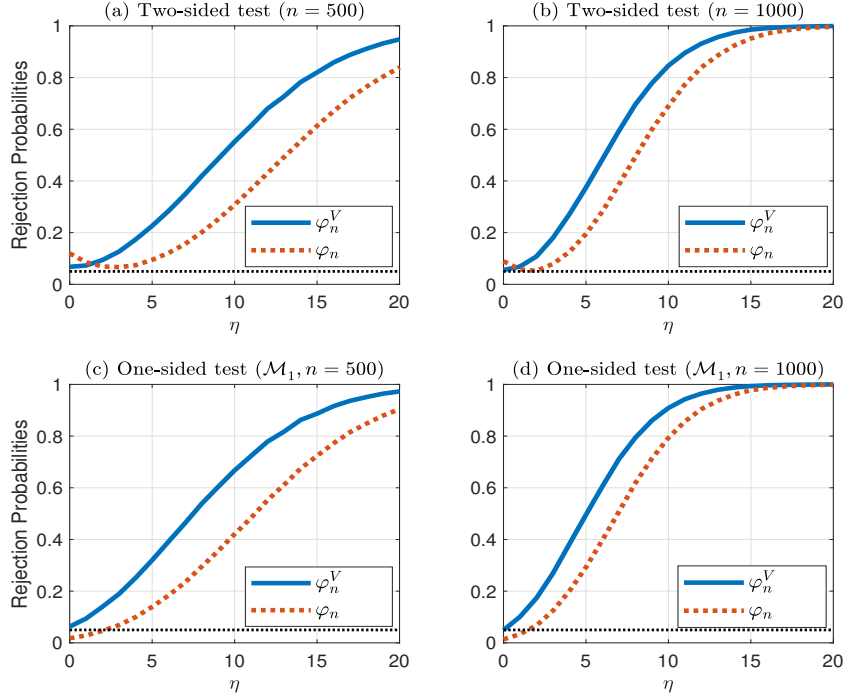
Independence between the regressors and the additive structure in the generation process of Y are not important for the performance of our test, but they allow us to derive an analytical form of

the fit measures and hence to conveniently characterize the null hypothesis. By exploiting them, we see that $u_1 = X_2b + c\sqrt{2}\sin(10\pi X_3) + \varepsilon$, and $u_2 = X_1a + \varepsilon$. Thus,

$$-2f(\mathcal{M}_1, F_0) = E_{F_0}[u_1^2] = b^2 + 1 + c^2 \text{ and } -2f(\mathcal{M}_2, F_0) = E_{F_0}[u_2^2] = a^2 + 1. \quad (7.3)$$

Therefore, the null hypothesis holds if and only if $a^2 = b^2 + c^2$, and when $a^2 > b^2 + c^2$, $f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0)$. When $a^2 = b^2 + c^2 = 0$, $u_1 = u_2$, in which case, $\omega_{F_0,*}^2 = 0$. Otherwise, $\omega_{F_0,*}^2 > 0$.

Figure 3: Null and Alternative Rejection Rates of the Tests



To evaluate the performance of the nondegenerate test, we consider two collections of DGPs. One collection sets $a^2 = b^2 + c^2$, $b = c$, and b (and c) to grid points in $[0, 0.4]$ with the spacing of 0.02 between adjacent points. This is the null collection in which, as b runs from 0 to 0.4, $\omega_{F_0,*}^2$ grows from zero up. The other collection sets $b = c = 0.2$, $a^2 = b^2 + c^2 + \eta$, and η to grid points in $[0, 0.2]$ with the spacing of 0.01 between adjacent points. This is the alternative collection in which, as η runs from 0 to 0.2, model \mathcal{M}_2 gets worse and worse relative to model \mathcal{M}_1 . We implement the nondegenerate test as well as the naive extension test as they are defined in Section 3. We use cubic spline to approximate $g(\cdot)$ in model 2.¹⁷

Selection of the number of series terms on approximating $g(\cdot)$ is important for the implementation of our nondegenerate test and conditional confidence intervals. For regression examples like

¹⁷Fourier series yields similar results.

the one considered in this section, we recommend using cross-validation with a slowly diverging lower bound imposed on the number of sieve terms. Cross-validation is a commonly used method in the semi/nonparametric regression literature for selecting smoothing parameters and has been shown to yield optimal rate of convergence in nonparametric series regression (ref. Li (1987) and Andrews (1991a)) as well as in nonparametric series quantile regression (ref. Chetverikov and Liao (2019)). The slowly diverging lower bound – we use $2 \log(\log(n))$ – ensures that the dimension of at least one model to diverge to infinity which is needed for our Assumption 4.3(b).¹⁸ ¹⁹

The finite sample rejection rates of the tests are calculated using 50,000 simulated samples. Figure 2 presents the rejection rates of the two-sided and one-sided tests under the first collection of DGPs—the collection of null DGPs. Graphs (a) and (b) show the tests for H_0 against $H_1 : f(\mathcal{M}_1, F_0) \neq f(\mathcal{M}_2, F_0)$ with sample size $n = 500$ and $n = 1000$ respectively. In graph (a), the naive extension test (dotted line) over-rejects noticeably when $\omega_{F_0,*}^2$ is zero or close to zero. On the other hand, the rejection rate of the nondegenerate test (solid line) never exceeds the nominal level by much, although there is some under-rejection at very small b 's and slight over-rejection at bigger b 's. When the sample size is increased from 500 to 1000, the rejection rate of the nondegenerate test gets closer to the nominal level while the naive extension test maintains overall over-rejection and under-rejection respectively. Graphs (c) and (d) show the one-sided tests for H_0 against $H_1 : f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0)$ with sample sizes $n = 500$ and $n = 1000$ respectively, and graphs (e) and (f) show the one-sided tests for H_0 against $H_1 : f(\mathcal{M}_1, F_0) < f(\mathcal{M}_2, F_0)$ with sample size $n = 500$ and $n = 1000$ respectively. Recall that model \mathcal{M}_1 is the more parsimonious one. As we can see, our robust test has a rejection rate of approximately 5% against both one-sided alternative hypotheses. The naive extension test has severe under-rejection when \mathcal{M}_1 is better under the alternative (graphs (c) and (d)) and severe over-rejection when \mathcal{M}_2 is better under the alternative (graphs (e) and (f)). This behavior is in line with our theoretical derivation.

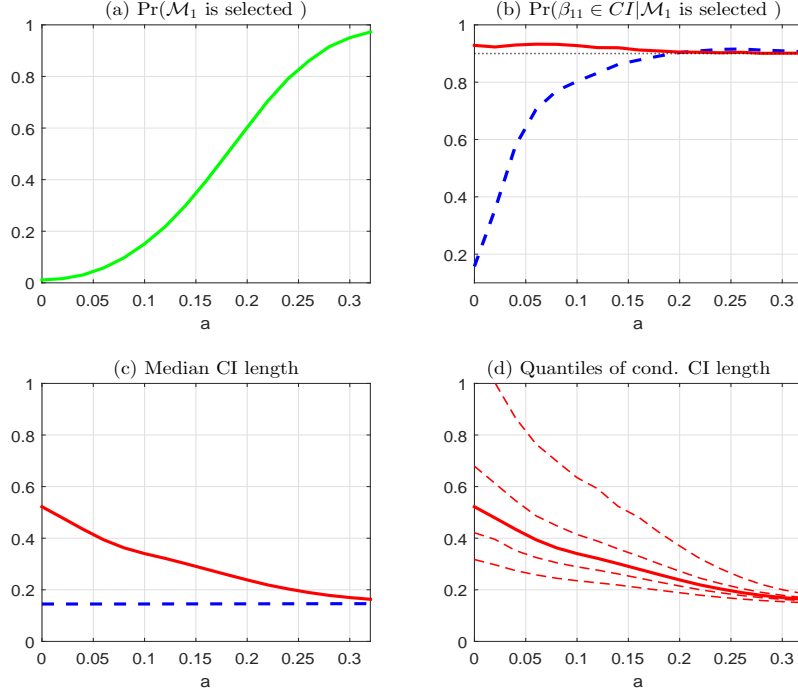
The rejection rates of the two-sided and one-sided tests under the second collection of DGPs—the collection of null and alternative DGPs are included in Figure 3. In this set of DGPs, the null hypothesis H_0 holds when $\eta = 0$ and the alternative hypothesis $H_1 : f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0)$ holds when $\eta \neq 0$. The model \mathcal{M}_2 becomes worse when the magnitude of η becomes large. Moreover, in this set of DGPs, $\omega_{F_0,*}^2 > 0$ since $b = c = 0.2$ for all different values of η . In Figure 3, we see that the nondegenerate test has rejection rates close to the nominal level 5% under the null H_0 (when $\eta = 0$), while the naive extension test over-rejects for the two-sided alternative (graphs (a) and (b)) and under-rejects for the one-sided alternative (graphs (c) and (d)). This is again in line with

¹⁸In our simulations, we also impose an upper bound of 15 on the cross-validation search range.

¹⁹Strictly speaking, the theory presented in earlier sections applies only to non-data-dependent choices of series terms. However, in practice, cross-validation is often employed, which is why we suggest it for empirical implementation of our tests and why we use it in this simulation example. The performance of our test with the cross-validated series terms is encouraging.

our theoretical results that the naive extension test favors large models. For the power properties, the nondegenerate test has the best power across most of the range of η in the two-sided test. It also has better power than the naive extension test in the one-sided test.

Figure 4: Performance of Conditional Confidence Interval for β_{11} .



7.2 Conditional Confidence Interval

In this subsection, we evaluate the performance of the conditional confidence interval $CI_{\psi_1}(1-p)$ with $p = 0.1$. Consider the parameters of interest β_{11} and β_{21} . Let model \mathcal{M}_1 be selected if $T_n > z_{0.95}$ and model \mathcal{M}_2 be selected otherwise. Consider the DGPs with $b = 0$, $c = 0$ and a running from 0 to 0.32. We report the probability of the model being selected, as well as the coverage probability, the median length, and other quantiles of the length of the conditional confidence interval. For comparison, we also report the performance of the naive confidence interval that ignores the model selection step, that is, for $j = 1, 2$,

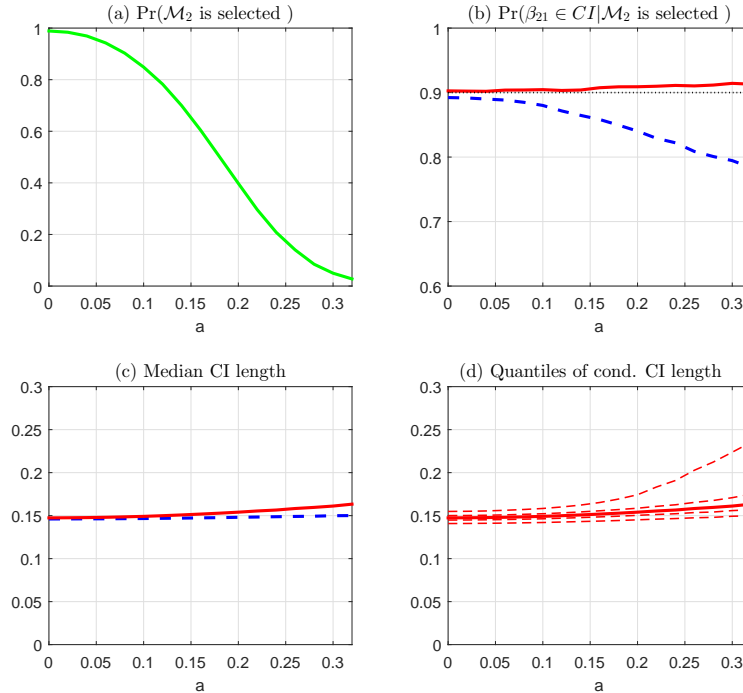
$$CI_j^{\text{naive}}(1-p) = [\psi_j(\hat{\alpha}_{k_j}) - n^{-1/2}z_{1-p/2}\hat{v}_{\psi,k_j}^*, \psi_j(\hat{\alpha}_{k_j}) - n^{-1/2}z_{p/2}\hat{v}_{\psi,k_j}^*], \quad (7.4)$$

where z_p stands for the p quantile of the standard normal distribution. Note that the conditional CI is only different from the naive CI in that it uses the critical value $c_{j,p}$ instead of z_p .

Figure 4 shows the results for β_{11} , and Figure 5 shows those for β_{21} . In graphs (b) and (c)

of both figures, the blue dotted lines are for the naive CIs and the red solid lines are for our conditional CIs; in graph (d), the five lines are respectively the 25%, 40%, 50%, 60%, and 75% quantile of the length of the conditional CI. As we can see, the naive CI may severely under-cover when the probability that the model is selected is small. On the other hand, the coverage probability of our conditional CI is always very close to the nominal level. In terms of length, our conditional CI is longer than the naive CI when the naive CI under-covers, and is about the same as the naive CI when the latter has good coverage properties.

Figure 5: Performance of Conditional Confidence Interval for β_{21}



By definition, the critical values of the conditional CI depends on T_n , and thus is random. As a result, the length of the conditional CI is also random. Part (d) of Figure 4 shows the variability of the length of the conditional CI. As we can see, the variability is small when the probability that the model under consideration is selected is large, and can be big otherwise. In light of the difficulties of post model selection inference pointed out by Leeb and Pötscher (2005), we view the variability and the extra length of the conditional CI as an inevitable price to pay for its good coverage property. It is encouraging to see that the conditional CI has similar length as the naive CI when the latter does not under-cover.

8 An Empirical Example

In this section we illustrate the use of our robust model selection test and the conditional confidence interval in the study of life-cycle schooling choices. We compare two models considered in Cameron and Heckman (1998) using our model selection test, and also report the conditional confidence intervals of some of the model specific parameters. The two models considered are parametric likelihood models. We consider our theory presented for the semi/non-parametric environment as reasonable approximation to this context since the number of parameters in each model is large.

8.1 Model Description

We apply our test on the comparison of two life cycle schooling models taken from Cameron and Heckman (1998). The paper is a classic piece of structural modeling, which is why we use it to illustrate our model selection and post model selection inference tools.

Consider an individual deciding how much schooling (S , number of years of schooling) to complete, and consider a vector of individual characteristics X that may be relevant for this decision. The first model (Model \mathcal{M}_1) is the *logit transition* model that Cameron and Heckman (1998) set up to formalize the statistical model prevalent in the political science literature at the time. To describe this model, define the binary variable $D_s = 1\{S \geq s\}$. This variable indicates whether or not the individual completed grade s or not. The model imposes a logit form on the transition probability from completing grade s to completing grade $s + 1$:

$$\Pr(D_{s+1} = 1|D_s = 1, X) = \frac{\exp(X'\beta_s)}{1 + \exp(X'\beta_s)},$$

where β_s is the grade-specific effect of X on the transition probability. This implies that the probability of s being the highest grade completed is given by

$$P_1(s|X, \theta_1) = \frac{1}{1 + \exp(X'\beta_s)} \times \frac{\exp(X'\beta_{s-1})}{1 + \exp(X'\beta_{s-1})} \times \cdots \times \frac{\exp(X'\beta_1)}{1 + \exp(X'\beta_1)} \quad (8.1)$$

where $\theta_1 = (\beta'_1, \beta'_2, \dots, \beta'_s)'$ with \bar{s} being the highest grade available. Note that this model contains many parameters since β_s is allowed to be different across s . However, it allows no individual heterogeneity other than the logit error, and thus effectively assumes that the population making the transition decision at different grade levels are the same. In technical terms, it rules out dynamic selection as the population move up grades. This is an important drawback of the model as discussed in Cameron and Heckman (1998).

The second model (Model \mathcal{M}_2) is an *ordered logit* model. Cameron and Heckman (1998) set

up this model as an economically well-grounded yet parsimonious contestant to the first model. In this model, the probability of s being the highest grade completed is given by

$$P_2(s|X, \theta_2) = \int_{\Omega} F(\alpha_{s-1} + y + X'\beta) - F(\alpha_s + y + X'\beta) dF_{\omega}(y), \quad (8.2)$$

where $\theta_2 = (\alpha_1, \dots, \alpha_{\bar{s}-1}, \beta)'$, $F(t) = \exp(t)(1 + \exp(t))^{-1}$, $\alpha_0 = +\infty$ for the highest possible grade \bar{s} , and ω is an unobservable individual type that has support Ω and distribution $F_{\omega}(\cdot)$. From the statistical point of view, the ordered logit aspect is not fundamentally different from the logit transition model since an ordered logit model can be written as a transition model with some (albeit non-logit) shocks in the transition decisions. However, this model adds the unobservable type ω , which makes sure that the dynamic selection effect is accounted for. The model further specifies that $\Omega = \{0, \omega_2\}$, and $F_{\omega}(y) = p_1 1(y \geq 0) + (1 - p_1) 1(y \geq \omega_2)$ for unknown parameters $\omega_2 > 0$ and $p_1 \in (0, 1)$. The model uses a parsimonious specification for the effect of X on the ordered logit cutoffs — the β is not indexed by s .

8.2 Implementation Details

We compare the models in terms of their population log-likelihood. We implement the two-sided version of both our robust test and the Vuong (1989) test. The detailed implementation steps are as follows:

1. Given the data set $(S_i, X_i)_{i=1}^n$, define the log-density functions for the two models respectively as $m_j(S_i, X_i, \theta_j) = \log P_j(S_i|X_i, \theta_j)$ for $j = 1, 2$.
2. Define the log-likelihoods of the two models as $\hat{f}(\mathcal{M}_j, \theta_j) = n^{-1} \sum_{i=1}^n m_j(S_i, X_i, \theta_j)$ for $j = 1, 2$.
3. Respectively for $j = 1, 2$, compute $\hat{\theta}_{n,j} = \arg \max_{\theta_j} \hat{f}(\mathcal{M}_j, \theta_j)$ using a suitable maximization algorithm, like the **fminunc** function in Matlab, or the **ml** package in Stata.
4. Compute $\bar{\ell}_n(\hat{\theta}_n) = \hat{f}(\mathcal{M}_1, \hat{\theta}_{n,1}) - \hat{f}(\mathcal{M}_2, \hat{\theta}_{n,2})$ and $\hat{\omega}_n^2(\hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^n (\ell_i(\hat{\theta}_n) - \bar{\ell}_n(\hat{\theta}_n))^2$, where $\ell_i(\theta) = m_1(S_i, X_i, \theta_1) - m_2(S_i, X_i, \theta_2)$ and $\hat{\theta}_n = (\hat{\theta}'_{n,1}, \hat{\theta}'_{n,2})'$.
5. Compute the score $\partial m_j(S_i, X_i, \hat{\theta}_{n,j}) / \partial \theta_j$ for each i and $j = 1, 2$ either by deriving and using the analytical formula for the first derivative function, or by numerical differentiation of the log-density function. Let $\hat{\ell}_{\theta,i} = \partial m_1(S_i, X_i, \hat{\theta}_{n,1}) / \partial \theta_1 - \partial m_2(S_i, X_i, \hat{\theta}_{n,2}) / \partial \theta_2$.
6. Compute $\hat{D}_n = n^{-1} \sum_{i=1}^n \hat{\ell}_{\theta,i} \hat{\ell}'_{\theta,i}$.

7. Compute $\widehat{H}_{n,j} = \partial^2 f(\mathcal{M}_j, \widehat{\theta}_{n,j}) / \partial \theta_j \partial \theta_j'$ for $j = 1, 2$ either numerically or using analytical formula of the second derivative. Let $\widehat{H}_n = \text{diag}(\widehat{H}_{n,1}, -\widehat{H}_{n,2})$.
8. Let $T_n^V = n^{1/2} \bar{\ell}_n(\widehat{\theta}_n) (\widehat{\omega}_n^2(\widehat{\theta}_n))^{-1/2}$ and let $T_n = \frac{n \bar{\ell}_n(\widehat{\theta}_n) + 2^{-1} \text{tr}(\widehat{D}_n \widehat{H}_n^{-1})}{\sqrt{\max\{n \widehat{\omega}_n^2(\widehat{\theta}_n) - \frac{1}{2} \text{tr}((\widehat{D}_n \widehat{H}_n^{-1})^2), 2^{-1} \text{tr}((\widehat{D}_n \widehat{H}_n^{-1})^2)\}}}$.
9. Compute the p-value of our robust test as $\text{p-value} = 2(1 - \Phi(T_n))$ and of the Vuong (1989) test as $\text{p-value}^V = 2(1 - \Phi(T_n^V))$.

8.3 Data and Results

We compare these models using data from the 1997 wave of the National Longitudinal Survey (NLSY 97). This is a newer wave of the NLSY 79 used in Cameron and Heckman (1998) that covers a sample of young men and women born between 1980 and 1984. Following Cameron and Heckman (1998), we use the male sample only and drop observations with missing values on the relevant variables. Our final sample contains 1938 individuals.²⁰

The X variables for models \mathcal{M}_1 contain a constant and 15 nonconstant variables including the number of siblings, highest grade completed by father, that by mother, broken family dummy, log family income, urban/rural residence dummy, etc. and interaction terms. The X variable for model \mathcal{M}_2 contains all those 15 nonconstant variables, but does not contain a constant term. We aggregate the grades (S) into four, following Cameron and Heckman (1998): completed high school ($s = 1$), attended college ($s = 2$), graduated college ($s = 3$) and attended 17 or more years of school ($s = 4$). As a result, Model \mathcal{M}_1 contains $4 \times 16 = 64$ parameters and Model \mathcal{M}_2 contains $4 + 15 + 2 = 21$ parameters. Clearly, Model \mathcal{M}_2 is much more parsimonious than Model \mathcal{M}_1 .²¹

Table 1: Model Selection Tests Based on NLSY 97

	Test Statistic	p-value
Robust Test	1.856	.063
Vuong (1989) Test	3.924	.000

Table 1 shows the value of the test statistics as well as p-values of both tests. The Vuong (1989) test strongly rejects the null in favor of the less parsimonious models \mathcal{M}_1 . However, we believe that the strong rejection is partly due to the bias in favor of large models. Indeed, the robust test that corrects the bias presents much weaker evidence against the parsimonious Model \mathcal{M}_2 . In particular, according to the robust test, we cannot reject the null that \mathcal{M}_2 is as good as

²⁰Results using reconstructed sample from the NLSY 79 are reported in Supplemental Appendix G.

²¹Parameter estimates are irrelevant for our analysis and thus are omitted. They are available upon request.

\mathcal{M}_1 at significance level 5%. Cameron and Heckman (1998) advocate for \mathcal{M}_2 for its simplicity and interpretability. Our robust test shows that it achieves the simplicity without sacrificing too much of its fit to the data. In contrast, the Vuong (1989) test tells a different story and can be misleading.²²

To illustrate our conditional confidence interval, we computed these intervals for the parameters in Model \mathcal{M}_2 conditional on the event that $T_n < z_{0.975} \approx 1.96$. It turns out that the conditional confidence intervals are the same as the naive CI's computed using the sandwich standard error formula. Upon further inspection, we find that the correlation coefficients of T_n and the parameter estimates of Model \mathcal{M}_2 are nearly zero, which causes $c_{2,p}$ to be the same as z_p up to at least the sixth digit. We believe that this is a special feature of this application and does not have general implication.

9 Conclusion

This paper studies the statistical comparison of semi/nonparametric models when the competing models are overlapping nonnested, strictly nonnested, or nested. We provide a new model selection test that achieves uniform asymptotic size control. The new test uses a critical value from standard normal distribution and employs a bias-corrected quasi-likelihood ratio statistic that is easy to compute in practice. This makes our test convenient for empirical implementation. Moreover, uniformly valid post model selection test inference procedures of model parameters are also provided. Simulation results show that our test and our post model selection test confidence interval perform well in finite samples.

At least two future research directions arise from the findings of this paper. First, the theory of this paper is established under the i.i.d. assumption of the data. It is important to extend it for the comparison of time series models with dependent data. Second, when there are many competing models to be compared, it shall be interesting to construct a model confidence set that covers the best model with valid asymptotic size. These directions of research form part of our ongoing work, during the course of which some preliminary results have been obtained.

²²Cameron and Heckman (1998) implemented the Vuong (1989) test with the Bayesian information criterion (BIC) penalty, and thus were effectively testing the null hypothesis that

$$H_0 : f(\mathcal{M}_1, F_0) - \frac{k_1 \log(n)}{2n} - \frac{\text{tr}(D_{F_0, k_1} H_{F_0, k_1}^{-1})}{2n} = f(\mathcal{M}_2, F_0) - \frac{k_2 \log(n)}{2n} - \frac{\text{tr}(D_{F_0, k_2} H_{F_0, k_2}^{-1})}{2n},$$

where $f(\mathcal{M}_j, F_0) \equiv \max_{\theta_j} E_{F_0} \log P(S|X; \theta_j)$ is the Kullback-Leibler distance from model \mathcal{M}_j to the data. Their test result strongly rejects the null in favor of the ordered logit model. The penalty would not matter asymptotically in the asymptotic framework assuming strict nonnestedness, as argued in Vuong (1989). Yet it clearly leads to a different testing conclusion here.

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Supplemental Appendix of “A Nondegenerate Vuong Test and A Post Selection Confidence Interval for Semi/Nonparametric Models”

In this supplemental appendix, we present supporting materials for the main paper:

- Section A presents the proofs for the results in Section 4 of the main paper.
- Section B proves the results in Section 6 of the main paper.
- Section C verifies the high-level assumptions in the main paper in a nonparametric mean-regression example.
- Section D includes the illustration of the high-level theory and verification of the high-level assumptions in the main paper in a nonparametric quantile-regression example.
- Section E includes some auxiliary results and their proofs.
- Section F conducts simulation studies to compare the finite sample properties of our nonparametric nondegenerate test with the test proposed in Shi (2015b).
- Section G includes additional results from our empirical studies in Section 8 of the main paper.

A Proofs for Results in Section 4

Proof of Theorem 4.1. First note that

$$\begin{aligned}
 tr(\widehat{D}_n(\alpha_{\mathbf{k}}^*)H_{F_n, \mathbf{k}}^{-1}) &= tr\left(n^{-1}\sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)\ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'H_{F_n, \mathbf{k}}^{-1}\right) \\
 &= n^{-1}\sum_{i=1}^n tr(\ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)\ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'H_{F_n, \mathbf{k}}^{-1}) \\
 &= n^{-1}\sum_{i=1}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'H_{F_n, \mathbf{k}}^{-1}\ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*), \tag{A.1}
 \end{aligned}$$

where the steps follow from properties of the trace operator. This and Assumption 4.3(a) together implies that

$$n\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) + (1/2)tr(\widehat{D}_n(\alpha_{\mathbf{k}}^*)H_{F_n, \mathbf{k}}^{-1})$$

$$= \sum_{i=1}^n \ell(Z_i; \alpha_n^*) - n^{-1} \sum_{i=2}^n \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} \left(\sum_{i'=1}^{i-1} \ell_{\alpha, \mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \right) + o_p(n^{1/2} \sigma_{F_n, n}). \quad (\text{A.2})$$

For the ease of notation, let

$$\begin{aligned} V_{i,n} &= n^{-1/2} \sigma_{F_n, n}^{-1} [\ell(Z_i; \alpha_n^*) - E_{F_n}(\ell(Z_i; \alpha_n^*))] \text{ for } i = 1, \dots, n, \text{ and} \\ U_{i,n} &= \begin{cases} n^{-3/2} \sigma_{F_n, n}^{-1} \left[\ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} \sum_{i'=1}^{i-1} \ell_{\alpha, \mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \right] & i \geq 2 \\ 0 & i = 1. \end{cases} \end{aligned} \quad (\text{A.3})$$

Let $W_{i,n} = V_{i,n} - U_{i,n}$. Then we have

$$\frac{n[\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}, n}) - E_{F_n}(\ell(Z_i; \alpha_n^*))] + (1/2) \text{tr}(\hat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1})}{n^{1/2} \sigma_{F_n, n}} = \sum_{i=1}^n W_{i,n} + o_p(1). \quad (\text{A.4})$$

Let $Z_0 = 0$ and $\mathcal{F}_{i,n}$ be the natural filtration generated by $\{Z_0, Z_1, \dots, Z_i\}$ under F_n for $i = 0, \dots, n$. Under Assumptions 4.1(a) and (c),

$$E_{F_n} [W_{i,n} | \mathcal{F}_{i-1,n}] = E_{F_n} [V_{i,n} | \mathcal{F}_{i-1,n}] - E_{F_n} [U_{i,n} | \mathcal{F}_{i-1,n}] = 0, \quad (\text{A.5})$$

for all $i = 1, \dots, n$. This implies that $W_{i,n}$ is a triangle array Martingale difference sequence (m.d.s.). Next, we use Hall and Heyde's (1980) Corollary 3.1—a Martingale central limit theorem—to show the desired convergence.

To use Hall and Heyde's (1980) Corollary 3.1, it suffices to verify two sufficient conditions:

$$\begin{aligned} \text{(a)} \quad & \sum_{i=1}^n E_{F_n} [W_{i,n}^2 | \mathcal{F}_{i-1,n}] \rightarrow_p 1, \text{ and} \\ \text{(b)} \quad & \sum_{i=1}^n E_{F_n} [W_{i,n}^2 1\{|W_{i,n}| > \varepsilon\}] \rightarrow 0, \forall \varepsilon > 0. \end{aligned} \quad (\text{A.6})$$

First, we verify condition (a). By Assumption 4.1(a),

$$\sum_{i=1}^n E_{F_n} [V_{i,n}^2 | \mathcal{F}_{i-1,n}] = \sigma_{F_n, n}^{-2} E_{F_n} [(\ell(Z_i; \alpha_n^*) - E_{F_n}(\ell(Z_i; \alpha_n^*)))^2] = \sigma_{F_n, n}^{-2} \omega_{F_n, *}. \quad (\text{A.7})$$

Also, when $\omega_{F_n, *}^2 = 0$, we have $V_{i,n} = 0$ a.s. for all i , and hence $E_{F_n} [V_{i,n} U_{i,n} | \mathcal{F}_{i-1,n}] = 0$ for all i and all n . When $\omega_{F_n, *}^2 > 0$ and $i = 1$, we have $U_{i,n} = 0$ by definition, which implies

$E_{F_n}[V_{i,n}U_{i,n}|\mathcal{F}_{i-1,n}] = 0$. When $\omega_{F_n,*}^2 > 0$, and $i \geq 2$,

$$E_{F_n}[V_{i,n}U_{i,n}|\mathcal{F}_{i-1,n}] = \frac{\omega_{F_n,*} \sum_{i'=1}^{i-1} \rho'_{F_n,\mathbf{k}} D_{F_n,\mathbf{k}}^{1/2} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)}{n^2 \sigma_{F_n,n}^2}, \quad (\text{A.8})$$

where $\rho_{F_n,\mathbf{k}} = \omega_{F_n,*}^{-1} (D_{F_n,\mathbf{k}}^{1/2})^+ \text{Cov}[\ell(Z; \alpha), \ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)]$ and $(D_{F_n,\mathbf{k}}^{1/2})^+$ denotes the Moore-Penrose inverse of $D_{F_n,\mathbf{k}}^{1/2}$. Note that in (A.8), we use

$$((D_{F_n,\mathbf{k}}^{1/2})^+ \text{Cov}[\ell(Z; \alpha_n^*), \ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)])' D_{F_n,\mathbf{k}}^{1/2} = \text{Cov}[\ell(Z; \alpha_n^*), \ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)]',$$

which follows by Lemma E.1(c). Thus,

$$\sum_{i=1}^n E_{F_n}[V_{i,n}U_{i,n}|\mathcal{F}_{i-1,n}] = \frac{\omega_{F_n,*} \sum_{i=1}^{n-1} (1 - i/n) \rho'_{F_n,\mathbf{k}} D_{F_n,\mathbf{k}}^{1/2} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)}{n \sigma_{F_n,n}^2}. \quad (\text{A.9})$$

Then, $\sum_{i=1}^n E_{F_n}[V_{i,n}U_{i,n}|\mathcal{F}_{i-1,n}] = o_p(1)$ follows from the derivation below:

$$\begin{aligned} & E_{F_n} \left[\sum_{i=1}^n E_{F_n}[V_{i,n}U_{i,n}|\mathcal{F}_{i-1,n}] \right]^2 \\ &= \frac{\omega_{F_n,*}^2 \sum_{i=1}^{n-1} (1 - i/n)^2 \rho'_{F_n,\mathbf{k}} D_{F_n,\mathbf{k}}^{1/2} H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}}^{1/2} \rho_{F_n,\mathbf{k}}}{n^2 \sigma_{F_n,n}^4} \\ &\leq \frac{\omega_{F_n,*}^2 \rho'_{F_n,\mathbf{k}n} (D_{F_n,\mathbf{k}}^{1/2} H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}}^{1/2})^2 \rho_{F_n,\mathbf{k}n}}{\sigma_{F_n,n}^2 n \sigma_{F_n,n}^2} \leq \frac{\max_{j=1,\dots,|\mathbf{k}|} \lambda_{F_n,j}^2}{n \sigma_{F_n,n}^2} = o(1), \end{aligned} \quad (\text{A.10})$$

where the first equality holds by (A.9) and Assumptions 4.1(a) and (c), the first inequality holds by $(1 - i/n)^2 \leq 1$, the second inequality holds by Lemma E.1(b) in Supplemental Appendix E and the definition of $\sigma_{F_n,n}^2$, and the last equality holds by Assumptions 4.1(f) and 4.3(b). Moreover, observe that, for $i \geq 2$,

$$\begin{aligned} n^3 \sigma_{F_n,n}^2 E_{F_n}[U_{i,n}^2|\mathcal{F}_{i-1,n}] &= E_{F_n} \left[\left(\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \sum_{i'=1}^{i-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \right)^2 \middle| \mathcal{F}_{i-1,n} \right] \\ &= \sum_{i'=1}^{i-1} \sum_{i''=1}^{i-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i''}; \alpha_{\mathbf{k}}^*). \end{aligned} \quad (\text{A.11})$$

Let $R_{i,i',n} = \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)$. Then,

$$\begin{aligned}
\sum_{i=1}^n E_{F_n}[U_{i,n}^2 | \mathcal{F}_{i-1,n}] &= \frac{\sum_{i=2}^n \sum_{i'=1}^{i-1} \sum_{i''=1}^{i'-1} R_{i',i'',n}}{n^3 \sigma_{F_n,n}^2} \\
&= \frac{\sum_{i=2}^n \sum_{i'=1}^{i-1} R_{i',i',n}}{n^3 \sigma_{F_n,n}^2} + 2 \frac{\sum_{i=3}^n \sum_{i'=2}^{i-1} \sum_{i''=1}^{i'-1} R_{i',i'',n}}{n^3 \sigma_{F_n,n}^2} \\
&= \frac{\sum_{i=1}^{n-1} (n-i) R_{i,i,n}}{n^3 \sigma_{F_n,n}^2} + 2 \frac{\sum_{i'=2}^{n-1} \sum_{i''=1}^{i'-1} (n-i') R_{i',i'',n}}{n^3 \sigma_{F_n,n}^2}. \tag{A.12}
\end{aligned}$$

Consider the derivation:

$$\begin{aligned}
E_{F_n} \left[\frac{\sum_{i=1}^{n-1} (n-i) R_{i,i,n}}{n^3 \sigma_{F_n,n}^2} \right] &= \frac{\sum_{i=1}^{n-1} (n-i) \text{tr}((D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1})^2)}{n^3 \sigma_{F_n,n}^2} \\
&= \frac{\sum_{i=1}^{n-1} (n-i) \sum_{j=1}^{|\mathbf{k}|} \lambda_{F_n,j}^2}{n^3 \sigma_{F_n,n}^2} = \frac{(n-1) \omega_{F_n,U,\mathbf{k}}^2}{2n^2 \sigma_{F_n,n}^2}. \tag{A.13}
\end{aligned}$$

Also consider

$$\begin{aligned}
&E_{F_n} \left[\frac{\sum_{i=1}^{n-1} (n-i) (R_{i,i,n} - E_{F_n}(R_{i,i,n}))}{n^3 \sigma_{F_n,n}^2} \right]^2 \\
&= \frac{\sum_{i=1}^{n-1} (n-i)^2 (E_{F_n}[R_{i,i,n}^2] - [E_{F_n}(R_{i,i,n})]^2)}{n^6 \sigma_{F_n,n}^4} \\
&\leq \frac{E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 \right]}{n^3 \sigma_{F_n,n}^4} \\
&\leq C \frac{E_{F_n} [|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2]}{n^3 \sigma_{F_n,n}^4} \leq C \frac{|\mathbf{k}| \xi_{\mathbf{k}}}{n^3 \sigma_{F_n,n}^4} = o(1), \tag{A.14}
\end{aligned}$$

where the first equality holds by expanding the square of the sum over i and by applying the i.i.d. assumption (Assumption 4.1(a)); the first inequality holds by the definition of $R_{i,i,n}$, $[E_{F_n}(R_{i,i,n})]^2 \geq 0$, and $\sum_{i=1}^{n-1} (n-i)^2 \leq n^3$; the second inequality holds by the property of eigenvalues and Assumption 4.1(f); the last inequality holds by Assumption 4.1(d); and the last equality holds by Assumption 4.3(b). Further consider

$$\begin{aligned}
&E_{F_n} \left[\left(\frac{\sum_{i'=2}^{n-1} \sum_{i''=1}^{i'-1} (n-i') R_{i',i'',n}}{n^3 \sigma_{F_n,n}^2} \right)^2 \right] = \frac{\sum_{i'=2}^{n-1} \sum_{i''=1}^{i'-1} (n-i')^2 E_{F_n}[R_{i',i'',n}^2]}{n^6 \sigma_{F_n,n}^4} \\
&\leq \frac{E_{F_n}[R_{i',i'',n}^2]}{n^2 \sigma_{F_n,n}^4} \leq \frac{\max_{j=1,\dots,|\mathbf{k}|} \lambda_{F_n,j}^2 \text{tr}((D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1})^2)}{n \sigma_{F_n,n}^2} = o(1), \tag{A.15}
\end{aligned}$$

where the first equality holds by Assumptions 4.1(a) and 4.1(c), the first inequality holds by Assumption 4.1(a) and $\sum_{i'=2}^{n-1} \sum_{i''=1}^{i'-1} (n-i')^2 \leq n^4$, the second inequality holds by the definition of $R_{i',i'',n}$, and the last equality holds by the definition of $\sigma_{F_n,n}^2$, and Assumptions 4.1(f) and 4.3(b). The results of (A.12)-(A.15) imply that

$$\sum_{i=1}^n E_{F_n}[U_{i,n}^2 | \mathcal{F}_{i-1,n}] = \frac{(n-1)\omega_{F_n,U,\mathbf{k}}^2}{2n^2\sigma_{F_n,n}^2} + o_p(1). \quad (\text{A.16})$$

Combining (A.7), (A.10) and (A.16), we get

$$\begin{aligned} \sum_{i=1}^n E_{F_n}[W_{i,n}^2 | \mathcal{F}_{i-1,n}] &= \sum_{i=1}^n E_{F_n}[V_{i,n}^2 | \mathcal{F}_{i-1,n}] - 2 \sum_{i=1}^n E_{F_n}[V_{i,n}U_{i,n} | \mathcal{F}_{i-1,n}] + \sum_{i=1}^n E_{F_n}[U_{i,n}^2 | \mathcal{F}_{i-1,n}] \\ &= \frac{\omega_{F_n,*}^2 + (2n^2)^{-1}(n-1)\omega_{F_n,U,\mathbf{k}}^2}{\sigma_{F_n,n}^2} + o_p(1) = 1 + o_p(1). \end{aligned} \quad (\text{A.17})$$

This verifies condition (a) in (A.6).

Now we verify condition (b) in (A.6). First, $W_{i,n}^2 1\{|W_{i,n}| > \varepsilon\} \leq \varepsilon^{-2}W_{i,n}^4$. Hence by the monotonicity of expectation, we have

$$\sum_{i=1}^n E_{F_n}[W_{i,n}^2 1\{|W_{i,n}| > \varepsilon\}] \leq \sum_{i=1}^n E_{F_n}[\varepsilon^{-2}|W_{i,n}|^4] \leq 8\varepsilon^{-2} \sum_{i=1}^n E_{F_n}[|V_{i,n}|^4 + |U_{i,n}|^4], \quad (\text{A.18})$$

where the last inequality holds by the convexity of the function $f(x) = |x|^4$. Consider that

$$\begin{aligned} E_{F_n}[|V_{i,n}|^4] &= n^{-2} \frac{E_{F_n}[(\ell(Z_i; \alpha_n^*) - E_{F_n}(\ell(Z_i; \alpha_n^*)))^4]}{\sigma_{F_n,n}^4} \\ &\begin{cases} \leq n^{-2} E_{F_n} \left[\left| \frac{\ell(Z_i; \alpha_n^*) - E_{F_n}(\ell(Z_i; \alpha_n^*))}{\omega_{F_n,*}} \right|^4 \right] \leq n^{-2} C & \text{if } \omega_{F_n,*} > 0 \\ = 0 & \text{if } \omega_{F_n,*} = 0 \end{cases} \\ &\leq Cn^{-2}, \end{aligned} \quad (\text{A.19})$$

where the first inequality holds because $\omega_{F_n,U,\mathbf{k}}^2 \geq 0$, and the second inequality holds by Assumption 4.1(e). Now consider that

$$E_{F_n}[|U_{i,n}|^4] \leq E_{F_n}[U_{i,n}^4] = n^{-6}\sigma_{F_n,n}^{-4} E_{F_n} \left[\left(\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \sum_{i'=1}^{i-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \right)^4 \right], \quad (\text{A.20})$$

for $i > 2$. Define $M_{i,i',n} = \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)$. Then the above display leads to:

$$\begin{aligned}
& E_{F_n} [|U_{i,n}|^4] \\
& \leq n^{-6} \sigma_{F_n,n}^{-4} E_{F_n} \left[\sum_{i'=1}^{i-1} M_{i,i',n} \right]^4 \\
& = n^{-6} \sigma_{F_n,n}^{-4} E_{F_n} \left[\sum_{i'=1}^{i-1} M_{i,i',n}^4 + c \sum_{i=2}^{i-1} \sum_{i''=1}^{i'-1} M_{i,i',n}^2 M_{i,i'',n}^2 \right], \tag{A.21}
\end{aligned}$$

where c is some appropriate positive integers whose exact value is not important for the purpose. The equality holds because $E[M_{i,i',n} M_{i,i'',n}^3] = 0$ for any $i' \neq i''$ by Assumptions 4.1(a) and 4.1(c). Observe that, for $i \neq i'$,

$$\begin{aligned}
E_{F_n} [M_{i,i',n}^4] & = E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)|^4 \right] \\
& = E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' [H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1}] \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 \right] \\
& \leq E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 |\ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-2} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)|^2 \right] \\
& \leq C \left| E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 \right] \right|^2 \leq C |\mathbf{k}|^2 \xi_{\mathbf{k}}^2, \tag{A.22}
\end{aligned}$$

where the first inequality holds by the inequality $a'Aa \leq (a'a)tr(A)$ for any semi-positive definite matrix A with $a = \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)$ and $A = H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1}$, and $tr(A) = \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-2} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)$; the second inequality holds by properties of eigenvalues and Assumptions 4.1(a) and 4.1(f); and the last inequality holds by Assumption 4.1(d). Moreover,

$$\begin{aligned}
E_{F_n} [M_{i,i',n}^2 M_{i,i'',n}^2] & = E_{F_n} \left[\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*) \ell_{\alpha,\mathbf{k}}(Z_{i'}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \right. \\
& \quad \left. \times \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_{i''}; \alpha_{\mathbf{k}}^*) \ell_{\alpha,\mathbf{k}}(Z_{i''}; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \right] \\
& = E_{F_n} \left[\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \right. \\
& \quad \left. \times \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) \right] \\
& = E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' (H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 \right] \\
& \leq C E_{F_n} \left[|\ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' \ell_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)|^2 \right] \leq C |\mathbf{k}| \xi_{\mathbf{k}}. \tag{A.23}
\end{aligned}$$

Therefore, from the results in (A.18)-(A.23), we deduce that

$$\sum_{i=1}^n E [W_{i,n}^2 1\{|W_{i,n}| > \varepsilon\}] \leq 8C\varepsilon^{-2} \left(n^{-1} + \frac{|\mathbf{k}|^2 \xi_{\mathbf{k}}^2 + n|\mathbf{k}| \xi_{\mathbf{k}}}{n^4 \sigma_{F_n,n}^4} \right) = o(1), \tag{A.24}$$

where the equality holds by Assumption 4.3(b). Thus, condition (b) in (A.6) is verified. ■

Proof of Lemma 4.1. First note that for any real symmetric matrices A and B with $B \geq 0$, Lemma 1 in Wang et al. (1986) implies that

$$|tr(AB)| \leq \|A\| tr(B). \quad (\text{A.25})$$

By the linearity of the trace operation,

$$tr(\widehat{D}_n \widehat{H}_n^{-1}) - tr(\widehat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1}) = tr\left((\widehat{D}_n - \widehat{D}_n(\alpha_{\mathbf{k}}^*)) \widehat{H}_n^{-1}\right) + tr\left((\widehat{H}_n^{-1} - H_{F_n, \mathbf{k}}^{-1}) \widehat{D}_n(\alpha_{\mathbf{k}}^*)\right). \quad (\text{A.26})$$

Let \widehat{D}_{n, k_1} , $\widehat{D}_{n, k_1}(\alpha_{\mathbf{k}}^*)$, and \widehat{H}_{n, k_1} be the leading $k_1 \times k_1$ submatrices of \widehat{D}_n , $\widehat{D}_n(\alpha_{\mathbf{k}}^*)$, and \widehat{H}_n , respectively. Let \widehat{D}_{n, k_2} , $\widehat{D}_{n, k_2}(\alpha_{\mathbf{k}}^*)$, and \widehat{H}_{n, k_2} be the last $k_2 \times k_2$ submatrices of \widehat{D}_n , $\widehat{D}_n(\alpha_{\mathbf{k}}^*)$, and \widehat{H}_n , respectively. Then by the definitions of \widehat{D}_n , $\widehat{D}_n(\alpha_{\mathbf{k}}^*)$, and \widehat{H}_n , the triangle inequality, (A.25), and Assumption 4.1(f),

$$\begin{aligned} n^{-1/2} \left| tr\left((\widehat{D}_n - \widehat{D}_n(\alpha_{\mathbf{k}}^*)) \widehat{H}_n^{-1}\right) \right| &\leq n^{-1/2} \sum_{j=1,2} \left| tr\left((\widehat{D}_{n, k_j} - \widehat{D}_{n, k_j}(\alpha_{\mathbf{k}}^*)) \widehat{H}_{n, k_j}^{-1}\right) \right| \\ &\leq n^{-1/2} \sum_{j=1,2} \left\| \widehat{D}_{n, k_j} - \widehat{D}_{n, k_j}(\alpha_{\mathbf{k}}^*) \right\| |tr((-\widehat{H}_{n, k_j})^{-1})| \\ &= n^{-1/2} O_p(n^{1/2} \sigma_{F_n, n} |\mathbf{k}|^{-1}) O_p(k_j) = o_p(\sigma_{F_n, n}), \end{aligned} \quad (\text{A.27})$$

where the first equality holds by Assumption 4.4(a) and by $|tr((-\widehat{H}_{n, k_j})^{-1})| = O_p(k_j)$ which holds by Assumptions 4.1(f) and 4.4(a).

Similarly, for the second term in the right-hand side of (A.26),

$$\begin{aligned} n^{-1/2} \left| tr\left((\widehat{H}_n^{-1} - H_{F_n, \mathbf{k}}^{-1}) \widehat{D}_n(\alpha_{\mathbf{k}}^*)\right) \right| &\leq n^{-1/2} \left\| \widehat{H}_n^{-1} - H_{F_n, \mathbf{k}}^{-1} \right\| tr(\widehat{D}_n(\alpha_{\mathbf{k}}^*)) \\ &\leq n^{-1/2} \left\| \widehat{H}_n^{-1} \right\| \left\| \widehat{H}_n - H_{F_n, \mathbf{k}} \right\| \left\| H_{F_n, \mathbf{k}}^{-1} \right\| tr(\widehat{D}_n(\alpha_{\mathbf{k}}^*)) \\ &= n^{-1/2} O_p(n^{1/2} \sigma_{F_n, n} |\mathbf{k}|^{-1}) O_p(|\mathbf{k}|) = o_p(\sigma_{F_n, n}). \end{aligned} \quad (\text{A.28})$$

Therefore, the lemma is proved. ■

Proof of Lemma 4.2. First note that

$$\begin{aligned} \widehat{\omega}_n^2(\widehat{\alpha}_{\mathbf{k}}) &= n^{-1} \sum_{i=1}^n \ell(Z_i, \widehat{\alpha}_{\mathbf{k}})^2 - \bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}})^2 \\ &= n^{-1} \sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 - (\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*))^2 \end{aligned}$$

$$+ n^{-1} \sum_{i=1}^n [(\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*))^2 - (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2]. \quad (\text{A.29})$$

Next we show the convergence of each of the three terms on the right-hand side of (A.29).

For the first term, consider the derivation:

$$\begin{aligned} & \frac{n^{-1} \sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 - \omega_{F_n, *}^2}{\sigma_{F_n, n}^2} \\ &= n^{-1} \sum_{i=1}^n \frac{(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 - \omega_{F_n, *}^2}{\sigma_{F_n, n}^2} = o_p(1), \end{aligned} \quad (\text{A.30})$$

where the last equality holds by the weak law of large numbers for row-wise i.i.d. triangular arrays. The law of large numbers applies because (i) $E_{F_n} \left[\frac{(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 - \omega_{F_n, *}^2}{\sigma_{F_n, n}^2} \right] = 0$, and

$$(ii) E_{F_n} \left[\left| \frac{(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 - \omega_{F_n, *}^2}{\sigma_{F_n, n}^2} \right| \right]^{1+\delta/2} \leq \begin{cases} 0 & \text{if } \omega_{F_n, *} = 0 \\ E_{F_n} \left[\left| \frac{\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)}{\omega_{F_n, *}} \right|^{2+\delta} \right] & \text{if } \omega_{F_n, *} > 0 \end{cases} < C \text{ by Assumption 4.1(e) with } \delta = 2 \text{ and the constant } C \text{ in that assumption.}$$

Next we consider the second summand on the right-hand side of (A.29): $(\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*))^2$. Theorem 4.1 implies that

$$\begin{aligned} \bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*) &= -\frac{\text{tr}(\hat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1})}{2n} + O_p(n^{-1/2} \sigma_{F_n, n}) \\ &= -\frac{\text{tr}((\hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}) H_{F_n, \mathbf{k}}^{-1})}{2n} - \frac{\text{tr}(D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1})}{2n} + O_p(n^{-1/2} \sigma_{F_n, n}). \end{aligned} \quad (\text{A.31})$$

The first summand in the last line of (A.31) can be bounded by similar arguments as (A.27):

$$\begin{aligned} \frac{|\text{tr}((\hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}) H_{F_n, \mathbf{k}}^{-1})|}{2n} &\leq \frac{\|\hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}\| [|\text{tr}(H_{F_n, k_1}^{-1})| + |\text{tr}(H_{F_n, k_2}^{-1})|]}{2n} \\ &= (2n)^{-1} [o_p(n \sigma_{F_n, k_n}^2 |\mathbf{k}|^{-1}) O(|\mathbf{k}|)] = o_p(\sigma_{F_n, n}), \end{aligned} \quad (\text{A.32})$$

where the inequality is by (A.25), the equality holds by Assumptions 4.1(f) and 4.4(a). The second summand in the last line of (A.31) can be bounded as follows:

$$\begin{aligned} \frac{|\text{tr}(D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1})|}{2n} &\leq \frac{\sum_{j=1}^{|\mathbf{k}|} |\lambda_{F_n, j}|}{2n} \leq \frac{|\mathbf{k}|^{\frac{1}{2}} \sqrt{\sum_{j=1}^{|\mathbf{k}|} \lambda_{F_n, j}^2}}{n} \\ &= \sqrt{|\mathbf{k}| n^{-1}} \sqrt{\frac{\omega_{F_n, U, \mathbf{k}}^2}{n}} = O(\sqrt{|\mathbf{k}| n^{-1}} \sigma_{F_n, n}) = o(\sigma_{F_n, n}), \end{aligned} \quad (\text{A.33})$$

where the first inequality is by the definition of $\lambda_{F_n,j}$ ($j = 1, \dots, |\mathbf{k}|$) and the triangle inequality, the second inequality holds by the Cauchy-Schwarz inequality, the first equality is by the definition of $\omega_{F_n,U,\mathbf{k}}^2$, the second equality is by $\omega_{F_n,U,\mathbf{k}}^2 n^{-1} \leq 4 \sigma_{F_n,n}^2$, and the last equality is by Assumption 4.4(d). Therefore, from (A.31), (A.32) and (A.33), we have

$$\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*) = o_p(\sigma_{F_n,n}). \quad (\text{A.34})$$

For the third term, consider the derivation:

$$\begin{aligned} & n^{-1} \sum_{i=1}^n [(\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_n^*))^2 - (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2] \\ &= n^{-1} \sum_{i=1}^n [|\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_n^*)|^2] \\ &\quad + 2n^{-1} \sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) [\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_n^*)] \\ &= \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) + o_p(\sigma_{F_n,n}^2), \end{aligned} \quad (\text{A.35})$$

where the second equality holds by Assumptions 4.4(b)-(c). Also consider:

$$\begin{aligned} & \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) - n^{-1} \omega_{F_n,U,n}^2 \\ &= n^{-2} \sum_{i=1}^n [\ell_{\alpha,\mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)'(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \ell_{\alpha,\mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*) - \text{tr}((H_{F_n,k_n}^{-1} D_{F_n,k_n})^2)] \\ &\quad + 2n^{-2} \sum_{i=2}^n \left[\ell_{\alpha,\mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)'(H_{F_n,k_n}^{-1} D_{F_n,k_n} H_{F_n,k_n}^{-1}) \sum_{i'=1}^{i-1} \ell_{\alpha,\mathbf{k}}(Z_{i'}, \alpha_{\mathbf{k}}^*) \right]. \end{aligned} \quad (\text{A.36})$$

By algebra, we see that the first summand on the right-hand side of (A.36) equals

$$\begin{aligned} & n^{-1} \left| \text{tr}(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} \hat{D}_n(\alpha_{\mathbf{k}}^*)) - \text{tr}(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}}) \right| \\ &= n^{-1} \left| \text{tr} \left(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} (\hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n,\mathbf{k}}) \right) \right| \\ &\leq n^{-1} \text{tr}(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \left\| \hat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n,\mathbf{k}} \right\| \\ &= n^{-1} O_p(|\mathbf{k}|) o_p(n \sigma_{F_n,n}^2 |\mathbf{k}|^{-1}) = o_p(\sigma_{F_n,n}^2), \end{aligned} \quad (\text{A.37})$$

where the inequality holds by (A.25), and the second equality holds by Assumptions 4.4(a) and 4.3(b) and by

$$\text{tr}(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \leq |\mathbf{k}| \rho_{\max}(H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}) \leq C |\mathbf{k}|, \quad (\text{A.38})$$

where the last inequality is by Assumption 4.1(f). Consider the second moment of the second summand on the right-hand-side of (A.36). By the Martingale property discussed in detail in the proof of Theorem 4.1, its second moment equals

$$\begin{aligned}
& \frac{4}{n^4} \sum_{i=1}^n E_{F_n} \left[\ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)' (H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}) \sum_{i'=1}^{i-1} \ell_{\alpha, \mathbf{k}}(Z_{i'}, \alpha_{\mathbf{k}}^*) \right]^2 \\
&= \frac{2(n-1) E_{F_n} \left[\ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)' (H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}) D_{F_n, \mathbf{k}} (H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*) \right]}{n^3} \\
&\leq 2C \frac{E_{F_n} \left[\ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1} \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*) \right]}{n^2} \\
&= 2C \frac{\text{tr}((D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1})^2)}{n^2} = \frac{2C \omega_{F_0, U, \mathbf{k}}^2}{n^2} = o(\sigma_{F_n, n}^4), \tag{A.39}
\end{aligned}$$

where the first equality holds due to the i.i.d. structure of the data, the first inequality holds by the property of eigenvalues and Assumption 4.1(f), the second and third equality hold by the definitions of $D_{F_n, \mathbf{k}}$ and $\omega_{F_0, U, \mathbf{k}}^2$ respectively, and the last equality is by $n^{-1} \omega_{F_0, U, \mathbf{k}}^2 \leq 4\sigma_{F_n, n}^2$ and Assumption 4.3(b). The result (A.39) implies that the second summand on the right-hand side of (A.36) is $o_p(\sigma_{F_n, n}^2)$. Combining this with (A.35), (A.36), and (A.37), we get

$$n^{-1} \sum_{i=1}^n \left[(\ell(Z_i, \hat{\alpha}_{\mathbf{k}}) - \ell_{F_n}(\alpha_{\mathbf{k}}^*))^2 - (\ell(Z_i, \alpha_{\mathbf{k}}^*) - \ell_{F_n}(\alpha_{\mathbf{k}}^*))^2 \right] = n^{-1} \omega_{F_n, U, \mathbf{k}}^2 + o_p(\sigma_{F_n, n}^2). \tag{A.40}$$

The lemma immediately follows from (A.29), (A.30), (A.34), and (A.40). ■

Proof of Lemma 4.3. The proof of part (a) is similar to that for Lemma 4.1 and is omitted for brevity. Part (b) follows from the derivation:

$$\begin{aligned}
& \hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) - 2^{-1} n^{-1} \text{tr}((\hat{D}_n \hat{H}_n^{-1})^2) - \sigma_{F_n, n}^2 \\
&= \hat{\omega}_n^2(\hat{\alpha}_{\mathbf{k}}) - (\omega_{F_n, * }^2 + n^{-1} \omega_{F_n, U, \mathbf{k}}^2) + \frac{(n+1) \omega_{F_n, U, \mathbf{k}}^2}{2n^2} - \frac{\text{tr}((\hat{D}_n \hat{H}_n^{-1})^2)}{2n} \\
&= o_p(\sigma_{F_n, n}^2) + \frac{\omega_{F_n, U, \mathbf{k}}^2}{2n^2} - \frac{\text{tr}((\hat{D}_n \hat{H}_n^{-1})^2) - \omega_{F_n, U, \mathbf{k}}^2}{2n} = o_p(\sigma_{F_n, n}^2), \tag{A.41}
\end{aligned}$$

where the first equality holds by the definition of $\sigma_{F_n, n}^2 (= \omega_{F_n, * }^2 + 2^{-1} n^{-2} (n-1) \omega_{F_n, U, \mathbf{k}}^2)$, the second by Lemma 4.2, and the last by part (a) and by $2^{-1} n^{-2} (n-1) \omega_{F_n, U, \mathbf{k}}^2 \leq \sigma_{F_n, n}^2$ by definition. ■

Proof of Theorem 4.2. Using Lemma 4.3, we deduce that, for any $\varepsilon > 0$,

$$\Pr_{F_n}((2n)^{-1} \text{tr}((\hat{D}_n \hat{H}_n^{-1})^2) - \sigma_{F_n, n}^2 > \varepsilon) = \Pr_{F_n}(-\omega_{F_n, * }^2 + o_p(\sigma_{F_n, n}^2) > \varepsilon) \rightarrow 0, \tag{A.42}$$

where the equality holds by the definition of $\sigma_{F_n, n}^2$, and the convergence holds by Assumption 4.3(b). Next observe that

$$T_n = \frac{n^{-1/2} \sigma_{F_n, n}^{-1} \left[n \bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) + (1/2) \text{tr}(\widehat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1}) \right]}{\sigma_{F_n, n}^{-1} \widehat{\sigma}_n} - \frac{\sigma_{F_n, n}^{-1} 2^{-1} n^{-1/2} (\text{tr}(\widehat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1}) - \widehat{D}_n \widehat{H}_n^{-1})}{\sigma_{F_n, n}^{-1} \widehat{\sigma}_n},$$

which together with Theorem 4.1, Lemma 4.1, and equation (A.42) imply that $T_n \rightarrow_d N(0, 1)$ under any sequence $\{F_n\}$ such that $F_n \in \mathcal{F}_0$ for all n . This implies part (a) of the theorem immediately. Part (b) follows similarly. ■

B Proofs for Results in Section 6

B.1 Proof of Lemma 6.1

As we mentioned in the main paper, Lemma 6.1 follows immediately from Lemmas 4.2 and 4.3 in the main paper and Lemmas B.1, and B.2 below.

The extra assumptions referenced in Lemma 6.1 are as follows. Assumption B.1 is regarding the plug-in estimator $\psi_j(\hat{\alpha}_{k_j})$ for $j = 1, 2$. Sufficient Conditions can be found in Chen et al. (2014) and Chen and Liao (2015). Assumption B.2 is regarding the consistency of $\widehat{v}_{\psi, k_j}^{*2}$, $\widehat{\rho}_{0j, n}$, and $\widehat{\rho}_{12, n}$. Assumption B.2 calls for some new notation: Let $\ell_{\alpha, k_j}(Z; \alpha_j) = \partial \ell_{\alpha, k_j}(Z; \alpha_j(\beta_{k_j})) / \partial \beta'_{k_j}$. Define

$$D_{\ell, k_j, n} = E_{F_n} \left[\ell^2(Z; \alpha_n^*) \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*)' \right] \text{ and } H_{\ell, k_j, n} = E_{F_n} \left[\ell(Z; \alpha_n^*) \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \right]. \quad (\text{B.1})$$

Assumption B.1 Under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have for $j = 1, 2$,

$$n^{1/2} \left[\psi_j(\widehat{\alpha}_{k_j}) - \psi_j(\alpha_{j, n}^*) \right] (v_{\psi, k_j}^*)^{-1} = n^{-1/2} \sum_{i=1}^n \psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1} \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) (v_{\psi, k_j}^*)^{-1} + o_p(1) \quad (\text{B.2})$$

and $E_{F_n} [|\psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1} \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) (v_{\psi, k_j}^*)^{-1}|^4] = o(n)$.

Assumption B.2 For any $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have for $j = 1, 2$:

- (a) $\frac{1}{v_{\psi, k_j}^*} \left\| \psi_{\alpha, k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right\| = o_p(|\mathbf{k}|^{-1/2})$;
- (b) $\rho_{\max}(D_{\ell, k_j, n}) \leq C$ and $\rho_{\max}(H_{\ell, k_j, n}) \leq C$;
- (c) the following expansion holds

$$n^{-1} \sum_{i=1}^n \left[\ell_{\alpha, k_j}(Z_i; \widehat{\alpha}_{k_j}) \ell(Z_i; \widehat{\alpha}_{\mathbf{k}}) - \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) \ell(Z_i; \alpha_n^*) \right]$$

$$\begin{aligned}
&= n^{-1} \sum_{i=1}^n \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_n^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) \\
&+ n^{-1} \sum_{i=1}^n \ell(Z_i; \alpha_n^*) \ell_{\alpha\alpha, k_j}(Z_i; \alpha_{k_j}^*) H_{F_n, k_j}^{-1} \bar{\ell}_{\alpha_j, n}(\alpha_{k_j}^*) + o_p(\sigma_{F_n, n});
\end{aligned}$$

- (d) $\left\| n^{-1} \sum_{i=1}^n \ell(Z_i; \alpha_n^*) \ell_{\alpha\alpha, k_j}(Z_i; \alpha_{k_j}^*) - H_{\ell, k_j, n} \right\| = o_p(|\mathbf{k}|^{-1/2});$
(e) $\left\| \psi_{\alpha, k_j}(\alpha_{k_j}^*) / v_{\psi, k_j}^* \right\| \leq C;$
(f) $\|\widehat{H}_n - H_{F_n, \mathbf{k}}\| = o_p(|\mathbf{k}|^{-1/2})$ and $\|\widehat{D}_n - D_{F_n, \mathbf{k}}\| = o_p(|\mathbf{k}|^{-1/2}).$

Lemma B.1 shows the joint convergence in distribution of the normalized log-likelihood ratio and the normalized parameter estimators.

Lemma B.1 *Suppose that Assumptions 4.1, 4.3 and B.1 hold. For any sequence $\{F_n\}_{n \geq 1}$ with $F_n \in \mathcal{F}$ for all n and any subsequence $\{u_n\}$ of $\{n\}$ such that $\rho_{0j, u_n} \rightarrow \rho_{0j}$ and $\rho_{12, u_n} \rightarrow \rho_{12}$ for some ρ_{0j} and $\rho_{12} \in [-1, 1]$, we have*

$$G_{u_n, F_{u_n}} = \begin{pmatrix} \frac{n [\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) - E_{F_{u_n}}[\ell(Z; \alpha_{F_{u_n}}^*)]] + (1/2) \text{tr}(\widehat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_0, \mathbf{k}}^{-1})}{n^{1/2} \sigma_{F_{u_n}, n}} \\ n^{1/2} \left[\psi_1(\widehat{\alpha}_{k_1}) - \psi_1(\alpha_{F_{u_n}, 1}^*) \right] (v_{\psi, k_1}^*)^{-1} \\ n^{1/2} \left[\psi_2(\widehat{\alpha}_{k_2}) - \psi_2(\alpha_{F_{u_n}, 2}^*) \right] (v_{\psi, k_2}^*)^{-1} \end{pmatrix} \rightarrow_d N(\mathbf{0}_3, \Sigma_G).$$

Proof of Lemma B.1. Let $\kappa = (\kappa_1, \kappa_2, \kappa_3)' \in R^3$ be any vector with $\kappa' \kappa = 1$. For ease of notations, let $\mathbf{W}_{i,n} = (W_{i,n}, W_{1,i,n}, W_{2,i,n})'$ where

$$W_{i,n} = V_{i,n} - U_{i,n}, \text{ and } W_{j,i,n} = \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1} \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*)}{n^{1/2} v_{\psi, k_j}^*} \text{ for } j = 1, 2, \quad (\text{B.3})$$

where $V_{i,n}$ and $U_{i,n}$ are defined in (A.3). By the definition of κ , and Assumptions 4.3(a) and B.1,

$$\kappa' G_{n, F_n} = \sum_{i=1}^n \kappa' \mathbf{W}_{i,n} + o_p(1). \quad (\text{B.4})$$

Let $Z_0 = 0$ and $\mathcal{F}_{i,n}$ be the natural filtration generated by $\{Z_0, Z_1, \dots, Z_i\}$ under F_n for $i = 0, \dots, n$. Under Assumptions 4.1(b) and 4.1(d),

$$E_{F_n} [\kappa' \mathbf{W}_{i,n} | \mathcal{F}_{i-1, n}] = 0,$$

for all $i = 1, \dots, n$, which implies that $\{\kappa' \mathbf{W}_{i,n}\}_{i \leq n}$ is a martingale difference array. We shall use the martingale central limit theorem (ref. Corollary 3.1 in Hall and Heyde (1980)) to show the

desired convergence. It suffices to verify the following two sufficient conditions:

$$\sum_{i=1}^n E_{F_{u_n}} \left[|\kappa' \mathbf{W}_{i,u_n}|^2 \middle| \mathcal{F}_{i-1,u_n} \right] \rightarrow_p \kappa' \Sigma_G \kappa, \text{ and} \quad (\text{B.5})$$

$$\sum_{i=1}^{u_n} E_{F_{u_n}} \left[|\kappa' \mathbf{W}_{i,u_n}|^2 1_{\{|\kappa' \mathbf{W}_{i,u_n}| > \varepsilon\}} \right] \rightarrow_p 0, \quad \forall \varepsilon > 0. \quad (\text{B.6})$$

In order to show (B.5), it suffices to show that

$$\sum_{i=1}^n E_{F_n} \left[\mathbf{W}_{i,n} \mathbf{W}'_{i,n} \middle| \mathcal{F}_{i-1,n} \right] - \Sigma_G = o_p(1). \quad (\text{B.7})$$

Note that in part (a) of equation (A.6) in the proof of Theorem 4.1, we have shown that

$$\sum_{i=1}^n E_{F_n} \left[W_{i,n}^2 \middle| \mathcal{F}_{i-1,n} \right] \rightarrow_p 1. \quad (\text{B.8})$$

Also, we have, for $j = 1, 2$,

$$\begin{aligned} & \sum_{i=1}^n E_{F_n} \left[W_{j,i,n}^2 \middle| \mathcal{F}_{i-1,n} \right] \\ &= n^{-1} \sum_{i=1}^n \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_n,k_j}^{-1}}{v_{\psi,k_j}^*} E_{F_n} \left[\ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*) \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)' \middle| \mathcal{F}_{i-1,n} \right] \frac{H_{F_n,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^*} \\ &= n^{-1} \sum_{i=1}^n \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_n,k_j}^{-1} D_{F_n,k_j} H_{F_n,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^{*2}} = 1, \end{aligned} \quad (\text{B.9})$$

where the second equality is by Assumption 4.1(b) and the definition of D_{F_n,k_j} , the last equality is by the definition of v_{ψ,k_j}^* . Next, note that, for $j = 1, 2$,

$$\begin{aligned} \sum_{i=1}^n E_{F_n} \left[W_{i,n} W_{j,i,n} \middle| \mathcal{F}_{i-1,n} \right] &= \sum_{i=1}^n E_{F_n} \left[V_{i,n} W_{j,i,n} \middle| \mathcal{F}_{i-1,n} \right] - \sum_{i=2}^n E_{F_n} \left[U_{i,n} W_{j,i,n} \middle| \mathcal{F}_{i-1,n} \right] \\ &= n^{-1} \sum_{i=1}^n E_{F_n} \left[\frac{\ell(Z_i; \alpha_n^*) \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)' H_{F_n,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{\sigma_{F_n,n} v_{\psi,k_j}^*} \middle| \mathcal{F}_{i-1,n} \right] \\ &= n^{-1} \sum_{i=1}^n \rho_{0j,n} = \rho_{0j,n}, \end{aligned} \quad (\text{B.10})$$

where the second equality is by the definitions of $V_{i,n}$, $U_{i,n}$ and $W_{j,i,n}$ and $E_{F_n} \left[U_{i,n} W_{j,i,n} \middle| \mathcal{F}_{i-1,n} \right] = 0$ which follows by the i.i.d. assumption and Assumption 4.1(c), the third equality is by the definition

of $\rho_{0j,n}$ and the i.i.d. assumption. Similarly

$$\begin{aligned}
& \sum_{i=1}^n E_{F_n} [W_{1,i,n} W_{2,i,n} | \mathcal{F}_{i-1,n}] \\
&= \sum_{i=1}^n E_{F_n} \left[\frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)' H_{F_n,k_1}^{-1} \ell_{\alpha,k_1}(Z_i; \alpha_{k_1}^*) \ell'_{\alpha,k_2}(Z_i; \alpha_{k_2}^*) H_{F_n,k_2}^{-1} \psi_{\alpha,k_2}(\alpha_{k_2}^*)}{n^{\frac{1}{2}} v_{\psi,k_1}^*} \frac{1}{n^{\frac{1}{2}} v_{\psi,k_2}^*} \middle| \mathcal{F}_{i-1,n} \right] \\
&= \frac{1}{n} \sum_{i=1}^n \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)' H_{F_n,k_1}^{-1}}{v_{\psi,k_1}^*} E_{F_n} [\ell_{\alpha,k_1}(Z; \alpha_{k_1}^*) \ell_{\alpha,k_2}(Z; \alpha_{k_2}^*)] \frac{H_{F_n,k_2}^{-1} \psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \\
&= \frac{1}{n} \sum_{i=1}^n \rho_{12,n} = \rho_{12,n}. \tag{B.11}
\end{aligned}$$

Collecting the results in (B.8), (B.9), (B.10) and (B.11), we get

$$\sum_{i=1}^n E_{F_n} [\mathbf{W}_{i,n} \mathbf{W}'_{i,n} | \mathcal{F}_{i-1,n}] - \Sigma_G = \begin{pmatrix} \sum_{i=1}^n E_{F_n} [W_{i,n}^2 | \mathcal{F}_{i-1,n}] - 1 & 0_{1 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix} = o_p(1)$$

which proves (B.7) and hence (B.5).

We next verify (B.6). First, by the monotonicity of expectation and the C_r inequality,

$$\begin{aligned}
\sum_{i=1}^n E_{F_n} [|\kappa' \mathbf{W}_{i,n}|^2 \mathbf{1}\{|\kappa' \mathbf{W}_{i,n}| > \varepsilon\}] &\leq \sum_{i=1}^n E_{F_n} [|\kappa' \mathbf{W}_{i,n}|^4 \varepsilon^{-2}] \\
&\leq \frac{C}{\varepsilon^2} \sum_{i=1}^n E_{F_n} [|\kappa_1 W_{i,n}|^4 + |\kappa_2 W_{1,i,n}|^4 + |\kappa_3 W_{2,i,n}|^4]. \tag{B.12}
\end{aligned}$$

In equation (A.24) in the proof of Theorem 4.1, we have shown that

$$\frac{1}{\varepsilon^2} \sum_{i=1}^n E_{F_n} [|W_{i,n}|^4] = o(1). \tag{B.13}$$

Under Assumptions 4.1(a)-(b) and B.1, we have

$$\begin{aligned}
& \frac{1}{\varepsilon^2} \sum_{i=1}^n E_{F_n} [|\kappa_2 W_{1,i,n}|^4 + |\kappa_3 W_{2,i,n}|^4] \\
&\leq \frac{1}{\varepsilon^2 n^2} \sum_{i=1}^n \sum_{j=1}^2 E_{F_n} \left[\left| \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'}{v_{\psi,k_j}^*} H_{F_n,k_j}^{-1} \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*) \right|^4 \right] \\
&= \frac{1}{\varepsilon^2 n} \sum_{j=1}^2 E_{F_n} \left[\left| \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'}{v_{\psi,k_j}^*} H_{F_n,k_j}^{-1} \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*) \right|^4 \right] = o(1) \tag{B.14}
\end{aligned}$$

which combined with (B.12) and (B.13) proves (B.6). ■

Lemma B.2 shows the consistency of the estimated variances and correlations.

Lemma B.2 *Suppose that Assumptions 4.1(b)-(g), 4.3(b), and B.2 hold. Then under all sequences $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

- (a) $\max_{j=1,2} \left[\widehat{v}_{\psi,k_j}^{*2} / v_{\psi,k_j}^{*2} - 1 \right] = o_p(1)$;
- (b) $\widehat{\rho}_{12,n} - \rho_{12,n} = o_p(1)$;
- (c) $\max_{j=1,2} \left| \widehat{\rho}_{0j,n} - \rho_{0j,n} \right| = o_p(1)$.

Proof of Lemma B.2. (a) Note that under Assumption B.2(f), we have

$$\|\widehat{H}_{k_j,n} - H_{F_n,k_j}\| = o_p(|\mathbf{k}|^{-1/2}) \text{ and } \|\widehat{D}_{k_j,n} - D_{F_n,k_j}\| = o_p(|\mathbf{k}|^{-1/2}), \quad (\text{B.15})$$

which together with Assumption 4.1(f) imply that

$$0 \leq \rho_{\max}(\widehat{D}_{k_j,n}) \leq 2C \text{ and } (2C)^{-1} \leq |\rho_{\min}(\widehat{H}_{k_j,n})| \leq |\rho_{\max}(\widehat{H}_{k_j,n})| \leq 2C \quad (\text{B.16})$$

with probability approaching 1. By definition, we can write

$$\begin{aligned} \frac{\widehat{v}_{\psi,k_j}^{*2} - v_{\psi,k_j}^{*2}}{v_{\psi,k_j}^{*2}} &= \left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right]' \frac{\widehat{H}_{k_j,n}^{-1} \widehat{D}_{k_j,n} \widehat{H}_{k_j,n}^{-1}}{v_{\psi,k_j}^{*2}} \left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right] \\ &+ 2 \left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right]' \frac{\widehat{H}_{k_j,n}^{-1} \widehat{D}_{k_j,n} \widehat{H}_{k_j,n}^{-1}}{v_{\psi,k_j}^{*2}} \psi_{\alpha,k_j}(\alpha_{k_j}^*) \\ &+ \psi_{\alpha,k_j}'(\alpha_{k_j}^*) H_{F_n,k_j}^{-1} \frac{\widehat{H}_{k_j,n} - H_{F_n,k_j}}{v_{\psi,k_j}^{*2}} \widehat{H}_{k_j,n}^{-1} \widehat{D}_{k_j,n} \widehat{H}_{k_j,n}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*) \\ &+ \psi_{\alpha,k_j}'(\alpha_{k_j}^*) H_{F_n,k_j}^{-1} \frac{\widehat{D}_{k_j,n} - D_{F_n,k_j}}{v_{\psi,k_j}^{*2}} \widehat{H}_{k_j,n}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*) \\ &+ \psi_{\alpha,k_j}'(\alpha_{k_j}^*) H_{F_n,k_j}^{-1} D_{F_n,k_j} H_{F_n,k_j}^{-1} \frac{\widehat{H}_{k_j,n} - H_{F_n,k_j}}{v_{\psi,k_j}^{*2}} \widehat{H}_{k_j,n}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*). \end{aligned} \quad (\text{B.17})$$

By (B.16) and Assumption B.2(a),

$$\begin{aligned} &\left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right]' \frac{\widehat{H}_{k_j,n}^{-1} \widehat{D}_{k_j,n} \widehat{H}_{k_j,n}^{-1}}{v_{\psi,k_j}^{*2}} \left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right] \\ &\leq \frac{\rho_{\max}(\widehat{D}_{k_j,n})}{\rho_{\min}^2(\widehat{H}_{k_j,n})} \frac{1}{v_{\psi,k_j}^{*2}} \left\| \psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right\|^2 = o_p(1). \end{aligned} \quad (\text{B.18})$$

By the Cauchy-Schwarz inequality, (B.16), Assumptions B.2(a) and B.2(e),

$$\begin{aligned}
& \left| \left[\psi_{\alpha, k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right]' \frac{\widehat{H}_{k_j, n}^{-1} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-1}}{v_{\psi, k_j}^{*2}} \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right|^2 \\
& \leq \frac{\left\| \psi_{\alpha, k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right\|^2}{v_{\psi, k_j}^{*2}} \psi_{\alpha, k_j}'(\alpha_{k_j}^*) \frac{\widehat{H}_{k_j, n}^{-1} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-2} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-1}}{v_{\psi, k_j}^{*2}} \psi_{\alpha, k_j}(\alpha_{k_j}^*) \\
& \leq \frac{\rho_{\max}^2(\widehat{D}_{k_j, n})}{\rho_{\min}^4(\widehat{H}_{k_j, n})} \frac{\psi_{\alpha, k_j}'(\alpha_{k_j}^*) \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^{*2}} \frac{\left\| \psi_{\alpha, k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right\|^2}{v_{\psi, k_j}^{*2}} = o_p(1). \tag{B.19}
\end{aligned}$$

By the Cauchy-Schwarz inequality, (B.15), (B.16), Assumptions B.2(a) and B.2(e),

$$\begin{aligned}
& \left| \psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1} \frac{\widehat{H}_{k_j, n} - H_{F_n, k_j}}{v_{\psi, k_j}^{*2}} \widehat{H}_{k_j, n}^{-1} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-1} \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right|^2 \\
& \leq \left\| \widehat{H}_{k_j, n} - H_{F_n, k_j} \right\|^2 \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-2} \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^{*2}} \\
& \quad \times \frac{\psi_{\alpha, k_j}'(\alpha_{k_j}^*) \widehat{H}_{k_j, n}^{-1} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-2} \widehat{D}_{k_j, n} \widehat{H}_{k_j, n}^{-1} \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^{*2}} \\
& \leq \frac{\rho_{\max}^2(\widehat{D}_{k_j, n}) \left\| \widehat{H}_{k_j, n} - H_{F_n, k_j} \right\|^2}{(\rho_{\min}((\widehat{H}_{k_j, n})^2))^2 \rho_{\min}(H_{F_0, k_j}^2)} \left| \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^{*2}} \right|^2 = o_p(1). \tag{B.20}
\end{aligned}$$

Using the same arguments for showing (B.20), we can prove that

$$\begin{aligned}
& \left| \psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_0, k_j}^{-1} \frac{\widehat{D}_{k_j, n} - D_{F_n, k_j}}{v_{\psi, k_j}^{*2}} \widehat{H}_{k_j, n}^{-1} \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right| = o_p(1) \text{ and} \\
& \left| \psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_0, k_j}^{-1} D_{F_0, k_j} H_{F_0, k_j}^{-1} \frac{\widehat{H}_{k_j, n} - H_{F_n, k_j}}{v_{\psi, k_j}^{*2}} \widehat{H}_{k_j, n}^{-1} \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right| = o_p(1). \tag{B.21}
\end{aligned}$$

Collecting the results in (B.17)-(B.21), we immediately get the claimed result.

(b) By the consistency of $\widehat{v}_{\psi, k_j}^{*2}$ and the fact that $|\rho_{12, n}| \leq 1$, it is sufficient to show that

$$\widehat{\rho}_{12, n}^* \equiv \frac{\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1})' \widehat{H}_{k_1, n}^{-1} \widehat{D}_{k_1, k_2, n}}{v_{\psi, k_1}^*} \frac{\widehat{H}_{k_2, n}^{-1} \psi_{\alpha, k_2}(\widehat{\alpha}_{k_2})}{v_{\psi, k_2}^*} = \rho_{12, n} + o_p(1). \tag{B.22}$$

By Assumption B.2(f), we have

$$\left\| \widehat{D}_{k_1, k_2, n} - D_{F_n, k_1, k_2} \right\| = o_p(|\mathbf{k}|^{-1/2}). \quad (\text{B.23})$$

Note that

$$\begin{aligned} \widehat{\rho}_{12, n}^* - \rho_{12, n} &= \frac{[\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)]'}{v_{\psi, k_1}^*} \widehat{H}_{k_1, n}^{-1} \widehat{D}_{k_1, k_2, n} \widehat{H}_{k_2, n}^{-1} \frac{[\psi_{\alpha, k_2}(\widehat{\alpha}_{k_2}) - \psi_{\alpha, k_2}(\alpha_{k_2}^*)]}{v_{\psi, k_2}^*} \\ &+ 2 \frac{[\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)]'}{v_{\psi, k_1}^*} \widehat{H}_{k_1, n}^{-1} \widehat{D}_{k_1, k_2, n} \widehat{H}_{k_2, n}^{-1} \frac{\psi_{\alpha, k_2}(\alpha_{k_2}^*)}{v_{\psi, k_2}^*} \\ &+ \frac{\psi_{\alpha, k_1}(\alpha_{k_1}^*)'}{v_{\psi, k_1}^*} H_{F_n, k_1}^{-1} (\widehat{H}_{k_1, n} - H_{F_n, k_1}) \widehat{H}_{k_1, n}^{-1} \widehat{D}_{k_1, k_2, n} \widehat{H}_{k_2, n}^{-1} \frac{\psi_{\alpha, k_2}(\alpha_{k_2}^*)}{v_{\psi, k_2}^*} \\ &+ \frac{\psi_{\alpha, k_1}(\alpha_{k_1}^*)'}{v_{\psi, k_1}^*} H_{F_n, k_1}^{-1} (\widehat{D}_{k_1, k_2, n} - D_{F_n, k_1, k_2}) \widehat{H}_{k_2, n}^{-1} \frac{\psi_{\alpha, k_2}(\alpha_{k_2}^*)}{v_{\psi, k_2}^*} \\ &+ \frac{\psi_{\alpha, k_1}(\alpha_{k_1}^*)'}{v_{\psi, k_1}^*} H_{F_n, k_1}^{-1} D_{F_n, k_1, k_2} H_{F_n, k_1}^{-1} (\widehat{H}_{k_1, n} - H_{F_n, k_1}) \widehat{H}_{k_1, n}^{-1} \frac{\psi_{\alpha, k_2}(\alpha_{k_2}^*)}{v_{\psi, k_2}^*}. \end{aligned} \quad (\text{B.24})$$

Lemma E.2 together with Assumption 4.1(f) implies that

$$\rho_{\max}(D'_{F_n, k_1, k_2} D_{F_n, k_1, k_2}) \leq C. \quad (\text{B.25})$$

Similarly, Lemma E.2 together with Assumptions 4.1(f) and (B.23) implies that

$$\rho_{\max}(\widehat{D}'_{k_1, k_2, n} \widehat{D}_{k_1, k_2, n}) \leq C \quad (\text{B.26})$$

with probability approaching 1. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\left| \frac{[\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)]'}{v_{\psi, k_1}^*} \widehat{H}_{k_1, n}^{-1} \widehat{D}_{k_1, k_2, n} \widehat{H}_{k_2, n}^{-1} \frac{[\psi_{\alpha, k_2}(\widehat{\alpha}_{k_2}) - \psi_{\alpha, k_2}(\alpha_{k_2}^*)]}{v_{\psi, k_2}^*} \right|^2 \\ &\leq \frac{[\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)]'}{v_{\psi, k_1}^*} \widehat{H}_{k_1, n}^{-2} \frac{[\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)]}{v_{\psi, k_1}^*} \\ &\times \frac{[\psi_{\alpha, k_2}(\widehat{\alpha}_{k_2}) - \psi_{\alpha, k_2}(\alpha_{k_2}^*)]'}{v_{\psi, k_2}^*} \widehat{H}_{k_2, n}^{-1} \widehat{D}'_{k_1, k_2, n} \widehat{D}_{k_1, k_2, n} \widehat{H}_{k_2, n}^{-1} \frac{[\psi_{\alpha, k_2}(\widehat{\alpha}_{k_2}) - \psi_{\alpha, k_2}(\alpha_{k_2}^*)]}{v_{\psi, k_2}^*} \\ &\leq \frac{\rho_{\max}(\widehat{D}'_{k_1, k_2, n} \widehat{D}_{k_1, k_2, n})}{\rho_{\min}((\widehat{H}_{k_1, n})^2) \rho_{\min}^2((\widehat{H}_{k_2, n})^2)} \frac{\|\psi_{\alpha, k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha, k_1}(\alpha_{k_1}^*)\|^2}{v_{\psi, k_1}^{*2}} \frac{\|\psi_{\alpha, k_2}(\widehat{\alpha}_{k_2}) - \psi_{\alpha, k_2}(\alpha_{k_2}^*)\|^2}{v_{\psi, k_2}^{*2}} = o_p(1), \end{aligned} \quad (\text{B.27})$$

where the last equality is by Assumption B.2(a), (B.16) and (B.26). Similarly,

$$\begin{aligned}
& \left| \frac{[\psi_{\alpha,k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha,k_1}(\alpha_{k_1}^*)]'}{v_{\psi,k_1}^*} \widehat{H}_{k_1,n}^{-1} \widehat{D}_{k_1,k_2,n} \widehat{H}_{k_2,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \right|^2 \\
& \leq \frac{[\psi_{\alpha,k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha,k_1}(\alpha_{k_1}^*)]'}{v_{\psi,k_1}^*} \widehat{H}_{k_1,n}^{-2} \frac{[\psi_{\alpha,k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha,k_1}(\alpha_{k_1}^*)]}{v_{\psi,k_1}^*} \\
& \times \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)'}{v_{\psi,k_2}^*} \widehat{H}_{k_2,n}^{-1} \widehat{D}'_{k_1,k_2,n} \widehat{D}_{k_1,k_2,n} \widehat{H}_{k_2,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \\
& \leq \frac{\rho_{\max}(\widehat{D}'_{k_1,k_2,n} \widehat{D}_{k_1,k_2,n})}{\rho_{\min}((\widehat{H}_{k_1,n})^2) \rho_{\min}^2((\widehat{H}_{k_2,n})^2)} \frac{\|\psi_{\alpha,k_1}(\widehat{\alpha}_{k_1}) - \psi_{\alpha,k_1}(\alpha_{k_1}^*)\|^2}{v_{\psi,k_1}^{*2}} \frac{\psi'_{\alpha,k_2}(\alpha_{k_2}^*) \psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^{*2}} = o_p(1) \quad (\text{B.28})
\end{aligned}$$

where the last equality is by Assumptions B.2(a), B.2(e), (B.16) and (B.26). Moreover,

$$\begin{aligned}
& \left| \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)'}{v_{\psi,k_1}^*} H_{F_n,k_1}^{-1} (\widehat{H}_{k_1,n} - H_{F_n,k_1}) \widehat{H}_{k_1,n}^{-1} \widehat{D}_{k_1,k_2,n} \widehat{H}_{k_2,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \right|^2 \\
& \leq \left\| \widehat{H}_{k_1,n} - H_{F_n,k_1} \right\|^2 \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)'}{v_{\psi,k_1}^*} H_{F_n,k_1}^{-2} \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)}{v_{\psi,k_1}^*} \\
& \times \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)'}{v_{\psi,k_2}^*} \widehat{H}_{k_2,n}^{-1} \widehat{D}'_{k_1,k_2,n} \widehat{H}_{k_1,n}^{-2} \widehat{D}_{k_1,k_2,n} \widehat{H}_{k_2,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \\
& \leq \frac{\left\| \widehat{H}_{k_1,n} - H_{F_n,k_1} \right\|^2 \rho_{\max}(\widehat{D}'_{k_1,k_2,n} \widehat{D}_{k_1,k_2,n})}{\rho_{\min}^2(H_{F_n,k_1}) \rho_{\min}^2(\widehat{H}_{k_1,n}) \rho_{\min}^2(\widehat{H}_{k_2,n})} \\
& \times \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)' \psi_{\alpha,k_1}(\alpha_{k_1}^*)}{v_{\psi,k_1}^{*2}} \frac{\psi'_{\alpha,k_2}(\alpha_{k_2}^*) \psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^{*2}} = o_p(1) \quad (\text{B.29})
\end{aligned}$$

where the last equality is by Assumptions B.2(a), B.2(e), (B.15), (B.16) and (B.26). Using the same arguments in showing (B.29), but replacing (B.26) with (B.25), we have

$$\left| \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)'}{v_{\psi,k_1}^*} H_{F_n,k_1}^{-1} D_{F_n,k_1,k_2} H_{F_n,k_1}^{-1} (\widehat{H}_{k_1,n} - H_{F_n,k_1}) \widehat{H}_{k_1,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \right|^2 = o_p(1). \quad (\text{B.30})$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)'}{v_{\psi,k_1}^*} H_{F_n,k_1}^{-1} (\widehat{D}_{k_1,k_2,n} - D_{F_n,k_1,k_2}) \widehat{H}_{k_2,n}^{-1} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^*} \right|^2 \\
& \leq \frac{\left\| \widehat{D}_{k_1,k_2,n} - D_{F_n,k_1,k_2} \right\|^2}{\rho_{\min}((H_{F_n,k_1})^2) \rho_{\min}((\widehat{H}_{k_2,n})^2)} \frac{\psi_{\alpha,k_1}(\alpha_{k_1}^*)' \psi_{\alpha,k_1}(\alpha_{k_1}^*)}{v_{\psi,k_1}^{*2}} \frac{\psi_{\alpha,k_2}(\alpha_{k_2}^*)' \psi_{\alpha,k_2}(\alpha_{k_2}^*)}{v_{\psi,k_2}^{*2}} = o_p(1), \quad (\text{B.31})
\end{aligned}$$

where the last equality is by Assumptions 4.1(f), B.2(a), B.2(e), (B.16) and (B.23). Collecting the results in (B.24), (B.27)-(B.31), we immediately prove (B.22).

(c) By the consistency of $\widehat{v}_{\psi,k_j}^{*2}$ and $\widehat{\sigma}_n$, and the fact that $|\rho_{0j,n}| \leq 1$, it is sufficient to show that

$$\widehat{\rho}_{0j,n}^* \equiv \frac{\psi'_{\alpha,k_j}(\widehat{\alpha}_{k_j})\widehat{H}_{k_j,n}^{-1}}{nv_{\psi,k_j}^*\sigma_{F_n,n}} \sum_{i=1}^n \ell_{\alpha,k_j}(Z_i; \widehat{\alpha}_{k_j})\ell(Z_i; \widehat{\alpha}_n) = \rho_{0j,n} + o_p(1). \quad (\text{B.32})$$

By definition, we can write

$$\begin{aligned} \widehat{\rho}_{0j,n}^* - \rho_{0j,n} &= \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'H_{F_n,k_j}^{-1}}{nv_{\psi,k_j}^*\sigma_{F_n,n}} \sum_{i=1}^n \left[\ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)\ell(Z_i; \alpha_n^*) - E_{F_n} \left[\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*)\ell(Z; \alpha_n^*) \right] \right] \\ &+ \frac{\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j})'\widehat{H}_{k_j,n}^{-1} - \psi_{\alpha,k_j}(\alpha_{k_j}^*)'H_{F_n,k_j}^{-1}}{nv_{\psi,k_j}^*\sigma_{F_n,n}} \sum_{i=1}^n \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)\ell(Z_i; \alpha_n^*) \\ &+ \frac{\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j})'\widehat{H}_{k_j,n}^{-1}}{nv_{\psi,k_j}^*\sigma_{F_n,n}} \sum_{i=1}^n \left[\ell_{\alpha,k_j}(Z_i; \widehat{\alpha}_{k_j})\ell(Z_i; \widehat{\alpha}_n) - \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)\ell(Z_i; \alpha_n^*) \right]. \end{aligned} \quad (\text{B.33})$$

Under the i.i.d assumption, Assumptions 4.1(f), 4.3(b) and B.2(e),

$$\begin{aligned} Var &\left[\frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'H_{F_n,k_j}^{-1}}{nv_{\psi,k_j}^*\sigma_{F_n,n}} \sum_{i=1}^n \left[\ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*)\ell(Z_i; \alpha_n^*) - E_{F_n} \left[\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*)\ell(Z; \alpha_n^*) \right] \right] \right] \\ &= \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'H_{F_n,k_j}^{-1}}{n^{\frac{1}{2}}v_{\psi,k_j}^*\sigma_{F_n,n}} E_{F_n} \left[\ell^2(Z; \alpha_n^*)\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*)\ell'_{\alpha,k_j}(Z; \alpha_{k_j}^*) \right] \frac{H_{F_n,k_j}^{-1}\psi_{\alpha,k_j}(\alpha_{k_j}^*)}{n^{\frac{1}{2}}v_{\psi,k_j}^*\sigma_{F_n,n}} \\ &\leq \frac{\rho_{\max}(D_{\ell,k_j,n})}{\rho_{\min}((H_{F_n,k_j})^2)} \frac{\psi'_{\alpha,k_j}(\alpha_{k_j}^*)\psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^{*2}} \frac{1}{n\sigma_{F_n,n}^2} = o_p(1), \end{aligned} \quad (\text{B.34})$$

which together with the Markov inequality implies that the first summand in (B.33) is $o_p(1)$. By the triangle inequality

$$\begin{aligned} &\left\| \frac{\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j})'\widehat{H}_{k_j,n}^{-1} - \psi'_{\alpha,k_j}(\alpha_{k_j}^*)H_{F_n,k_j}^{-1}}{v_{\psi,k_j}^{*2}} \right\|^2 \\ &\leq \frac{\left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right]' \widehat{H}_{k_j,n}^{-2} \left[\psi_{\alpha,k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha,k_j}(\alpha_{k_j}^*) \right]}{v_{\psi,k_j}^{*2}} \\ &+ \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)'H_{F_n,k_j}^{-1} [\widehat{H}_{k_j,n} - H_{F_n,k_j}] \widehat{H}_{k_j,n}^{-2} [\widehat{H}_{k_j,n} - H_{F_n,k_j}] H_{F_n,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^{*2}} \end{aligned}$$

$$\leq \frac{\left\| \psi_{\alpha, k_j}(\widehat{\alpha}_{k_j}) - \psi_{\alpha, k_j}(\alpha_{k_j}^*) \right\|^2}{\rho_{\min}((\widehat{H}_{k_j, n})^2) v_{\psi, k_j}^{*2}} + \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' \psi_{\alpha, k_j}(\alpha_{k_j}^*) \left\| \widehat{H}_{k_j, n} - H_{F_n, k_j} \right\|^2}{\rho_{\min}((\widehat{H}_{k_j, n})^2) \rho_{\min}((H_{F_n, k_j})^2) v_{\psi, k_j}^{*2}} = o_p(|\mathbf{k}|^{-1}), \quad (\text{B.35})$$

where the last equality is by (B.15), (B.16), Assumptions 4.1(f), B.2(a) and B.2(e). By the i.i.d. assumption,

$$\begin{aligned} & E_{F_n} \left[\left\| n^{-1} \sum_{i=1}^n \left[\ell(Z_i; \alpha_n^*) \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) - E_{F_n} \left[\ell(Z; \alpha_n^*) \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \right] \right] \right\|^2 \right] \\ & \leq \frac{1}{n} E_{F_n} \left[\ell^2(Z; \alpha_n^*) \left\| \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \right\|^2 \right] = O(k_j n^{-1}). \end{aligned} \quad (\text{B.36})$$

where the equality is by Assumption B.2(b). Also note that

$$\left\| E_{F_n} \left[\ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \ell(Z; \alpha_n^*) \right] \right\|^2 \leq E_{F_n} \left[|\ell(Z; \alpha_n^*)|^2 \right] E_{F_n} \left[\ell_{\alpha, k_j}(Z; \alpha_{k_j}^*)' \ell_{\alpha, k_j}(Z; \alpha_{k_j}^*) \right] = o_p(k_j \sigma_{F_n, n}^2)$$

The above two displays together with the triangle inequality imply that

$$\left\| n^{-1} \sum_{i=1}^n \ell(Z_i; \alpha_n^*) \ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) \right\|^2 \leq O_p(k_j n^{-1}) + o_p(k_j \sigma_{F_n, n}^2). \quad (\text{B.37})$$

Equations (B.35) and (B.37) imply that the second summand in (B.33) is bounded from above by

$$o_p(|\mathbf{k}|^{-1/2}) (O(k_j^{1/2} n^{-1/2} \sigma_{F_n, n}^{-1}) + o_p(k_j^{1/2} \sigma_{F_n, n} \sigma_{F_n, n}^{-1})) = o_p(1),$$

where the equality is by Assumption 4.3(b). Thus, the second summand in (B.33) is also $o_p(1)$.

To show that the last summand in (B.33) is $o_p(1)$, let

$$\begin{aligned} \widehat{A}_n &= \frac{\psi_{\alpha, k_j}(\widehat{\alpha}_{k_j})' \widehat{H}_{k_j, n}^{-1}}{v_{\psi, k_j}^* \sigma_{F_n, n}}, \quad A_n = \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1}}{v_{\psi, k_j}^* \sigma_{F_n, n}}, \\ \widehat{B}_n &= n^{-1} \sum_{i=1}^n [\ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) \ell'_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)] H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) \\ B_n &= E_{F_n} [\ell_{\alpha, k_j}(Z_i; \alpha_{k_j}^*) \ell'_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)] H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) \\ \widehat{C}_n &= n^{-1} \sum_{i=1}^n \ell(Z_i; \alpha_{k_j}^*) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{k_j}^*) H_{F_n, k_j}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{k_j}^*), \quad \text{and} \quad C_n = H_{\ell, k_j, n} H_{F_n, k_j}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{k_j}^*), \end{aligned} \quad (\text{B.38})$$

Using Assumption B.2(c), we can write the last summand in (B.33) as

$$\widehat{A}_n(\widehat{B}_n + \widehat{C}_n + o_p(\sigma_{F_n,n})).$$

Next, note that (B.35) implies that

$$\widehat{A}_n = A_n + \widehat{A}_n - A_n = \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_n,k_j}^{-1}}{v_{\psi,k_j}^* \sigma_{F_n,n}} + o_p(|\mathbf{k}|^{-1/2} \sigma_{F_n,n}^{-1}) = O_p(\sigma_{F_n,n}^{-1}),$$

where the last equality is by Assumptions 4.1(f) and B.2(e). Under Assumptions 4.1(a) and (f)

$$E_{F_n} [\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-2} \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)] = \frac{1}{n} \text{tr} (H_{F_n,\mathbf{k}}^{-2} D_{F_n,\mathbf{k}}) = O(|\mathbf{k}| n^{-1}), \quad (\text{B.39})$$

which together with the Markov inequality implies that

$$\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)' H_{F_n,\mathbf{k}}^{-2} \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) = O_p(|\mathbf{k}| n^{-1}). \quad (\text{B.40})$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \|\widehat{B}_n - B_n\| \\ & \leq \left\| \frac{1}{n} \sum_{i=1}^n \ell_{\alpha,k_j}(Z_i; \alpha_{k_j}^*) \ell'_{\alpha,\mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*) - E_{F_n} [\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell'_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] \right\| \left\| \sqrt{\bar{\ell}'_{\alpha,n}(\alpha_{\mathbf{k}}^*) H_{F_n,\mathbf{k}}^{-2} \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)} \right\| \\ & = o_p(n^{-\frac{1}{2}}), \end{aligned} \quad (\text{B.41})$$

where the equality is by (B.40) and Assumption B.2(f). Note that Lemma E.2 combined with Assumptions 4.1(f) implies that

$$\rho_{\max} \left(E_{F_n} [\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell'_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] E_{F_n} [\ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \ell'_{\alpha,k_j}(Z; \alpha_{k_j}^*)] \right) \leq C. \quad (\text{B.42})$$

Under the i.i.d. assumption,

$$\begin{aligned} E_{F_n} [\|B_n\|^2] & = \text{tr} \left(E_{F_n} [\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell'_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] \frac{H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}}{n} E_{F_n} [\ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \ell'_{\alpha,k_j}(Z; \alpha_{k_j}^*)] \right) \\ & \leq \frac{\rho_{\max}(D_{F_n,\mathbf{k}})}{n \rho_{\min}((H_{F_n,\mathbf{k}})^2)} \text{tr} \left(E_{F_n} [\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell'_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] E_{F_n} [\ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \ell'_{\alpha,k_j}(Z; \alpha_{k_j}^*)] \right) \\ & = O(|\mathbf{k}| n^{-1}) \end{aligned} \quad (\text{B.43})$$

where the last equality is by Assumptions 4.1(f) and (B.42). (B.43) combined with the Markov

inequality yields

$$B_n = O_p(|\mathbf{k}|^{\frac{1}{2}} n^{-\frac{1}{2}}). \quad (\text{B.44})$$

B.2 Proof of Theorem 6.1

Proof of Theorem 6.1. First note that

$$\liminf_{n \rightarrow \infty} \inf_{F_0 \in \mathcal{F}_n} \Pr_{F_0}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_n \geq t) = \liminf_{n \rightarrow \infty} \Pr_{F_n}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_n \geq t) \quad (\text{B.45})$$

for some sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}_n$ for all n . And by the definition of \liminf , there exists a subsequence $\{a_n\}$ of n such that

$$\liminf_{n \rightarrow \infty} \Pr_{F_n}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_n \geq t) = \lim_{n \rightarrow \infty} \Pr_{F_{a_n}}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_{a_n} \geq t). \quad (\text{B.46})$$

By the completeness of the Euclidean space, there exists a subsequence $\{u_n\}$ of $\{a_n\}$ such that

$$\rho_{01, u_n} \rightarrow \rho_{01} \text{ and } n^{1/2} E_{F_{u_n}} [\ell(Z; \alpha_{u_n}^*)] / \sigma_{u_n} \rightarrow r \quad (\text{B.47})$$

for some $\rho_{01} \in [-1, 1]$ and $r \in [-C, \infty]$. Also since $\{u_n\}$ is a subsequence of $\{a_n\}$, we have

$$\lim_{n \rightarrow \infty} \Pr_{F_{u_n}}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_{u_n} \geq t) = \lim_{n \rightarrow \infty} \Pr_{F_{a_n}}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_{a_n} \geq t). \quad (\text{B.48})$$

Next, we show that

$$\lim_{n \rightarrow \infty} \Pr_{F_{u_n}}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_{u_n} \geq t) = 1-p, \quad (\text{B.49})$$

which combined with the above displays shows the desired result.

Let $\tilde{X}_{1,n} = n^{1/2}(\psi_1(\hat{\alpha}_{1,n}) - \psi_1(\alpha_1^*)) / \hat{v}_{\psi, k_1}^*$. Then

$$\begin{aligned} & \psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) \\ & \Leftrightarrow c_{1,p/2}(t) \leq \tilde{X}_{1,n} \leq c_{1,1-p/2}(t) \\ & \Leftrightarrow p/2 \leq \Psi(\tilde{X}_{1,n}, T_n - t, \hat{\rho}_{01,n}) \leq 1-p/2, \end{aligned} \quad (\text{B.50})$$

where the last lines follows from the strict monotonicity (increasing) of $\Psi(\cdot, T_n - t, \hat{\rho}_{01,n})$ and this monotonicity can be shown by taking the first derivative and using the strict monotonicity of the hazard ratio of the standard normal distribution. Therefore,

$$\Pr_{F_{u_n}}(\psi_1(\alpha_1^*) \in CI_{\psi_1}(1-p, t) | T_{u_n} \geq t)$$

$$\begin{aligned}
&= \frac{\Pr_{F_{u_n}}(p/2 \leq \Psi(\tilde{X}_{1,u_n}, T_{u_n} - t, \hat{\rho}_{01,u_n}) \leq 1 - p/2, T_{u_n} - t > 0)}{\Pr_{F_{u_n}}(T_{u_n} - t > 0)} \\
&= \frac{\Pr_{F_{u_n}}(p/2 \leq \Psi(\tilde{X}_{1,u_n}, |T_{u_n} - t|, \hat{\rho}_{01,u_n}) \leq 1 - p/2, T_{u_n} t > 0)}{\Pr_{F_{u_n}}(T_{u_n} - t > 0)}. \tag{B.51}
\end{aligned}$$

Lemma 6.1 implies that

$$\begin{pmatrix} \tilde{X}_{1,u_n} \\ T_{u_n} - t \end{pmatrix} \rightarrow_d \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_0 + r - t \end{pmatrix}, \tag{B.52}$$

where $(\tilde{X}_0, \tilde{X}_1) \sim N\left(\mathbf{0}_2, \begin{pmatrix} 1 & \rho_{01} \\ \rho_{01} & 1 \end{pmatrix}\right)$.

Elementary algebra show that $\Psi(c, h, \rho)$ is continuous on $R \times (0, \infty] \times [-1, 1]$. With this continuity, we can conclude that the last line of (B.51) converges to

$$\Pr(p/2 \leq \Psi(\tilde{X}_1, \tilde{X}_0 + r - t, \rho_{01}) \leq 1 - p/2 | \tilde{X}_0 + r - t > 0). \tag{B.53}$$

Let $X_0^* = \rho_{01}\tilde{X}_1 - \tilde{X}_0 - r + t$. Then X_0^* is independent of \tilde{X}_1 and the above expression can be written as

$$\Pr(p/2 \leq \Psi(\tilde{X}_1, \rho_{01}\tilde{X}_1 - X_0^*, \rho_{01}) \leq 1 - p/2 | \rho_{01}\tilde{X}_1 > X_0^*). \tag{B.54}$$

Note that

$$\Psi(x_1, \rho_{01}x_1 - x_0, \rho_{01}) = \begin{cases} [\Phi(x_1) - \Phi(x_0/\rho_{01})]/[1 - \Phi(x_0/\rho_{01})] & \text{if } \rho_{01} > 0 \text{ and } x_0 \in R \\ \Phi(x_1) & \text{if } \rho_{01} = 0 \text{ or } \rho_{01}x_1 - x_0 = \infty \\ \Phi(x_1)/\Phi(x_0/\rho_{01}) & \text{if } \rho_{01} < 0 \text{ and } x_0 \in R \end{cases}. \tag{B.55}$$

If $\rho_{01} \neq 0$, $\Psi(x_1, \rho_{01}x_1 - x_0, \rho_{01})$ is the conditional CDF of \tilde{X}_1 given $\rho_{01}\tilde{X}_1 > x_0$ for any $x_0 \in R$. Thus, $\Psi(\tilde{X}_1, \rho_{01}\tilde{X}_1 - x_0, \rho_{01}) | \rho_{01}\tilde{X}_1 > x_0 \sim \text{Uniform}[0, 1]$. But since X_0^* is independent of \tilde{X}_1 , that implies

$$\Psi(\tilde{X}_1, \rho_{01}\tilde{X}_1 - X_0^*, \rho_{01}) | \rho_{01}\tilde{X}_1 > X_0^* \sim \text{Uniform}[0, 1]. \tag{B.56}$$

On the other hand, if $\rho_{01} = 0$, by the independence between X_0^* and \tilde{X}_1 , $\Psi(x_1, \rho_{01}x_1 - x_0, \rho_{01})$ is the conditional CDF of \tilde{X}_1 given $X_0^* < 0$, which means that (B.56) also holds for $\rho_{01} = 0$. Therefore, the expression in (B.54) equals $1 - p$. This concludes the proof of (B.49). ■

B.3 Proof of Theorem 6.2

Proof of Theorem 6.2. Consider an arbitrary sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n . It is sufficient to show that

$$\liminf_{n \rightarrow \infty} \Pr_{F_n}(\theta^* \in CI_{\theta^*}(1-p)) \geq 1-p. \quad (\text{B.57})$$

Consider the derivation

$$\begin{aligned} \Pr_{F_n}(\theta^* \in CI_{\theta^*}(1-p)) &\leq \Pr_{F_n}(\widehat{G}_n(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0^*)' \widehat{\Sigma}_G^{-1} \widehat{G}_n(\psi_1(\alpha_1^*), \psi_2(\alpha_2^*), \ell_0^*) \leq \chi_3^2(1-p)) \\ &= \Pr_{F_n}(\widehat{G}'_{n,F_n} \widehat{\Sigma}_G^{-1} \widehat{G}_{n,F_n} \leq \chi_3^2(1-p)) \\ &\rightarrow \Pr(\chi_3^2 \leq \chi_3^2(1-p)) = 1-p, \end{aligned} \quad (\text{B.58})$$

where the convergence holds by Lemmas 6.1 and B.2 and $\rho_{\min}(\Sigma_G) > C^{-1}$. This shows (B.57) and in turn proves the theorem. ■

C Verification of the High-level Assumptions in the Mean-Regression Example

The assumptions needed for verifying the high-level assumptions in Section 4 of the main paper in the mean-regression example in Section 5 of the main paper are given below. Assumption C.1 imposes conditions on the data structure and the finite-dimensional approximation. These are commonly used conditions in the literature (see, e.g., Andrews (1991b), Newey (1997), Chen (2007), and Belloni et al. (2015)).²³

Assumption C.1 *There exist positive constants C_1, r_1, r_2 and nondecreasing sequences $\{\xi_{k_1}\}_{k_1 \geq 1}$ and $\{\xi_{k_2}\}_{k_2 \geq 1}$, such that, for any $F_0 \in \mathcal{F}$ and for $j = 1, 2$:*

- (i) $\{Z_i\}_{i \geq 1}$ are i.i.d. draws from F_0 ;
- (ii) $\sup_{x_j \in \mathcal{X}_j} \left| \alpha_j^*(x_j) - P_{k_j}(x_j)' \beta_{k_j, F_0}^* \right| \leq C_1 k_j^{-r_j}$ where $r_j > 0$ and β_{k_j, F_0}^* is defined in equation (5.4) in the main paper;
- (iii) $\sup_{x_j \in \mathcal{X}_j} \|P_{k_j}(x_j)\|^2 \leq \xi_{k_j}$; and
- (iv) the eigenvalues of $E_{F_0} [P_{k_j}(X_j) P_{k_j}(X_j)']$ lie in the interval $[C_1^{-1}, C_1]$.

Assumption C.2 below imposes condition on the error terms. For $j = 1, 2$, define $u_j = Y - \alpha_j^*(X_j)$. It is useful to note here that $\omega_{F_0, * }^2 = \text{Var}_{F_0}(u_1^2 - u_2^2)/4$ and $\sigma_{F_0, n}^2 = \omega_{F_0, * }^2 + (2n^2)^{-1}(n -$

²³For some approximating functions such as power series, Assumptions C.1(ii-iv) hold under certain nonsingular transformation on the vector of approximating functions, i.e., $BP(\cdot)$, where B is some non-singular constant matrix. Since the nonparametric series estimator is invariant to any nonsingular transformation of $P(\cdot)$, we do not distinguish between $BP(\cdot)$ and $P(\cdot)$ throughout this paper for notational convenience.

1) $\text{tr}((H_{F_0, \mathbf{k}}^{-1} D_{F_0, \mathbf{k}})^2)$ for this example.

Assumption C.2 *There exist constants $b \geq 4$ and $C_2 > 0$ such that for any $F_0 \in \mathcal{F}$:*

- (i) $E_{F_0} \left[\left| \omega_{F_0, *}^{-1} (u_1^2 - u_2^2 - E_{F_0}(u_1^2 - u_2^2)) \right|^4 \right] < C_2$ whenever $\omega_{F_0, *}^2 > 0$; and
- (ii) $E_{F_0} \left[|u_j|^b \mid X_j = x_j \right] \leq C_2$ for all $x_j \in \mathcal{X}_j$ for $j = 1, 2$.

Assumption C.3 *For any $F_0 \in \mathcal{F}_0$, we have $F_0 \in \mathcal{F}$ and $E_{F_0} [u_1^2 - u_2^2] = 0$.*

Let $\xi_{\mathbf{k}} = \xi_{k_1} + \xi_{k_2}$. Assumption C.4 imposes conditions on the numbers of series terms in the finite-dimensional approximation, and on the divergence of $n\sigma_{F_n, n}^2$.

Assumption C.4 *For any sequence of DGP's $\{F_n\}_{n \geq 1}$ such that $F_n \in F$, we have, for the constant b in Assumption C.2 and the constants r_1, r_2 in Assumption C.1,*

- (i) $(n^{1/2} \sigma_{F_n, n})^{-1} (1 + |\mathbf{k}| (\xi_{\mathbf{k}} \log(n))^{1/2} n^{-(1/2-1/b)}) = o(1)$;
- (ii) $\max_{j=1,2} (n\sigma_{F_n, n}^2)^{-1} k_j (\xi_{\mathbf{k}} \log(n))^{1/2} \left(k_j^{1/2} n^{-1/2} + n^{-(1/2-1/b)} \right) = o(1)$;
- (iii) $\xi_{\mathbf{k}} \log(n) n^{-(1-2/b)} = o(1)$;
- (iv) $\max_{j=1,2} \left(n k_j^{-2r_j} + \xi_{k_j} k_j \log(n) n^{-1} \right) = O(1)$.

The following theorem summarizes the main results in this section.

Theorem C.1 *Assumptions C.1-C.4 together imply Assumptions 4.1-4.4 in the main paper.*

This theorem is an immediate consequence of the following Lemmas C.1-C.3 below. Each lemma is proved immediately after it is stated.

Lemma C.1 *Under Assumptions C.1 and C.2, Assumption 4.1 in the main paper holds with $C = \max\{8C_2, 2C_1C_2\}$.*

Proof of Lemma C.1. Assumption 4.1(a) is implied by Assumption C.1(i). Assumption 4.1(b) holds because $E_{F_0}[\ell(Z; \alpha(\beta_{\mathbf{k}}))]$ is a quadratic function of $\beta_{\mathbf{k}}$ for any F_0 . Assumption 4.1(c) holds by the first order condition of $\alpha_{\mathbf{k}}^*$. For Assumption 4.1(d), first note that

$$E_{F_0} [\ell(Z; \alpha_{F_0}^*)^2] = E_{F_0} [(u_2^2 - u_1^2)^2] / 4 \leq E_{F_0} [u_1^4] / 2 + E_{F_0} [u_2^4] / 2 \leq C_2 \quad (\text{C.1})$$

where the second inequality is by Assumption C.2(ii) with C_2 specified in that assumption. Since $u_{k_j} = u_j + \alpha_j^*(x_j) - \alpha_{k_j}^*(x_j)$, by Assumptions C.2(ii) and C.1(ii),

$$\begin{aligned} E_{F_0} \left[u_{k_j}^4 \mid X_j = x_j \right] &\leq 8E_{F_0} [u_j^4 \mid X_j = x_j] + 8 \left| \alpha_j^*(x_j) - \alpha_{k_j}^*(x_j) \right|^4 \\ &\leq 8C_2 + 8C_1^4 k_j^{-4r_j} \leq C \end{aligned} \quad (\text{C.2})$$

for $j = 1, 2$ and uniformly over $x_j \in \mathcal{X}_j$. Moreover,

$$\begin{aligned} E_{F_0} [\|\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)\|^4] &\leq 2 \sum_{j=1,2} E_{F_0} \left[u_{k_j}^4 |P_{k_j}(X_j)' P_{k_j}(X_j)|^2 \right] \\ &\leq C \sum_{j=1,2} \xi_{k_j} \text{tr}(-H_{F_0, k_j}) \leq C \sum_{j=1,2} \xi_{k_j} k_j = C \xi_{\mathbf{k}} |\mathbf{k}|, \end{aligned} \quad (\text{C.3})$$

where the second inequality is by (C.2) and Assumption C.1(iii), and the third inequality is by Assumption C.1(iv). Hence Assumption 4.1(d) is also satisfied. Assumption 4.1(e) holds by Assumption C.2(i). For Assumption 4.1(f), the first part of it holds by Assumption C.1(iv) and equation (5.7). To show the second part, first note that, under Assumptions C.1(iv) and (C.2),

$$\rho_{\max} \left(E_{F_0} \left[u_{k_j}^2 P_{k_j}(X_j) P_{k_j}'(X_j) \right] \right) \leq C \rho_{\max}(-H_{F_0, k_j}) \leq CC_1 \text{ for } j = 1, 2, \quad (\text{C.4})$$

which together with the form of $D_{F_0, \mathbf{k}}$ in (5.6) and the Aronszajn's Inequality (see, e.g., Theorem III.2.9 in Bhatia (1997)) implies that $\rho_{\max}(D_{F_0, \mathbf{k}}) \leq 2CC_1 \leq C$. This verifies Assumption 4.1(f). ■

Lemma C.2 *Under Assumptions C.1, C.2, C.4(i) and C.4(iv), Assumption 4.3 of the main paper holds.*

Lemma C.3 *Under Assumptions C.1, C.2 and C.4, Assumption 4.4 of the main paper holds.*

The proof of Lemmas C.2 and C.3 makes use of the following three lemmas. The first one of which follows from Lemma 6.2 of Belloni et al. (2015), and the other two are proved at the end of this section.

In the rest of this section, we use the new notation $\mathbf{U}_{j,n} = [u_{j,1}, \dots, u_{j,n}]'$ for $j = 1, 2$. Let β_{k_j, F_n}^* be abbreviated as $\beta_{k_j}^*$. Let $\alpha_{k_j}^*(\cdot) = P_{k_j}(\cdot)' \beta_{k_j}^*$.

Lemma C.4 *Suppose that Assumptions C.1(i), (iii) and (iv) holds. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n*

$$\left\| n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n} - E_{F_n} [P_{k_j}(X_j) P_{k_j}'(X_j)] \right\|^2 = O_p(\xi_{k_j} \log(k_j) n^{-1}). \quad (\text{C.5})$$

Using Lemma C.4, Assumptions C.1(iv) and C.4(iv), we have

$$C_1^{-1}/2 \leq \rho_{\min}(n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n}) \leq \rho_{\max}(n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n}) \leq 2C_1 \quad (\text{C.6})$$

for $j = 1, 2$ with probability approaching 1.

Lemma C.5 *Suppose that Assumptions C.1, C.2, C.4(ii) and C.4(iv) hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n and for $j = 1, 2$, we have*

- (a) $\|\widehat{\beta}_{k_j} - \beta_{k_j}^*\|^2 = O_p(k_j n^{-1})$, and
- (b) $\gamma'_n(\widehat{\beta}_{k_j} - \beta_{k_j}^*) = O_p(n^{-1/2}\|\gamma_n\|)$ for any deterministic $k_j \times 1$ vector sequence $\{\gamma_n\}_{n \geq 1}$
- (c) $(\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)' D_{F_n, \mathbf{k}}(\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) = \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)'(H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}) \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) + o_p(\sigma_{F_n, n}^2)$.

Lemma C.6 *Suppose that Assumptions C.1, C.2 and C.4(iv) hold. Then under any sequence $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for all n , we have*

- (a) for $j = 1, 2$, $n^{-1} \sum_{i=1}^n \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^2 = O_p(k_j n^{-1})$, and
- (b) for $\delta_{n, \mathbf{k}} = (k_1 \sqrt{\xi_{k_1}} + k_2 \sqrt{\xi_{k_2}}) \sqrt{\log(n)} n^{-2/3} + (k_1^{-r_1} + k_2^{-r_2}) n^{-1/2} + k_1^{-2r_1} + k_2^{-2r_2}$,

$$\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}}) = (2n)^{-1} \sum_{i=1}^n (u_{1,i}^2 - u_{2,i}^2) - \frac{\bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)}{2} + O_p(\delta_{n, \mathbf{k}}).$$

Now we present the proofs of the lemmas above.

Proof of Lemma C.2. First we verify Assumption 4.3(a), which is a quadratic expansion of $\bar{\ell}_n(\widehat{\alpha}_{\mathbf{k}})$. In Lemma C.6(b), we have derived the second order expansion where the remainder term is of the order

$$\delta_{n, \mathbf{k}} = (k_1 \sqrt{\xi_{k_1}} + k_2 \sqrt{\xi_{k_2}}) (\log(n))^{1/2} n^{-3/2} + (k_1^{-r_1} + k_2^{-r_2}) n^{-1/2} + k_1^{-2r_1} + k_2^{-2r_2}. \quad (\text{C.7})$$

Observe that

$$\begin{aligned} \frac{n^{1/2} \delta_{n, \mathbf{k}}}{\sigma_{F_n, n}} &\leq 2 \max_{j=1,2} \left[\frac{k_j (\xi_{k_j} \log(n))^{1/2}}{n \sigma_{F_n, n}} + \frac{1}{k_j^{r_j} \sigma_{F_n, n}} (1 + k_j^{-r_j} n^{1/2}) \right] \\ &= \frac{1}{n^{1/2} \sigma_{F_n, n}} \max_{j=1,2} \left[\frac{k_j (\xi_{k_j} \log(n))^{1/2}}{n^{1/2}} + (n^{1/2} k_j^{-r_j} + n k_j^{-2r_j}) \right] = o(1), \end{aligned} \quad (\text{C.8})$$

where the last equality holds by Assumptions C.4(i) and (iv). This combined with Lemma C.6(b) implies Assumption 4.3(a).

Assumption 4.3(b) is directly implied by Assumptions C.4(i). ■

Proof of Lemma C.3. We start with Assumption 4.4(a). First, Lemma C.4 implies that $\|\widehat{H}_n - H_{F_n, \mathbf{k}}\|^2 = O_p(\xi_{\mathbf{k}} \log(n) n^{-1})$, where $\xi_{\mathbf{k}} = \xi_{k_1} + \xi_{k_2}$. Then for δ_n in Assumption 4.4(a), the first part of Assumption 4.4(a) follows the derivation: for $j = 1, 2$

$$\frac{\xi_{k_j} \ln(n) n^{-1}}{\delta_n^2} = \max \left\{ \frac{\xi_{k_j} \log(n) |\mathbf{k}|^2}{n^2 \sigma_{F_n, n}^2}, \frac{\xi_{k_j} \log(n)}{n} \right\}$$

$$= \max \left\{ \frac{1}{n\sigma_{F_n,n}^2} \frac{\xi_{k_j} |\mathbf{k}|^2 \log(n)}{n}, \frac{\xi_{k_j} \log(n)}{n} \right\} = o(1), \quad (\text{C.9})$$

where the last equality holds by Assumptions C.4(i) and (iii).

For the second and the third parts of Assumption 4.4(a), note that by arguments similar to those in the proof of Theorem 4.6 in Belloni et al. (2015), we have

$$\|\widehat{D}_n - \widehat{D}_n(\alpha_{\mathbf{k}}^*)\|^2 = O_p \left(\frac{\xi_{\mathbf{k}} \log(n)}{n^{1-2/b}} \right) \text{ and } \|\widehat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}\|^2 = O_p \left(\frac{\xi_{\mathbf{k}} \log(n)}{n^{1-2/b}} \right). \quad (\text{C.10})$$

The second and the third parts of Assumption 4.4(a) follows from the derivation:

$$\frac{\xi_{\mathbf{k}} \log(n)}{n^{1-2/b} \delta_n^2} = \max \left\{ \frac{1}{n\sigma_{F_n,n}^2} \frac{\xi_{\mathbf{k}} |\mathbf{k}|^2 \log(n)}{n^{1-2/b}}, \frac{\xi_{\mathbf{k}} \log(n)}{n^{1-2/b}} \right\} = o(1),$$

where the last equality holds by Assumption C.4(i) and (iii). Thus Assumption 4.4(a) is verified.

Next, we verify Assumption 4.4(b). By Lemma C.5(a) and Assumption C.1(iii), we have

$$\sup_{x_j \in \mathcal{X}_j} \left| \widehat{\alpha}_{k_j}(x_j) - \alpha_{k_j}^*(x_j) \right|^2 = O_p(\xi_{k_j} k_j n^{-1}). \quad (\text{C.11})$$

Using Assumption C.1(ii) and Lemma C.6(a) and (C.11), we get

$$\begin{aligned} & n^{-1} \sum_{i=1}^n \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^4 \\ & \leq \frac{\sup_{x_j \in \mathcal{X}_j} \left| \widehat{\alpha}_{k_j}(x_j) - \alpha_{k_j}^*(x_j) \right|^2 \sum_{i=1}^n \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^2}{n} \\ & = O_p(\xi_{k_j} k_j n^{-1}) O_p(k_j n^{-1}) \\ & = \sigma_{F_n,n}^2 O_p \left(\frac{1}{n\sigma_{F_n,n}^2} \frac{\xi_{k_j} k_j^2}{n} \right) = o_p(\sigma_{F_n,n}^2), \end{aligned} \quad (\text{C.12})$$

where the last equality holds by Assumption C.4(i). Using Assumptions 4.1(f), C.4(iii) and (C.10), we have

$$0 \leq \rho_{\max}(\widehat{D}_n(\alpha_{\mathbf{k}}^*)) \leq 2C \text{ and } 0 \leq \rho_{\max}(\widehat{D}_n(\alpha_n^*)) \leq 2C \quad (\text{C.13})$$

with probability approaching 1. By Lemma C.5(a) and (C.13), we have

$$n^{-1} \sum_{i=1}^n u_{k_j,i}^2 \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^2 = O_p(k_j n^{-1}). \quad (\text{C.14})$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \left| n^{-1} \sum_{i=1}^n u_{k_j, i} \left[\widehat{\alpha}_{k_j}(X_{j, i}) - \alpha_{k_j}^*(X_{j, i}) \right]^3 \right|^2 \\
& \leq n^{-1} \sum_{i=1}^n \left[u_{k_j, i}^2 \left| \widehat{\alpha}_{k_j}(X_{j, i}) - \alpha_{k_j}^*(X_{j, i}) \right|^2 \right] \\
& \times n^{-1} \sum_{i=1}^n \left[\left| \widehat{\alpha}_{k_j}(X_{j, i}) - \alpha_{k_j}^*(X_{j, i}) \right|^4 \right] \\
& = O_p(k_j n^{-1}) O_p(\xi_{k_j} k_j^2 n^{-2}) = O_p(\xi_{k_j} k_j^3 n^{-3}) = o_p(\sigma_{F_n, n}^4) \tag{C.15}
\end{aligned}$$

where the first equality is by the third line of (C.12) and by (C.14), the second equality is by the last line of (C.12), and the last equality holds by Assumptions C.4(ii). Similarly, we can show that

$$\left| n^{-1} \sum_{i=1}^n u_{k_j, i} \left[\widehat{\alpha}_{k_j}(X_{j, i}) - \alpha_{k_j}^*(X_{j, i}) \right] \left[\widehat{\alpha}_{k_{-j}}(X_{-j, i}) - \alpha_{k_{-j}}^*(X_{-j, i}) \right]^2 \right|^2 = o_p(\sigma_{F_n, n}^4) \tag{C.16}$$

for $(j, -j) = (1, 2)$ or $(2, 1)$. Also, by definition,

$$\begin{aligned}
\ell(Z, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z, \alpha_{\mathbf{k}}^*) &= \left| \widehat{\alpha}_{k_2}(X_2) - \alpha_{k_2}^*(X_2) \right|^2 / 2 - \left| \widehat{\alpha}_{k_1}(X_1) - \alpha_{k_1}^*(X_1) \right|^2 / 2 \\
&+ u_{k_1} \left[\widehat{\alpha}_{k_1}(X_1) - \alpha_{k_1}^*(X_1) \right] - u_{k_2} \left[\widehat{\alpha}_{k_2}(X_2) - \alpha_{k_2}^*(X_2) \right]. \tag{C.17}
\end{aligned}$$

which together with (C.12), (C.15) and (C.16) implies that

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n \left| \ell(Z, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z, \alpha_{\mathbf{k}}^*) \right|^2 \\
&= n^{-1} \sum_{i=1}^n \left| u_{k_1} \left[\widehat{\alpha}_{k_1}(X_1) - \alpha_{k_1}^*(X_1) \right] - u_{k_2} \left[\widehat{\alpha}_{k_2}(X_2) - \alpha_{k_2}^*(X_2) \right] \right|^2 + o_p(\sigma_{F_n, n}^2). \tag{C.18}
\end{aligned}$$

By (C.14),

$$n^{-1} \sum_{i=1}^n \left| u_{k_1} \left[\widehat{\alpha}_{k_1}(X_1) - \alpha_{k_1}^*(X_1) \right] - u_{k_2} \left[\widehat{\alpha}_{k_2}(X_2) - \alpha_{k_2}^*(X_2) \right] \right|^2 = O_p((k_1 + k_2)n^{-1}). \tag{C.19}$$

Next note that

$$\begin{aligned}
\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*) &= \frac{u_1^2 - u_{k_1}^2}{2} - \frac{u_2^2 - u_{k_2}^2}{2} \\
&= \frac{(u_1 - u_{k_1})^2}{2} + (u_1 - u_{k_1})u_{k_1}
\end{aligned}$$

$$-\frac{(u_2 - u_{k_2})^2}{2} - (u_2 - u_{k_2})u_{k_2}. \quad (\text{C.20})$$

By Assumption C.1(ii),

$$n^{-1} \sum_{i=1}^n (u_{j,i} - u_{k_j,i})^4 = n^{-1} \sum_{i=1}^n \left| \alpha_{k_j}^*(X_{j,i}) - \alpha_{j,n}^*(X_{j,i}) \right|^4 = O_p(k_j^{-4r_j}). \quad (\text{C.21})$$

By Assumption C.1(ii), (C.2) and the Markov inequality,

$$n^{-1} \sum_{i=1}^n (u_{j,i} - u_{k_j,i})^2 u_{k_j,i}^2 = n^{-1} \sum_{i=1}^n u_{k_j,i}^2 \left| \alpha_{k_j}^*(X_{j,i}) - \alpha_{j,n}^*(X_{j,i}) \right|^2 = O_p(k_j^{-2r_j}). \quad (\text{C.22})$$

Combining the results in (C.20), (C.21) and (C.22), we get

$$n^{-1} \sum_{i=1}^n |\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)|^2 = O_p(k_1^{-2r_1} + k_2^{-2r_2}). \quad (\text{C.23})$$

By Assumptions C.1(i), C.1(ii) and C.1(iv), (C.2) and (C.20)

$$\begin{aligned} & E_{F_n} \left[\left\| n^{-1} \sum_{i=1}^n [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j) - E_{F_n}((\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j))] \right\|^2 \right] \\ & \leq n^{-2} \sum_{i=1}^n E_{F_n} [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*))^2 P_{k_j}(X_j)' P_{k_j}(X_j)] \\ & \leq C n^{-1} (k_1^{-2r_1} + k_2^{-2r_2}) E_{F_n} [P_{k_j}(X_j)' P_{k_j}(X_j)] = O((k_1^{-2r_1} + k_2^{-2r_2}) k_j n^{-1}). \end{aligned} \quad (\text{C.24})$$

By Assumptions C.1(ii) and C.1(iv), (C.2) and (C.20)

$$\begin{aligned} & \left\| E_{F_n} [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j)] \right\|^2 \\ & = \rho_{\max}(E_{F_n} [P_{k_j}(X_j) P_{k_j}(X_j)']) E_{F_n} [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*))^2] \leq C(k_1^{-2r_1} + k_2^{-2r_2}). \end{aligned} \quad (\text{C.25})$$

Using Lemma C.5(a)-(b), (C.24), (C.25), the Markov inequality, the Cauchy-Schwarz inequality and the triangle inequality, we get

$$\begin{aligned} & \left| n^{-1} \sum_{i=1}^n (\ell(Z, \hat{\alpha}_{\mathbf{k}}) - \ell(Z, \alpha_{\mathbf{k}}^*)) (\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) \right| \\ & = \left| n^{-1} \sum_{i=1}^n (\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j)' (\hat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| E_{F_n} [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j)'] (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) \right| + O((k_1^{-r_1} + k_2^{-r_2}) |\mathbf{k}| n^{-1}) \\
&= O_p(\|E_{F_n} [(\ell(Z, \alpha_{\mathbf{k}}^*) - \ell(Z, \alpha_n^*)) P_{k_j}(X_j)']\| n^{-1/2}) + O((k_1^{-r_1} + k_2^{-r_2}) |\mathbf{k}| n^{-1}) \\
&= O_p((k_1^{-r_1} + k_2^{-r_2})(n^{-1/2} + |\mathbf{k}| n^{-1})) = o_p(\sigma_{F_n, n}^2) \tag{C.26}
\end{aligned}$$

where the last equality is by Assumptions C.4(i) and C.4(iv). By Assumption C.4, (C.18), (C.23) and (C.24),

$$\begin{aligned}
&n^{-1} \sum_{i=1}^n |\ell(Z, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z, \alpha_n^*)|^2 \\
&= n^{-1} \sum_{i=1}^n |u_{k_1} [\widehat{\alpha}_{k_1}(X_1) - \alpha_{k_1}^*(X_1)] - u_{k_2} [\widehat{\alpha}_{k_2}(X_2) - \alpha_{k_2}^*(X_2)]|^2 + o_p(\sigma_{F_n, n}^2) \\
&= (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)' \widehat{D}_n(\alpha_{\mathbf{k}}^*) (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) + o_p(\sigma_{F_n, n}^2). \tag{C.27}
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)' (\widehat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}}) (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) \right| \\
&\leq \left\| \widehat{\beta}_{\mathbf{k}, n} - \beta_{\mathbf{k}, n}^* \right\|^2 \left\| \widehat{D}_n(\alpha_{\mathbf{k}}^*) - D_{F_n, \mathbf{k}} \right\| \\
&= O_p \left(\frac{(\xi_{\mathbf{k}} \log(n))^{1/2}}{n^{1/2-1/b}} \right) O_p((k_1 + k_2) n^{-1}) = o_p(\sigma_{F_n, n}^2) \tag{C.28}
\end{aligned}$$

where the first equality holds by (C.10) and Lemma C.5(a), and the second equality holds by Assumptions C.4(ii). Combining (C.27) with (C.28), we have

$$\frac{1}{n} \sum_{i=1}^n |\ell(Z_i, \widehat{\alpha}_n) - \ell(Z_i, \alpha_n^*)|^2 = (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)' D_{F_n, \mathbf{k}} (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) + o_p(\sigma_{F_n, n}^2), \tag{C.29}$$

which together with Lemma C.5(c) implies that

$$\frac{1}{n} \sum_{i=1}^n |\ell(Z_i, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_n^*)|^2 = \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) + o_p(\sigma_{F_n, n}^2). \tag{C.30}$$

This verifies Assumption 4.4(b).

We next verify Assumption 4.4(c). First, notice that by the Cauchy-Schwarz inequality,

$$\left| \frac{\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) (\widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}))^2}{n \sigma_{F_n, n}^2} \right|^2$$

$$\leq \frac{\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2 \sum_{i=1}^n \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^4}{n\sigma_{F_n,n}^2 n\sigma_{F_n,n}^2} = o_p(1) \quad (\text{C.31})$$

where the equality holds by the Markov inequality, Assumption C.1(i), $E_{F_n}[(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2] \leq \sigma_{F_n,n}^2$, and (C.12). That together with (C.17) implies that

$$\begin{aligned} & \frac{\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) (\ell(Z_i, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_{\mathbf{k}}^*))}{n\sigma_{F_n,n}^2} \\ &= \frac{\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)' (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)}{n\sigma_{F_n,n}^2} + o_p(1) \\ &= \frac{E_{F_n}[(\ell(Z_i; \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'] (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)}{\sigma_{F_n,n}^2} + \\ & \quad \frac{\sum_{i=1}^n [(\ell(Z_i; \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)' - E_{F_n} \ell(Z_i; \alpha_n^*) \ell_{\alpha, \mathbf{k}}(Z_i; \alpha_{\mathbf{k}}^*)'] (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}})}{n\sigma_{F_n,n}^2} + \\ & \quad o_p(1). \end{aligned} \quad (\text{C.32})$$

Let $\gamma_n = E_{F_n}[\ell(Z; \alpha_n^*) \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] / \sigma_{F_n,n}$. Then

$$\begin{aligned} \gamma_n' \gamma_n &= \frac{\omega_{F_n,n}^2}{\sigma_{F_n,n}^2} E_{F_n} \left[\frac{\ell(Z; \alpha_n^*)}{\omega_{F_n,n}} \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)' (D_{F_n, \mathbf{k}}^{1/2})^+ \right] D_{F_n, \mathbf{k}} E_{F_n} \left[(D_{F_n, \mathbf{k}}^{1/2})^+ \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \frac{\ell(Z; \alpha_n^*)}{\omega_{F_n,n}} \right] \\ &\leq C E_{F_n} \left[\frac{\ell(Z; \alpha_n^*)}{\omega_{F_n,n}} \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)' (D_{F_n, \mathbf{k}}^{1/2})^+ \right] E_{F_n} \left[\frac{\ell(Z; \alpha_n^*)}{\omega_{F_n,n}} \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) (D_{F_n, \mathbf{k}}^{1/2})^+ \right] \leq C, \end{aligned} \quad (\text{C.33})$$

where the first inequality holds by Assumption 4.1(f) which has been verified in Lemma C.1, and the second inequality holds by Lemma E.1. Equation (C.33) together with Lemma C.5(b) implies that

$$E_{F_n}[\ell(Z; \alpha_n^*) \ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)'] (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}) / \sigma_{F_n,n}^2 = O_p(n^{-1/2} \sigma_{F_n,n}^{-1}) = o(1). \quad (\text{C.34})$$

where the second equality holds by Assumption C.4(i). Now consider,

$$\begin{aligned} & \frac{E_{F_n} [\|(\ell(Z, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z, \alpha_{\mathbf{k}}^*)\|^2]}{\sigma_{F_n,n}^2} \\ &= E_{F_n} \left[\frac{(\ell(Z, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2}{\sigma_{F_n,n}^2} \sum_{j=1,2} u_{k_j}^2 P_{k_j}(X_j)' P_{k_j}(X_j) \right]. \end{aligned} \quad (\text{C.35})$$

Using the Hölder inequality, Assumptions C.1(iii)-(iv) and (C.2), we get

$$\begin{aligned}
& E_{F_n} \left[\frac{(\ell(Z, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2}{\sigma_{F_n, n}^2} u_{k_j}^2 P_{k_j}(X_j)' P_{k_j}(X_j) \right] \\
& \leq |E_{F_n} [|(\ell(Z, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) / \sigma_{F_n, n}|^4]|^{\frac{1}{2}} \left| E_{F_n} \left[\left| u_{k_j}^2 P_{k_j}(X_j)' P_{k_j}(X_j) \right|^2 \right] \right|^{\frac{1}{2}} \\
& \leq C_2^{\frac{1}{2}} \left| E_{F_n} \left[\left| u_{k_j}^2 P_{k_j}(X_j)' P_{k_j}(X_j) \right|^2 \right] \right|^{\frac{1}{2}} \\
& \leq C_2 |\xi_{k_j} E_{F_n} [P_{k_j}(X_j)' P_{k_j}(X_j)]|^{\frac{1}{2}} = O(\xi_{k_j}^{\frac{1}{2}} k_j^{\frac{1}{2}}), \tag{C.36}
\end{aligned}$$

which together with the Markov inequality implies that

$$\frac{\| \sum_{i=1}^n [(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*) - E_{F_n} [\ell(Z_i, \alpha_n^*) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)]] \|}{n^{1/2} \sigma_{F_n, n}} = O_p \left(\sum_{j=1,2} \xi_{k_j}^{\frac{1}{4}} k_j^{\frac{1}{4}} \right). \tag{C.37}$$

Then we have

$$\begin{aligned}
& \frac{\left| \sum_{i=1}^n [(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*) - E_{F_n} [\ell(Z_i, \alpha_n^*) \ell_{\alpha, \mathbf{k}}(Z_i, \alpha_{\mathbf{k}}^*)]]' (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}) \right|}{n \sigma_{F_n, n}^2} \\
& \leq O_p \left(\sum_{j=1,2} \xi_{k_j}^{\frac{1}{4}} k_j^{\frac{1}{4}} \right) \frac{\| \widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}} \|}{n^{1/2} \sigma_{F_n, n}} \\
& = O_p \left(\sum_{j=1,2} \xi_{k_j}^{\frac{1}{4}} k_j^{\frac{1}{4}} \right) O_p (|\mathbf{k}|^{1/2} \sigma_{F_n, n}^{-1} n^{-1}) \\
& = O_p \left(\frac{1}{n^{1/2} \sigma_{F_n, n}} \right) O_p \left(\frac{\sum_{j=1,2} \xi_{k_j}^{\frac{1}{4}} |\mathbf{k}|^{\frac{3}{4}}}{n^{1/2}} \right) = o_p(1), \tag{C.38}
\end{aligned}$$

where the first inequality holds by the Cauchy-Schwarz inequality and (C.37), the first equality holds by Lemma C.5(a), the second equality holds by arranging terms and the last equality holds by Assumptions C.4(i). By (C.32), (C.34) and (C.38),

$$\frac{\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) (\ell(Z_i, \widehat{\alpha}_{\mathbf{k}}) - \ell(Z_i, \alpha_{\mathbf{k}}^*))}{n \sigma_{F_n, n}^2} = o_p(1). \tag{C.39}$$

By the Cauchy-Schwarz inequality,

$$\frac{|\sum_{i=1}^n (\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*)) (\ell(Z_i, \alpha_{\mathbf{k}}^*) - \ell(Z_i, \alpha_n^*))|^2}{n^2 \sigma_{F_n, n}^4}$$

$$\begin{aligned}
&\leq n^{-1} \sum_{i=1}^n \frac{(\ell(Z_i, \alpha_n^*) - \ell_{F_n}(\alpha_n^*))^2}{\sigma_{F_n, n}^2} \times n^{-1} \sum_{i=1}^n \frac{(\ell(Z_i, \alpha_k^*) - \ell(Z_i, \alpha_n^*))^2}{\sigma_{F_n, n}^2} \\
&= O_p((k_1^{-2r_1} + k_2^{-2r_2})\sigma_{F_n, n}^{-2}) = o_p(1)
\end{aligned} \tag{C.40}$$

where the equality is by (C.23), Assumptions C.2(i) and C.4(iv), and the Markov inequality. Assumption 4.4(c) is implied by (C.39) and (C.40).

Assumption 4.4(d) is implied by Assumption C.4(iv). ■

Proof of Lemma C.5. Proof of the result in part (a) is standard and follows the similar arguments in Newey (1997). We include it for completeness. By the definition of $\widehat{\beta}_{k_j}$

$$\widehat{\beta}_{k_j} - \beta_{k_j}^* = (\mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \tag{C.41}$$

where $\mathbf{U}_{k_j, n} = [u_{k_j, 1}, \dots, u_{k_j, n}]'$. Therefore

$$\begin{aligned}
\|\widehat{\beta}_{k_j} - \beta_{k_j}^*\|^2 &= \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} (\mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-2} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \\
&\leq (\rho_{\min}(n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n}))^{-2} n^{-2} \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n}
\end{aligned} \tag{C.42}$$

By Assumptions C.1(i) and (C.2),

$$E_{F_n} \left[\sum_{l=1}^{k_j} \left(\sum_{i=1}^n p_{j, l}(X_{j, i}) u_{k_j, i} \right)^2 \right] = \sum_{l=1}^{k_j} \sum_{i=1}^n E_{F_n} [(p_{j, l}(X_{j, i}) u_{k_j, i})^2] \leq n \text{Ctr}(-H_{F_0, k_j}), \tag{C.43}$$

which together with the Markov inequality and Assumptions C.1(iv) implies that

$$n^{-2} \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} = O_p(k_j n^{-1}). \tag{C.44}$$

The result in part (a) follows by (C.6), (C.42) and (C.44).

For part (b), first observe that

$$\begin{aligned}
\gamma'_n(\widehat{\beta}_{k_j} - \beta_{k_j}^*) &= \gamma'_n(\mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \\
&= n^{-1} \gamma'_n(-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \\
&\quad + n^{-1} \gamma'_n((n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-1} - (-H_{F_n, k_j})^{-1}) \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n}.
\end{aligned} \tag{C.45}$$

By the first order condition of $\alpha_{k_j}^*$,

$$E_{F_n} [u_{k_j} P_{k_j}(X_j)] = \mathbf{0}_{k_j \times 1}. \tag{C.46}$$

By the definition of H_{F_n, k_j} , (C.46), Assumptions C.1(i) and C.1(iv), and (C.2)

$$\begin{aligned}
E_{F_n} \left[\left| \gamma'_n (-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \right|^2 \right] &= E_{F_n} \left[\left| \sum_{i=1}^n \gamma'_n (-H_{F_n, k_j})^{-1} P_{k_j}(X_{j, i}) u_{k_j, i} \right|^2 \right] \\
&= \sum_{i=1}^n E_{F_n} \left[\left| \gamma'_n (-H_{F_n, k_j})^{-1} P_{k_j}(X_{j, i}) u_{k_j, i} \right|^2 \right] \\
&\leq n C E_{F_n} \left[\left| \gamma'_n (-H_{F_n, k_j})^{-1} P_{k_j}(X_{j, i}) \right|^2 \right] \\
&= n C \gamma'_n (-H_{F_n, k_j})^{-1} \gamma_n = O(n \|\gamma_n\|^2), \tag{C.47}
\end{aligned}$$

which together with the Markov inequality implies that

$$n^{-1} \gamma'_n (-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} = O_p(n^{-1/2} \|\gamma_n\|). \tag{C.48}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| n^{-1} \gamma'_n ((n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-1} - (-H_{F_n, k_j})^{-1}) \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \right| \\
&\leq \frac{\|\gamma_n\| \left\| n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n} + H_{F_n, k_j} \right\|}{(\rho_{\min}(n^{-1} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n}))^{1/2} \rho_{\min}(-H_{F_n, k_j})} \frac{(\mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} (\mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n})^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n})^{1/2}}{n^{1/2}} \\
&= O_p((\xi_{k_j} k_j \log(k_j))^{1/2} n^{-1} \|\gamma_n\|) = O_p(n^{-1/2} \|\gamma_n\|), \tag{C.49}
\end{aligned}$$

where the first equality is by Assumption C.1(iv), (C.5), (C.6) and (C.44), the last equality is by Assumption C.4(iv). Combining the results in (C.45), (C.48) and (C.49), we prove the result in part (b).

For part (c), by the definitions of $\widehat{\beta}_{\mathbf{k}}$ and $\beta_{\mathbf{k}}^*$, we have

$$\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^* = -(\widehat{H}_n)^{-1} \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*). \tag{C.50}$$

Hence, we can write

$$(\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*)' D_{F_n, \mathbf{k}} (\widehat{\beta}_{\mathbf{k}} - \beta_{\mathbf{k}}^*) = \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)' ((\widehat{H}_n)^{-1} D_{F_n, \mathbf{k}} (\widehat{H}_n)^{-1}) \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) \tag{C.51}$$

Using (C.44) and the Markov inequality, we get

$$\bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*)' \bar{\ell}_{\alpha, n}(\alpha_{\mathbf{k}}^*) = \sum_{j=1, 2} n^{-2} \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} = O_p((k_1 + k_2) n^{-1}). \tag{C.52}$$

Note that

$$\begin{aligned}
& \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'((\widehat{H}_n)^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1} - H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}H_{F_n,\mathbf{k}}^{-1})\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \\
& = \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(\widehat{H}_n)^{-1}(H_{F_n,\mathbf{k}} - \widehat{H}_n)H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \\
& \quad + \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1}(H_{F_n,\mathbf{k}} - \widehat{H}_n)H_{F_n,\mathbf{k}}^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*).
\end{aligned} \tag{C.53}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(\widehat{H}_n)^{-1}(H_{F_n,\mathbf{k}} - \widehat{H}_n)H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \right| \\
& \leq \left| \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(\widehat{H}_n)^{-2}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \right|^{1/2} \times \|H_{F_n,\mathbf{k}} - \widehat{H}_n\| \times \\
& \quad \left| \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'(\widehat{H}_n)^{-1}D_{F_n,\mathbf{k}}H_{F_n,\mathbf{k}}^{-2}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \right|^{1/2} \\
& \leq \rho_{\max}(D_{F_n,\mathbf{k}})\rho_{\max}^{1/2}(H_{F_n,\mathbf{k}}^{-2})\rho_{\max}((\widehat{H}_n)^{-2}) \left| \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \right| \times \|H_{F_n,\mathbf{k}} - \widehat{H}_n\| \\
& = O_p((k_1 + k_2)n^{-1})O_p((\xi_{\mathbf{k}} \log(n)n^{-1})^{1/2}) = o_p(\sigma_{F_n,n}^2)
\end{aligned} \tag{C.54}$$

where the first equality is by Assumption 4.1(f) (which has been verified in Lemma C.1), Lemma C.4, (C.6) and (C.52), the last equality is by Assumptions C.4(i). Similarly, we can show that

$$\left| \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1}(H_{F_n,\mathbf{k}} - \widehat{H}_n)H_{F_n,\mathbf{k}}^{-1}\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) \right| = o_p(\sigma_{F_n,n}^2) \tag{C.55}$$

which together with (C.53) and (C.55) implies that

$$\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)'((\widehat{H}_n)^{-1}D_{F_n,\mathbf{k}}(\widehat{H}_n)^{-1} - H_{F_n,\mathbf{k}}^{-1}D_{F_n,\mathbf{k}}H_{F_n,\mathbf{k}}^{-1})\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*) = o_p(\sigma_{F_n,n}^2). \tag{C.56}$$

Then part(c) is proved by collecting the results in (C.51) and (C.56). \blacksquare

Proof of Lemma C.6. Proof of the results in part (a) and part (b) is standard and follows the similar arguments in Newey (1997). We include it for completeness. Let $\mathbf{U}_{k_j,n} = [u_{k_j,1}, \dots, u_{k_j,n}]'$. By the definition of $\widehat{\alpha}_{k_j}$ and $\alpha_{k_j}^*$,

$$\widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) = P_{k_j}(X_{j,i})'(\widehat{\beta}_{k_j} - \beta_{k_j}^*) \tag{C.57}$$

which implies that

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n \left| \widehat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^2 \\
& = (\widehat{\beta}_{k_j} - \beta_{k_j}^*)'(n^{-1}\mathbf{P}'_{k_j,n}\mathbf{P}_{k_j,n})(\widehat{\beta}_{k_j} - \beta_{k_j}^*) \\
& \leq \rho_{\max}(n^{-1}\mathbf{P}'_{k_j,n}\mathbf{P}_{k_j,n}) \left\| \widehat{\beta}_{k_j} - \beta_{k_j}^* \right\|^2 = O_p(k_j n^{-1})
\end{aligned} \tag{C.58}$$

where the second equality is by Assumption C.1(iv), (C.6) and Lemma C.5(a). This shows part (a).

Now we show part (b). By (C.41) and (C.57),

$$\begin{aligned}
n^{-1} \sum_{i=1}^n \left[|Y_i - \hat{\alpha}_{k_j}(X_{j,i})|^2 \right] &= n^{-1} \sum_{i=1}^n \left[\left| u_{k_j,i} - (\hat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i})) \right|^2 \right] \\
&= n^{-1} \sum_{i=1}^n u_{k_j,i}^2 + n^{-1} \sum_{i=1}^n \left| \hat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i}) \right|^2 \\
&\quad - 2n^{-1} \sum_{i=1}^n (\hat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i})) u_{k_j,i} \\
&= n^{-1} \sum_{i=1}^n u_{k_j,i}^2 - n^{-1} \mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}. \tag{C.59}
\end{aligned}$$

Since $u_{k_j,i} - u_i = \alpha_{k_j}^*(X_{j,i}) - \alpha_j^*(X_{j,i})$,

$$\begin{aligned}
&n^{-1} \sum_{i=1}^n u_{k_j,i}^2 - n^{-1} \sum_{i=1}^n u_i^2 \\
&= n^{-1} \sum_{i=1}^n (u_{k_j,i} - u_i)^2 - 2n^{-1} \sum_{i=1}^n (u_{k_j,i} - u_i) u_i \\
&= n^{-1} \sum_{i=1}^n (\alpha_{k_j}^*(X_{j,i}) - \alpha_j^*(X_{j,i}))^2 + 2n^{-1} \sum_{i=1}^n (\alpha_{k_j}^*(X_{j,i}) - \alpha_j^*(X_{j,i})) u_i. \tag{C.60}
\end{aligned}$$

The pseudo-true value of the parameter can be written as the limit of a sequence of sieve approximation: $\alpha_j^*(x_j) = \sum_{l=1}^{\infty} p_{j,l}(x_j) \beta_{j,l}^*$, where $(\beta_{j,l}^*)_{l=1}^{\infty} = \arg \min_{\beta_{j,l} \in R, \forall l} E_{F_n} \left[|Y - \sum_{l=1}^{\infty} p_{j,l}(X_j) \beta_{j,l}|^2 \right]$. By the definition of u_j and the first order optimality condition of $(\beta_{j,l}^*)_{l=1}^{\infty}$,

$$E_{F_n} [u_j p_{j,l}(X_j)] = 0 \tag{C.61}$$

for $j = 1, 2$ and for any $l = 1, 2, \dots$, which implies that $E_{F_n} [u_j \alpha_j^*(X_j)] = 0$ and $E_{F_n} [u_j \alpha_{k_j}^*(X_j)] = 0$ for $j = 1, 2$. Therefore

$$\begin{aligned}
&E_{F_n} \left[\left| n^{-1} \sum_{i=1}^n (\alpha_{k_j}^*(X_{j,i}) - \alpha_j^*(X_{j,i})) u_i \right|^2 \right] \\
&= n^{-2} \sum_{i=1}^n E_{F_n} \left[(\alpha_{k_j}^*(X_{j,i}) - \alpha_j^*(X_{j,i}))^2 u_i^2 \right] = O(n^{-1} k_j^{-2r_j}), \tag{C.62}
\end{aligned}$$

where the first equality is by Assumption C.1(i), the second equality is by Assumptions C.1(ii) and C.2(ii). Combining the results in (C.59), (C.60), (C.62) and applying Assumption C.1(ii) and the Markov inequality, we deduce that

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^n \left[|Y_i - \hat{\alpha}_{k_j}(X_{j,i})|^2 - |Y_i - \alpha_j^*(X_{j,i})|^2 \right] \\ &= -\frac{1}{2} \mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n} + O_p(k_j^{-r_j} n^{1/2}) + O(nk_j^{-2r_j}). \end{aligned} \quad (\text{C.63})$$

By (C.6) and (C.44).

$$\begin{aligned} & \mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-2} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n} \\ & \leq \rho_{\max}((n^{-1} \mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-2}) n^{-2} \mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n} = O_p(k_j n^{-1}) \end{aligned} \quad (\text{C.64})$$

and similarly

$$n^{-1} \mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} H_{F_n, k_j}^{-2} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n} = O_p(k_j). \quad (\text{C.65})$$

Using the Cauchy-Schwarz inequality, Lemma C.4, (C.64) and (C.65), we have

$$\begin{aligned} & \left| \frac{\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}}{n} - \frac{\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}}{n^2} \right|^2 \\ &= \left| \frac{\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-1} (n^{-1} \mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n} + H_{F_n, k_j}) (-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}}{n} \right|^2 \\ &\leq \left\| n^{-1} \mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n} + H_{F_n, k_j} \right\|^2 \times (\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (\mathbf{P}'_{k_j,n} \mathbf{P}_{k_j,n})^{-2} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}) \times \\ & \quad \frac{\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} H_{F_n, k_j}^{-2} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}}{n^2} = O_p(\log(k_j) \xi_{k_j} k_j^2 n^{-3}). \end{aligned} \quad (\text{C.66})$$

That together with (C.63) implies that

$$\begin{aligned} & \frac{1}{2n} \sum_{i=1}^n \left[|Y_i - \hat{\alpha}_{k_j}(X_{j,i})|^2 - |Y_i - \alpha_j^*(X_{j,i})|^2 \right] \\ &= -\frac{\mathbf{U}'_{k_j,n} \mathbf{P}_{k_j,n} (-H_{F_n, k_j})^{-1} \mathbf{P}'_{k_j,n} \mathbf{U}_{k_j,n}}{2n^2} + O_p((\log(k_j) \xi_{k_j})^{1/2} k_j n^{-3/2} + k_j^{-r_j} n^{-1/2} + k_j^{-2r_j}). \end{aligned} \quad (\text{C.67})$$

Now observe that $\bar{\ell}_n(\hat{\alpha}_{\mathbf{k},n})$ can be decomposed as

$$\bar{\ell}_n(\hat{\alpha}_{\mathbf{k},n}) = \frac{1}{2n} \sum_{i=1}^n \left[|Y_i - \alpha_1^*(X_{1,i})|^2 - |Y_i - \alpha_2^*(X_{2,i})|^2 \right]$$

$$\begin{aligned}
& + \frac{1}{2n} \sum_{i=1}^n [|Y_i - \widehat{\alpha}_{k_1}(X_{1,i})|^2 - |Y_i - \alpha_1^*(X_{1,i})|^2] \\
& - \frac{1}{2n} \sum_{i=1}^n [|Y_i - \widehat{\alpha}_{k_2}(X_{2,i})|^2 - |Y_i - \alpha_2^*(X_{2,i})|^2]
\end{aligned} \tag{C.68}$$

which combined with (C.67) proves the lemma. ■

D Example II: Nonparametric Quantile-Regression

In this section, we illustrate the use of our test for comparing two nonparametric quantile regression models based on the expected check function criterion.

For $j = 1, 2$, consider the model

$$Y = \alpha_j^*(X_j) + u_j, \quad E_{F_0}[I\{u_j \leq 0\}|X_j] = \tau, \tag{D.1}$$

where $\tau \in (0, 1)$, u_j is a unobservable error term, $\alpha_j^*(x)$ is a unknown function and F_0 denotes the joint distribution of $Z \equiv (Y, X_1, X_2)$. The regressors X_1 and X_2 in two equations may be nested, over-lapped or strictly non-nested. Note that even when the regressors are strictly non-nested in the intuitive sense (i.e., X_1 and X_2 do not contain any common variable), models represented by the two equations in general are still overlapping according to the definitions in Section 2.2 of the main paper because it may be possible that $\alpha_1^*(X_1) = \alpha_2^*(X_2) = \text{Constant}$.

Suppose that the unknown function $\alpha_j^*(\cdot)$ of model j belongs to the set \mathcal{A}_j . We use the finite-dimensional approximations:

$$\mathcal{A}_{j,k_j} = \{ \alpha_{k_j}(\cdot) : \alpha_{k_j}(\cdot) = \alpha_j(\beta_{k_j}) = P_{k_j}(\cdot)' \beta_{k_j} : \beta_{k_j} \in R^{k_j} \} \tag{D.2}$$

where $P_{k_j}(\cdot) = [p_{j,1}(\cdot), \dots, p_{j,k_j}(\cdot)]'$ is a (k_j -dimensional) vector of basis functions. Let $\rho_\tau(u) = (I\{u \leq 0\} - \tau)u$. Define

$$\beta_{k_j, F_0}^* = \arg \min_{\beta_{k_j} \in R^{k_j}} E_{F_0} [\rho_\tau(Y - P_{k_j}(X_j)' \beta_{k_j})] \quad \text{for } j = 1, 2. \tag{D.3}$$

We use $\widehat{\beta}_{k_j}^*$ to denote the counterpart of β_{k_j, F_0}^* , when the expectation in (D.3) is taken with respect to any DGP $F \in \mathcal{F}$. We suppress the dependence of $\widehat{\beta}_{k_j}^*$ on F for notational convenience. The approximate M-estimator in this example is defined as

$$\widehat{\beta}_{k_j}^* = \arg \max_{\beta_{k_j} \in R^{k_j}} n^{-1} \sum_{i=1}^n \rho_\tau(Y_i - P_{k_j}(X_{j,i})' \beta_{k_j}) \quad \text{for } j = 1, 2. \tag{D.4}$$

In this example, the pseudo-density ratio function is

$$\ell(Z; \alpha) = \rho_\tau(Y - \alpha_1(X_1)) - \rho_\tau(Y - \alpha_2(X_2)). \quad (\text{D.5})$$

The score function $\ell_{\alpha, \mathbf{k}}(Z; \alpha)$ in this example is

$$\ell_{\alpha, \mathbf{k}}(Z; \alpha) = \begin{pmatrix} (\tau - I\{Y \leq \alpha_1(X_1)\}) P_{k_1}(X_1) \\ (I\{Y \leq \alpha_2(X_2)\} - \tau) P_{k_2}(X_2) \end{pmatrix}, \quad (\text{D.6})$$

which combined with the formula in (3.11) defines

$$D_{F_0, \mathbf{k}} = E_{F_0} \begin{bmatrix} u_{k_1}^2 P_{k_1}(X_1) P_{k_1}(X_1)' & -u_{k_1} u_{k_2} P_{k_1}(X_1) P_{k_2}(X_2)' \\ -u_{k_1} u_{k_2} P_{k_2}(X_2) P_{k_1}(X_1)' & u_{k_2}^2 P_{k_2}(X_2) P_{k_2}(X_2)' \end{bmatrix}, \quad (\text{D.7})$$

where $u_{k_j} = I\{Y \leq \alpha_{k_j}^*(X_j)\} - \tau$. The Hessian matrix H_{F_0, k_j} is

$$H_{F_0, k_j} = -E_{F_0} \left[f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) P_{k_j}(X_j) P_{k_j}(X_j)' \right] \text{ for } j = 1, 2 \quad (\text{D.8})$$

where $\Delta(\alpha_j) = \alpha_j(X_j) - \alpha_j^*(X_j)$ and $f_{u_j}(\cdot | X_j)$ denotes the conditional density of u_j given X_j . In this example, $H_{F_0, \mathbf{k}}$ can be constructed using (3.10) and (D.8).

We next provide the sufficient conditions for Theorem 4.1 in this example.

Assumption D.1 For any $F_0 \in \mathcal{F}$ and for $j = 1, 2$: (i) $f_{u_j}(u_j | x_j) < C$ uniformly in u_j and x_j ; (ii) $\partial f_{u_j}(u_j | x_j) / \partial u_j < C$ uniformly in u_j and x_j ; (iii) $f_{u_j}(0 | x_j) > C^{-1}$ uniformly in x_j ; (iv) $\xi_{k_j} k_j (\log(n))^2 n^{-1} + k_j^{-r_j} \log(n) = o(1)$; (v) $E_{F_0} [\rho_\tau(u_1) - \rho_\tau(u_2)] = 0$ when $F_0 \in \mathcal{F}_0$; (vi) $E_{F_0} [u_j^2] < C$ for $j = 1, 2$; (vii) there exists a $\delta > 0$ such that $E \left[|\omega_{F_0, * }^{-1} (\rho_\tau(u_1) - \rho_\tau(u_2))|^{2+\delta} \right] < C$ whenever $\omega_{F_0, * }^2 > 0$.

Let $\delta_{0, j, n} = (k_j \log(n) n^{-1})^{1/2}$, $\delta_{1, j, n} = \delta_{0, j, n} + k_j^{-r_j}$ and $\delta_{2, j, n} = \delta_{0, j, n} \xi_{k_j}^{1/2} + k_j^{-r_j}$.

Theorem D.1 Suppose that Assumptions C.1 and D.1 hold, where β_{k_j, F_0}^* in Assumption C.1(ii) is defined in (D.3). If for any sequence of DGP's $\{F_n\}_{n \geq 1}$ such that $F_n \in \mathcal{F}_0$, we also have

$$\frac{1}{n \sigma_{F_n, n}^2} + \max_{j=1, 2} \left\{ \frac{\delta_{1, j, n} (k_j \delta_{2, j, n} \log(n))^{1/2}}{\sigma_{F_n, n}} + \frac{1 + k_j^{-r_j} n^{1/2}}{k_j^{r_j} \sigma_{F_n, n}} \right\} = o(1), \quad (\text{D.9})$$

then under F_n for all n ,

$$\frac{n \bar{\ell}_n(\hat{\alpha}_{\mathbf{k}}) + (1/2) \text{tr}(\hat{D}_n(\alpha_{\mathbf{k}}^*) H_{F_n, \mathbf{k}}^{-1})}{\sqrt{n \omega_{F_0, * }^2 + \frac{n-1}{2n} \omega_{F_n, U, \mathbf{k}}^2}} \rightarrow_d N(0, 1), \quad (\text{D.10})$$

where $\widehat{D}_n(\alpha_{\mathbf{k}}^*) = n^{-1} \sum_{i=1}^n P_{u,\mathbf{k}}(X_i)P_{u,\mathbf{k}}(X_i)'$ and $P_{u,\mathbf{k}}(X_i)' = [u_{k_1}P_{k_1}(X_{1,i})', u_{k_2}P_{k_2}(X_{2,i})']$.

To prove Theorem D.1, we need to derive some useful lemmas. For this purpose, we introduce some notations and preliminary results in the literature.

Under Assumptions C.1 and D.1, we can invoke Theorem 1 in Belloni et al. (2011) to get

$$\left\| \widehat{\beta}_{k_j} - \beta_{k_j}^* \right\| = O_p \left(k_j^{1/2} n^{-1/2} \right) \text{ for } j = 1, 2. \quad (\text{D.11})$$

Let $\mathcal{N}_{j,n} = \left\{ \beta_{k_j} \in R^{k_j} : \left\| \beta_{k_j} - \beta_{k_j}^* \right\| \leq C\delta_{0,j,n} \right\}$ where C is some fixed and large constant. Then we have $\widehat{\beta}_{k_j} \in \mathcal{N}_{j,n}$ with probability approaching 1. Using (D.11), Assumptions C.1(ii) and (iv), we have

$$\sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} E_{F_0} \left[\left| P'_{k_j}(X_j)\beta_{k_j} - \alpha_j^*(X_j) \right|^2 \right] \leq C \left[\sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left\| \beta_{k_j} - \beta_{k_j}^* \right\|^2 + k_j^{-2r_j} \right] = O(\delta_{1,j,n}^2) \quad (\text{D.12})$$

for any $F_0 \in \mathcal{F}$. Moreover, using (D.11), Assumptions C.1(ii), (iii) and (iv), we get

$$\sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| P'_{k_j}(x_j)\beta_{k_j} - \alpha_j^*(x_j) \right| \leq \sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left[\left\| \beta_{k_j} - \beta_{k_j}^* \right\| \xi_{k_j}^{1/2} \right] + Ck_j^{-r_j} = O(\delta_{2,j,n}) \quad (\text{D.13})$$

uniformly over x_j .

For ease of notations, we define $\Delta_i(\alpha_j) = \alpha_j(X_{j,i}) - \alpha_j^*(X_{j,i})$ and $\Delta(\alpha_j) = \alpha_j(X_j) - \alpha_j^*(X_j)$ for $j = 1, 2$. Then by definition,

$$\begin{aligned} \rho_\tau(Y - \alpha_j(X_j)) &= (I\{u_j \leq \Delta(\alpha_j)\} - \tau)(u_j - \Delta(\alpha_j)) \\ &= (I\{u_j \leq 0\} - \tau)u_j - (I\{u_j \leq 0\} - \tau)\Delta(\alpha_j) \\ &\quad - (I\{u_j \leq 0\} - I\{u_j \leq \Delta(\alpha_j)\})(u_j - \Delta(\alpha_j)), \end{aligned} \quad (\text{D.14})$$

which together with $\rho_\tau(Y - \alpha_j^*(X_j)) = (I\{u_j \leq 0\} - \tau)u$ implies that

$$\begin{aligned} &\rho_\tau(Y - \alpha_j^*(X_j)) - \rho_\tau(Y - \alpha_j(X_j)) - (I\{u_j \leq 0\} - \tau)\Delta(\alpha_j) \\ &= (I\{u_j \leq 0\} - I\{u_j \leq \Delta(\alpha_j)\})(u_j - \Delta(\alpha_j)). \end{aligned} \quad (\text{D.15})$$

The above expression is useful to derive the expansion of the QLR statistic.

Let $\mu_n[g(Z)] = n^{-1} \sum_{i=1}^n [g(Z_i) - E_{F_0}[g(Z_i)]]$ denote the empirical process indexed by function g .

Lemma D.1 Under Assumptions C.1 and D.1, we have

$$\mu_n \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) u_j \right] = O_p \left(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} \right). \quad (\text{D.16})$$

Proof of Lemma D.1. Define $\mathcal{F}_{1,j,n} = \{ (I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\}) u_j : \beta_{k_j} \in \mathcal{N}_{j,n} \}$ for $j = 1, 2$. It is clear that the VC-dimension of $\mathcal{F}_{1,j,n}$ is $O(k_j)$. By (D.13), we have

$$\sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| (I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\}) u_j \right| \leq \sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| \Delta(P_{k_j}(X_j)' \beta_{k_j}) \right| \leq C \delta_{2,j,n}. \quad (\text{D.17})$$

By definition,

$$\begin{aligned} & E_{F_n} \left[\left| (I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\}) u_j \right|^2 \right] \\ &= E_{F_n} \left[I\{0 \leq u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} u_j^2 \right] + E_{F_n} \left[I\{\Delta(P_{k_j}(X_j)' \beta_{k_j}) \leq u_j \leq 0\} u_j^2 \right]. \end{aligned} \quad (\text{D.18})$$

Using Assumption D.1(i), we get

$$E_{F_n} \left[I\{0 \leq u_j \leq \Delta(\alpha_j)\} u_j^2 \mid X_j \right] \leq \left| \int_0^{\Delta(\alpha)} u^2 f_{u_j}(u \mid X_j) du \right| \leq C |\Delta(\alpha_j)|^3 \quad (\text{D.19})$$

which together with (D.12) and (D.13) implies that for any $\beta_{k_j} \in \mathcal{N}_{j,n}$,

$$\begin{aligned} & E_{F_n} \left[I\{0 \leq u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} u_j^2 \right] \\ &\leq C E_{F_n} \left[\left| \Delta(P_{k_j}(X_j)' \beta_{k_j}) \right|^3 \right] \leq C E_{F_n} \left[\Delta(P_{k_j}(X_j)' \beta_{k_j})^2 \right] \delta_{2,j,n} \leq C \delta_{1,j,n}^2 \delta_{2,j,n}. \end{aligned} \quad (\text{D.20})$$

Similarly, we have

$$E_{F_n} \left[I\{\Delta(P_{k_j}(X_j)' \beta_{k_j}) \leq u_j \leq 0\} u_j^2 \right] \leq C \delta_{1,j,n}^2 \delta_{2,j,n} \quad (\text{D.21})$$

which together with (D.18) and (D.20) implies that

$$E_{F_n} \left[\left| (I\{u_j \leq 0\} - I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\}) u_j \right|^2 \right] \leq C \delta_{1,j,n}^2 \delta_{2,j,n}. \quad (\text{D.22})$$

Combining the results in (D.17), (D.22) and the VC-dimension of $\mathcal{F}_{1,j,n}$ is $O(k_j)$, we can invoke Lemma 22.3 and Lemma 23 in Belloni et al. (2011) to show that

$$\mu_n \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) u_j \right] = O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + \delta_{0,j,n}^2 \delta_{2,j,n}). \quad (\text{D.23})$$

That together with $\delta_{0,j,n}^2 \delta_{2,j,n} \leq \delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} = o(1)$ (which is implied by Assumption D.1(iv)) implies (D.16). ■

Lemma D.2 Under Assumptions C.1 and D.1, we have

$$\mu_n \left[(I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) \Delta(\alpha_{k_j}^*) \right] = O_p \left(\delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j} \right). \quad (\text{D.24})$$

Proof of Lemma D.2. For $j = 1, 2$, we define

$$\mathcal{F}_{2,j,n} = \left\{ (I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\}) \Delta(\alpha_{k_j}^*) : \beta_{k_j} \in \mathcal{N}_{j,n} \right\}.$$

It is clear that the VC-dimension of $\mathcal{F}_{2,j,n}$ is $O(k_j)$. By Assumption C.1(ii), we have

$$\sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| (I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\}) \Delta(\alpha_{k_j}^*) \right| \leq C k_j^{-r_j}. \quad (\text{D.25})$$

Under Assumption D.1(i),

$$\begin{aligned} & E_{F_n} \left[|I\{u_j \leq \Delta(\alpha_j)\} - I\{u_j \leq 0\}|^2 \middle| X_j \right] \\ &= E_{F_n} [I\{0 \leq u_j \leq \Delta(\alpha_j)\} \middle| X_j] + E_{F_n} [I\{\Delta(\alpha_j) \leq u_j \leq 0\} \middle| X_j] \\ &\leq 2 \left| \int_0^{\Delta(\alpha_j)} f_{u_j}(u | X_j) du \right| \leq C |\Delta(\alpha_j)|, \end{aligned} \quad (\text{D.26})$$

which together with Assumption C.1(ii) and (D.12) implies that for any $\beta_{k_j} \in \mathcal{N}_{j,n}$

$$E_{F_n} \left[\left| I\{u_j \leq \Delta(P_{k_j}(X_j)' \beta_{k_j})\} - I\{u_j \leq 0\} \right|^2 \Delta^2(\alpha_{k_j,n}^*) \right] \leq C k_j^{-2r_j} \delta_{1,j,n}. \quad (\text{D.27})$$

Combining the results in (D.25), (D.27) and the VC-dimension of $\mathcal{F}_{2,j,n}$ is $O(k_j)$, we can invoke Lemma 22.3 and Lemma 23 in Belloni et al. (2011) to get

$$\mu_n \left[(I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) \Delta(\alpha_{k_j}^*) \right] = O_p \left(\delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j} + \delta_{0,j,n}^2 k_j^{-r_j} \right). \quad (\text{D.28})$$

That together with $\delta_{0,j,n} = o(1)$ (which is implied by Assumption D.1(iv)) implies (D.24). ■

Lemma D.3 Under Assumptions C.1 and D.1, we have

$$\sup_{\lambda_j \in S_{k_j}} \mu_n \left[(I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) \right] = O_p(\delta_{0,j,n} \delta_{2,j,n}^{1/2}), \quad (\text{D.29})$$

where $S_{k_j} = \{\lambda_j \in R^{k_j} : \|\lambda_j\| = 1\}$.

Proof of Lemma D.3. Using the same arguments in the proof of Lemma 33 in Belloni et al.

(2011), we have

$$\sup_{\lambda_j \in S_{k_j}} \mu_n \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq \Delta(\alpha_{k_j}^*)\}) \lambda_j' P_{k_j}(X_j) \right] = O_p(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 \xi_{k_j}^{1/2}). \quad (\text{D.30})$$

As $\delta_{0,j,n}^2 \xi_{k_j} = o(1)$ by Assumption D.1(iv), we get

$$\delta_{0,j,n}^2 \xi_{k_j}^{1/2} / (\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4}) = \delta_{0,j,n}^{1/2} \xi_{k_j}^{1/4} = o(1) \quad (\text{D.31})$$

which together with (D.30) implies that

$$\sup_{\lambda_j \in S_{k_j}} \mu_n \left[\left(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq \Delta(\alpha_{k_j}^*)\} \right) \lambda_j' P_{k_j}(X_j) \right] = O_p(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4}). \quad (\text{D.32})$$

For $j = 1, 2$, we define

$$\mathcal{F}_{3,j,n} = \left\{ (I\{u_j \leq \Delta(\alpha_{k_j}^*)\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) : \lambda_j \in S_{k_j} \right\}.$$

It is clear that the VC-dimension of $\mathcal{F}_{3,j,n}$ is $O(k_j)$. By Assumption C.1(iii), we have

$$\sup_{\lambda_j \in S_{k_j}} \left| (I\{u_j \leq \Delta(\alpha_{k_j}^*)\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) \right| \leq \xi_{k_j}^{1/2}. \quad (\text{D.33})$$

Under Assumptions C.1(ii), C.1(iv) and D.1(i),

$$\begin{aligned} & E_{F_n} \left[\left| (I\{u_j \leq \Delta(\alpha_{k_j}^*)\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) \right|^2 \right] \\ & \leq E_{F_n} \left[\left| (\lambda_j' P_{k_j}(X_j))^2 \int_0^{\Delta(\alpha_{k_j}^*)} f_{u_j}(u|X_j) du \right| \right] \leq C k_j^{-r_j}. \end{aligned} \quad (\text{D.34})$$

Combining the results in (D.33), (D.34) and the VC-dimension of $\mathcal{F}_{3,j,n}$ is $O(k_j)$, we can invoke Lemma 22.3 and Lemma 23 in Belloni et al. (2011) to get

$$\sup_{\lambda_j \in S_{k_j}} \mu_n \left[(I\{u_j \leq \Delta(\alpha_{k_j}^*)\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) \right] = O_p(\delta_{0,j,n} k_j^{-r_j/2} + \delta_{0,j,n}^2 \xi_{k_j}^{1/2}), \quad (\text{D.35})$$

which together with (D.32) and $\delta_{0,j,n}^2 \xi_{k_j}^{1/2} (\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4})^{-1} = (\delta_{0,j,n} \xi_{k_j}^{1/2})^{1/2} = o(1)$ (which is implied by Assumption D.1(iv)) implies that

$$\sup_{\lambda_j \in S_{k_j}} \mu_n \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) \lambda_j' P_{k_j}(X_j) \right] = O_p(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n} k_j^{-r_j/2}). \quad (\text{D.36})$$

Since $\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n} k_j^{-r_j/2} = O(\delta_{0,j,n} \delta_{2,j,n}^{1/2})$, (D.29) follows by (D.36). \blacksquare

Lemma D.4 *Under Assumptions C.1 and D.1, we have*

$$\begin{aligned} & E_{Z, F_n} \left[(I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) [u_j - (\widehat{\alpha}_{k_j} - \alpha_{k_j}^*)] \right] \\ &= \frac{(\widehat{\beta}_{k_j} - \beta_{k_j}^*)' H_{F_n, k_j} (\widehat{\beta}_{k_j} - \beta_{k_j}^*)}{2} \\ &+ O_p(\delta_{1,j,n}^2 \delta_{2,j,n} + \delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}), \end{aligned} \quad (\text{D.37})$$

where $E_{Z, F_n} [\cdot]$ denotes expectation taken on Z under the joint distribution F_n .

Proof of Lemma D.4. Let $H_{F_n, k_j}^* = -E_{F_n} [f_{u_j}(0|X_j) P_{k_j}(X_j) P_{k_j}(X_j)']$. By Assumptions C.1(iv), D.1(i) and D.1(iii),

$$-C \leq \rho_{\min}(H_{F_n, k_j}^*) \leq \rho_{\max}(H_{F_n, k_j}^*) \leq -C^{-1} \quad (\text{D.38})$$

uniformly in n for $j = 1, 2$. By Assumptions C.1(ii), C.1(iv) and D.1(ii),

$$\begin{aligned} & \left\| E_{F_n} \left[\left(f_{u_j}(0|X_j) - f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) \right) \lambda' P_{k_j}(X_j) P_{k_j}(X_j)' \right] \right\|^2 \\ & \leq C E_{F_n} \left[\left(f_{u_j}(0|X_j) - f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) \right)^2 |\lambda' P_{k_j}(X_j)|^2 \right] \\ & \leq C E_{F_n} \left[\Delta^2(\alpha_{k_j}^*) |\lambda' P_{k_j}(X_j)|^2 \right] \leq C \|\lambda\|^2 k_j^{-2r_j} \end{aligned} \quad (\text{D.39})$$

which together with Assumption D.1(iv) implies that

$$\left\| H_{F_n, k_j} - H_{F_n, k_j}^* \right\| = O(k_j^{-r_j}) = o(1). \quad (\text{D.40})$$

Combining the results in (D.38) and (D.40), we get

$$-C \leq \rho_{\min}(H_{F_n, k_j}) \leq \rho_{\max}(H_{F_n, k_j}) \leq -C^{-1}. \quad (\text{D.41})$$

Let $\mathbf{U}_{k_j, n} = (u_{k_j, 1}, \dots, u_{k_j, n})'$ where $u_{k_j, i} = I\{Y \leq \alpha_{k_j}^*(X_j)\} - \tau$. Under Assumptions C.1 and D.1, we can invoke Lemma 4 in Belloni et al. (2011) to deduce that

$$\lambda'_{k_j} (\widehat{\beta}_{k_j} - \beta_{k_j}^*) = \frac{\lambda'_{k_j} H_{F_n, k_j}^{*-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n}}{n} + O_p(\epsilon_{1,j,n} + \epsilon_{2,j,n} + \epsilon_{3,j,n}) \quad (\text{D.42})$$

uniformly over $\lambda_{k_j} \in S_{k_j} = \{\lambda_j \in R^{k_j} : \|\lambda_j\| = 1\}$, where

$$\epsilon_{1,j,n} = \delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 \xi_{k_j}^{1/2}, \quad \epsilon_{2,j,n} = \delta_{0,j,n}^2 \xi_{k_j}^{1/2} + \delta_{0,j,n} k_j^{-r_j} \quad \text{and} \quad \epsilon_{3,j,n} = k_j \xi_{k_j}^{1/2} n^{-1}. \quad (\text{D.43})$$

As $k_j n^{-1} = O(\delta_{0,j,n}^2)$ and $\delta_{0,j,n}^{1/2} \xi_{k_j}^{1/4} = o(1)$ by Assumption D.1(iv), we have

$$(k_j \xi_{k_j}^{1/2} n^{-1})(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4})^{-1} = \delta_{0,j,n}^{1/2} \xi_{k_j}^{1/4} O(1) = o(1), \quad (\text{D.44})$$

which together with (D.31), (D.42) and (D.43) implies that

$$\lambda'_{k_j} (\widehat{\beta}_{k_j} - \beta_{k_j}^*) = n^{-1} \lambda'_{k_j} H_{F_n, k_j}^{*-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} + O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n} k_j^{-r_j} \right). \quad (\text{D.45})$$

By the first order optimality condition of $\beta_{k_j}^*$,

$$E_{F_n} [P_{k_j}(X_j) u_{k_j}] = \mathbf{0}_{k_j \times 1}. \quad (\text{D.46})$$

By Assumptions C.1(i) and C.1(iv), (D.38), (D.40) and (D.41),

$$\begin{aligned} & E_{F_n} \left[\left| \lambda'_{k_j} (H_{F_n, k_j}^{*-1} - H_{F_n, k_j}^{-1}) \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \right|^2 \right] \\ &= E_{F_n} \left[\left| \sum_{i=1}^n \lambda'_{k_j} (H_{F_n, k_j}^{*-1} - H_{F_n, k_j}^{-1}) P_{k_j}(X_{j,i}) u_{k_j, i} \right|^2 \right] \\ &= \sum_{i=1}^n E_{F_n} \left[\left| \lambda'_{k_j} H_{F_n, k_j}^{*-1} (H_{F_n, k_j}^* - H_{F_n, k_j}) H_{F_n, k_j}^{-1} P_{k_j}(X_{j,i}) \right|^2 u_{k_j, i}^2 \right] \\ &\leq Cn \left\| \lambda'_{k_j} H_{F_n, k_j}^{*-1} (H_{F_n, k_j}^* - H_{F_n, k_j}) H_{F_n, k_j}^{-1} \right\|^2 = O(n k_j^{-2r_j}) \end{aligned} \quad (\text{D.47})$$

which together with the Markov inequality implies that

$$n^{-1} \lambda'_{k_j} (H_{F_n, k_j}^{*-1} - H_{F_n, k_j}^{-1}) \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} = O(n^{-1/2} k_j^{-r_j}). \quad (\text{D.48})$$

Combining the results in (D.45) and (D.48), and then applying Assumption D.1(iv), we deduce that

$$\lambda'_{k_j} (\widehat{\beta}_{k_j} - \beta_{k_j}^*) = n^{-1} \lambda'_{k_j} H_{F_n, k_j}^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} + O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} + \delta_{0,j,n} k_j^{-r_j} \right). \quad (\text{D.49})$$

By Assumptions C.1(ii) and D.1(i),

$$\begin{aligned} & f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) (\alpha_{k_j} - \alpha_{k_j}^*)^2 - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) (\alpha_{k_j} - \alpha_{k_j}^*)^2 \\ &= 2f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) (\alpha_{k_j} - \alpha_{k_j}^*) \Delta(\alpha_{k_j}^*) + f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\alpha_{k_j}^*)^2 \end{aligned}$$

$$= 2(\beta_{k_j} - \beta_{k_j}^*)' \left[f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) P_{k_j}(X_j) \Delta(\alpha_{k_j}^*) \right] + O(k_j^{-2r_j}). \quad (\text{D.50})$$

Let $\lambda_{*,j,n} = E_{F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) P_{k_j}(X_j) \Delta(\alpha_{k_j}^*) \right]$. Under Assumptions C.1(iv)

$$\lambda'_{*,j,n} \lambda_{*,j,n} \leq C \lambda'_{*,j,n} \left(E_{F_n} [P_{k_j}(X_j) P'_{k_j}(X_j)] \right)^{-1} \lambda_{*,j,n}, \quad (\text{D.51})$$

which together with the same arguments in the proof of Lemma E.1, Assumptions C.1(ii) and D.1(i) implies that

$$\lambda'_{*,j,n} \lambda_{*,j,n} \leq E_{F_n} \left[f_{u_j}^2(\Delta(\alpha_{k_j}^*)|X_j) \Delta(\alpha_{k_j}^*)^2 \right] = O(k_j^{-2r_j}). \quad (\text{D.52})$$

Using (D.49) and (D.52), we get

$$\begin{aligned} & (\widehat{\beta}_{k_j} - \beta_{k_j}^*)' E_{F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) P_{k_j}(X_j) \Delta(\alpha_{k_j}^*) \right] \\ &= \frac{\lambda'_{*,j,n} H_{F_n, k_j}^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n}}{n} + O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} \right). \end{aligned} \quad (\text{D.53})$$

Under the i.i.d. assumption,

$$\begin{aligned} E_{F_n} \left[\mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} \right] &= E_{F_n} \left[\sum_{i_1=1}^n \sum_{i_2=1}^n P_{k_j}(X_{j,i_1}) P_{k_j}(X_{j,i_2})' u_{k_j, i_1} u_{k_j, i_2} \right] \\ &= E_{F_n} \left[\sum_{i=1}^n P_{k_j}(X_{j,i}) P_{k_j}(X_{j,i})' u_{k_j, i}^2 \right] \end{aligned} \quad (\text{D.54})$$

which together with (D.41), Assumption C.1(iv) and (D.52) implies that

$$\frac{E_{F_n} \left[\lambda'_{*,j,n} H_{F_n, k_j}^{-1} \mathbf{P}'_{k_j, n} \mathbf{U}_{k_j, n} \mathbf{U}'_{k_j, n} \mathbf{P}_{k_j, n} H_{F_n, k_j}^{-1} \lambda_{*,j,n} \right]}{n^2} \leq C \frac{\lambda'_{*,j,n} \lambda_{*,j,n}}{n} = O(k_j^{-2r_j} n^{-1}), \quad (\text{D.55})$$

which together with the Markov inequality and (D.53) implies that

$$(\widehat{\beta}_{k_j, n} - \beta_{k_j, n}^*)' \lambda_{*,j,n} = O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right). \quad (\text{D.56})$$

Combining the results in (D.50) and (D.56), we get

$$\begin{aligned} & \left| f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) (\widehat{\alpha}_{k_j} - \alpha_{k_j}^*)^2 - f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) (\widehat{\alpha}_{k_j} - \alpha_{k_j}^*)^2 \right| \\ &= O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right), \end{aligned} \quad (\text{D.57})$$

where we note that $\delta_{0,j,n}k_j^{-2r_j} = o(k_j^{-2r_j})$ as $\delta_{0,j,n} = o(1)$. Let $E_{X_j, F_n}[\cdot]$ denote the expectation taken with respect to X_j . Note that using (D.56), we also have

$$\begin{aligned} & E_{X_j, F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j)\Delta(\widehat{\alpha}_{k_j})(\widehat{\alpha}_{k_j} - \alpha_{k_j}^*) - f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j)(\widehat{\alpha}_{k_j} - \alpha_{k_j}^*)^2 \right] \\ &= E_{X_j, F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j)\Delta(\alpha_{k_j}^*)(\widehat{\alpha}_{k_j} - \alpha_{k_j}^*) \right] \\ &= (\widehat{\beta}_{k_j} - \beta_{k_j}^*)' \lambda_{*,j,n} = O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right). \end{aligned} \quad (\text{D.58})$$

Using integration by parts, we can write

$$\begin{aligned} \int_0^{\Delta(\alpha_j)} u f_{u_j}(u|x_j) du &= \frac{1}{2} \int_0^{\Delta(\alpha_j)} f_{u_j}(u|x_j) du^2 \\ &= \frac{f_{u_j}(\Delta(\alpha_j)|x_j)\Delta^2(\alpha_j)}{2} - \frac{\int_0^{\Delta(\alpha_j)} u^2 \frac{\partial f_{u_j}(u|x_j)}{\partial u} du}{2}, \end{aligned} \quad (\text{D.59})$$

which together with the triangle inequality and Assumption D.1(ii) implies that

$$\begin{aligned} & \left| E_{F_n} \left[(I\{u_j \leq \Delta(\alpha_j)\} - I\{u_j \leq 0\}) u_j | X_j \right] - \frac{f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j)\Delta^2(\alpha_j)}{2} \right| \\ & \leq \frac{\left| f_{u_j}(\Delta(\alpha_j)|X_j) - f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j) \right| \Delta^2(\alpha_j)}{2} + \frac{\left| \int_0^{\Delta(\alpha_j)} u^2 \frac{\partial f_{u_j}(u|x_j)}{\partial u} du \right|}{2} \\ & \leq C \left(|\Delta(\alpha_j)|^2 |\alpha_j - \alpha_{k_j}^*| + |\Delta(\alpha_j)|^3 \right) \end{aligned} \quad (\text{D.60})$$

uniformly over $\beta_{k_j} \in \mathcal{N}_{j,n}$. Using (D.12), (D.13), (D.57), (D.60) and the triangle inequality, we immediately get

$$\begin{aligned} & \left| E_{Z, F_n} \left[(I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) u_j - \frac{f_{u_j}(\Delta(\alpha_{k_j}^*)|X_j)(\widehat{\alpha}_{k_j} - \alpha_{k_j}^*)^2}{2} \right] \right| \\ & \leq C E_{X_j, F_n} \left[|\Delta(\widehat{\alpha}_{k_j})|^2 |\widehat{\alpha}_{k_j} - \alpha_{k_j}^*| + |\Delta(\widehat{\alpha}_{k_j})|^3 \right] + O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right) \\ & = O_p \left(\delta_{1,j,n}^2 \delta_{2,j,n} + \delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right) \end{aligned} \quad (\text{D.61})$$

Similarly, we have

$$\int_0^{\Delta(\alpha_j)} f_{u_j}(u|X_j) du = \Delta(\alpha_j) f_{u_j}(\Delta(\alpha_j)|X_j) - \int_0^{\Delta(\alpha_j)} \frac{\partial f_{u_j}(u|x_j)}{\partial u} u du \quad (\text{D.62})$$

which together with the triangle inequality and Assumption D.1(ii) implies that for any α_j ,

$$\begin{aligned}
& \left| E_{F_n} [(I\{u_j \leq \Delta(\alpha_j)\} - I\{u_j \leq 0\}) | X_j] - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\alpha_j) \right| \\
& \leq \left| f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) - f_{u_j}(0 | X_j) \right| |\Delta(\alpha_j)| + C |\Delta(\alpha_j)|^2 \\
& \leq C \left(|\Delta(\alpha_j)|^2 + |\Delta(\alpha_j)| |\Delta(\alpha_{k_j}^*)| \right). \tag{D.63}
\end{aligned}$$

Using (D.12), (D.13), (D.58), (D.63) and the triangle inequality, we immediately get

$$\begin{aligned}
& \left| E_{Z, F_n} \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) (\hat{\alpha}_{k_j} - \alpha_{k_j}^*)^2 \right] \right| \\
& \leq \left| E_{Z, F_n} \left[\left((I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\hat{\alpha}_{k_j}) \right) (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) \right] \right| \\
& \quad + \left| E_{X_j, F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\hat{\alpha}_{k_j}) (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) (\hat{\alpha}_{k_j} - \alpha_{k_j}^*)^2 \right] \right| \\
& \leq C E_{X_j, F_n} \left[|\Delta(\hat{\alpha}_{k_j})|^2 + |\Delta(\hat{\alpha}_{k_j})| |\Delta(\alpha_{k_j}^*)| \right] + O_p \left(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right) \\
& = O_p \left(\delta_{1,j,n}^2 \delta_{2,j,n} + \delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right). \tag{D.64}
\end{aligned}$$

Combining the results in (D.61) and (D.64) and applying the triangle inequality, we immediately get the claimed result. ■

Lemma D.5 *Under Assumptions C.1 and D.1, we have*

$$(\hat{\beta}_{k_j} - \beta_{k_j}^*)' H_{F_n, k_j} (\hat{\beta}_{k_j} - \beta_{k_j}^*) = \frac{\mathbf{U}'_{k_j, n} \mathbf{P}'_{k_j, n} H_{F_n, k_j}^{-1} \mathbf{P}_{k_j, n} \mathbf{U}_{k_j, n}}{n^2} + O_p(\delta_{0,j,n}^{5/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 k_j^{-r_j}). \tag{D.65}$$

Proof of Lemma D.5. From (D.11) and (D.49), we get

$$(\hat{\beta}_{k_j} - \beta_{k_j}^*)' H_{F_n, k_j} (\hat{\beta}_{k_j} - \beta_{k_j}^*) = \frac{(\hat{\beta}_{k_j} - \beta_{k_j}^*)' \mathbf{P}_{k_j, n} \mathbf{U}_{k_j, n}}{n} + O_p \left(\delta_{0,j,n}^{5/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 k_j^{-r_j} \right). \tag{D.66}$$

Under the i.i.d. assumption, (D.46) and Assumptions C.1(iv), we have

$$\begin{aligned}
E_{F_n} \left[\frac{\mathbf{U}'_{k_j, n} \mathbf{P}'_{k_j, n} \mathbf{P}_{k_j, n} \mathbf{U}_{k_j, n}}{n^2} \right] &= E_{F_n} \left[n^{-2} \left\| \sum_{i=1}^n P_{k_j}(X_j) u_{k_j} \right\|^2 \right] \\
&= E_{F_n} \left[n^{-2} \sum_{l=1}^{k_j} \left\| \sum_{i=1}^n p_{l,j}(X_j) u_{k_j} \right\|^2 \right] \\
&\leq n^{-2} \sum_{l=1}^{k_j} \sum_{i=1}^n E_{F_n} [p_{l,j}(X_j)^2] = O(k_j n^{-1}) \tag{D.67}
\end{aligned}$$

which together with (D.45) and (D.66) immediately implies (D.65). ■

Lemma D.6 *Under Assumptions C.1 and D.1, we have*

$$\begin{aligned} n^{-1} \sum_{i=1}^n u_{k_j,i} \Delta_i(\hat{\alpha}_{k_j}) &= \frac{\mathbf{U}'_{k_j,n} \mathbf{P}'_{k_j,n} H_{F_n,k_j}^{-1} \mathbf{P}_{k_j,n} \mathbf{U}_{k_j,n}}{n^2} \\ &+ O_p(\delta_{0,j,n}^{5/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 k_j^{-r_j} + k_j^{-r_j} n^{-\frac{1}{2}} + k_j^{-2r_j}). \end{aligned} \quad (\text{D.68})$$

Proof of Lemma D.6. First, we note that

$$\begin{aligned} n^{-1} \sum_{i=1}^n u_{k_j,i} \Delta_i(\hat{\alpha}_{k_j}) &= n^{-1} \sum_{i=1}^n u_{k_j,i} (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) + n^{-1} \sum_{i=1}^n u_{j,i}^* \Delta_i(\alpha_{k_j}^*) \\ &+ n^{-1} \sum_{i=1}^n (u_{k_j,i} - u_{j,i}^*) \Delta_i(\alpha_{k_j}^*) \end{aligned} \quad (\text{D.69})$$

where $u_{j,i}^* = I\{u_{j,i} \leq 0\} - \tau$. We can write

$$n^{-1} \sum_{i=1}^n u_{k_j,i} (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) = \frac{\mathbf{U}'_{k_j,n} \mathbf{P}'_{k_j,n}}{n} (\hat{\beta}_{k_j} - \beta_{k_j}^*), \quad (\text{D.70})$$

which together with (D.49) and (D.67) implies that

$$n^{-1} \sum_{i=1}^n u_{k_j,i} (\hat{\alpha}_{k_j} - \alpha_{k_j}^*) = \frac{\mathbf{U}'_{k_j,n} \mathbf{P}'_{k_j,n} H_{F_n,k_j}^{-1} \mathbf{P}_{k_j,n} \mathbf{U}_{k_j,n}}{n^2} + O_p(\delta_{0,j,n}^{5/2} \xi_{k_j}^{1/4} + \delta_{0,j,n}^2 k_j^{-r_j}). \quad (\text{D.71})$$

By the definition of $u_{j,i}$

$$E_{F_n} [u_j^* | X_j] = 0. \quad (\text{D.72})$$

By the i.i.d. assumption, (D.72) and Assumption C.1(ii), we get

$$E_{F_n} \left[\left| n^{-1} \sum_{i=1}^n u_{j,i}^* (\alpha_{k_j}^* - \alpha_j^*) \right|^2 \right] = n^{-1} E_{F_n} \left[(u_{j,i}^*)^2 (\alpha_{k_j}^* - \alpha_j^*)^2 \right] \leq C k_j^{-2r_j} n^{-1} \quad (\text{D.73})$$

which together with the Markov inequality implies

$$n^{-1} \sum_{i=1}^n u_{j,i}^* (\alpha_{k_j}^* - \alpha_j^*) = O_p(k_j^{-r_j} n^{-\frac{1}{2}}). \quad (\text{D.74})$$

By the i.i.d. assumption, the triangle inequality and Assumption C.1(ii),

$$\begin{aligned}
& E_{F_n} \left[\left| n^{-1} \sum_{i=1}^n (u_{k_j,i} - u_{j,i}^*) \Delta_i(\alpha_{k_j}^*) \right| \right] \\
& \leq E_{F_n} \left[\left| (I\{u_j \leq 0\} - I\{u_j \leq \Delta(\alpha_{k_j}^*)\}) \Delta(\alpha_{k_j}^*) \right| \right] \\
& \leq E_{F_n} \left[\left| \Delta(\alpha_{k_j}^*) \int_0^{\Delta(\alpha_{k_j}^*)} f_{u_j}(u|X_j) du \right| \right] \leq C E_{F_n} \left[\Delta(\alpha_{k_j}^*)^2 \right] = C k_j^{-2r_j} \quad (\text{D.75})
\end{aligned}$$

which together with the Markov inequality implies

$$n^{-1} \sum_{i=1}^n (u_{k_j,i} - u_{j,i}^*) \Delta_i(\alpha_{k_j}^*) = O_p(k_j^{-2r_j}). \quad (\text{D.76})$$

Collecting the results in (D.69), (D.71), (D.74) and (D.76), we immediately get (D.68). ■

Lemma D.7 *Under Assumptions C.1 and D.1, we have*

$$\begin{aligned}
& E_{u_j, F_n} \left[(I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\} - I\{u_j \leq 0\}) \Delta(\alpha_{k_j}^*) \right] \\
& = O_p(\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} + \delta_{1,j,n}^2 k_j^{-r_j}). \quad (\text{D.77})
\end{aligned}$$

Proof of Lemma D.7. From (D.63), we see that

$$\left| E_{F_n} [I\{u_j \leq \Delta(\alpha_j)\} - I\{u_j \leq 0\} | X_j] - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\alpha_j) \right| \leq C \left(|\Delta^2(\alpha_j)| + |\Delta(\alpha_j)| |\Delta(\alpha_{k_j}^*)| \right) \quad (\text{D.78})$$

for any $\beta_{k_j} \in \mathcal{N}_{j,n}$, which combined with the triangle inequality implies that

$$\begin{aligned}
& \sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| E_{F_n} \left[\left[I\{u_j \leq \Delta(\alpha_j)\} - I\{u_j \leq 0\} - f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\alpha_j) \right] \Delta(\alpha_{k_j}^*) \right] \right| \\
& \leq \sup_{\beta_{k_j} \in \mathcal{N}_{j,n}} \left| E_{F_n} \left[(|\Delta^2(\alpha_j)| + |\Delta(\alpha_j)| |\Delta(\alpha_{k_j}^*)|) \Delta(\alpha_{k_j}^*) \right] \right| = O(\delta_{1,j,n}^2 k_j^{-r_j}). \quad (\text{D.79})
\end{aligned}$$

Invoking (D.56), we have

$$\begin{aligned}
& E_{X_j, F_n} \left[f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) \Delta(\hat{\alpha}_{k_j}) \Delta(\alpha_{k_j}^*) \right] \\
& = (\hat{\beta}_{k_j} - \beta_{k_j}^*)' E \left[f_{u_j}(\Delta(\alpha_{k_j}^*) | X_j) P_{k_j}(X_j) (\alpha_{k_j}^* - \alpha_j^*) \right] \\
& = O_p \left(\delta_{0,j,n}^{3/2} \xi_j^{1/4} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right). \quad (\text{D.80})
\end{aligned}$$

Collecting the results in (D.79) and (D.80), we immediately get (D.77). ■

Lemma D.8 Under Assumptions C.1 and D.1, we have

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n [\rho_\tau(Y_i - \hat{\alpha}_{k_1}(X_{1,i})) - \rho_\tau(Y_i - \hat{\alpha}_{k_2}(X_{2,i}))] \\
&= \frac{1}{n} \sum_{i=1}^n [\rho_\tau(u_{1,i}) - \rho_\tau(u_{2,i})] - \frac{\bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)' H_{F_n, \mathbf{k}}^{-1} \bar{\ell}_{\alpha,n}(\alpha_{\mathbf{k}}^*)}{2} + O_p(\delta_{n, \mathbf{k}})
\end{aligned} \tag{D.81}$$

where $\delta_{n, \mathbf{k}} = \max_{j=1,2} \left\{ \delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right\}$.

Proof of Lemma D.8. Using (D.15), we can write

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n [\rho_\tau(u_{j,i}) - \rho_\tau(Y - \hat{\alpha}_{k_j}(X_{j,i}))] \\
&= n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - \tau) \Delta_i(\hat{\alpha}_{k_j}) \\
&+ n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}) (u_{j,i} - \Delta_i(\hat{\alpha}_{k_j})).
\end{aligned} \tag{D.82}$$

Using (D.16) in Lemma D.1, we have

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}) u_{j,i} \\
&= E_{Z, F_n} [(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\}) u_j] + O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2}).
\end{aligned} \tag{D.83}$$

Using (D.24) in Lemma D.2, and (D.77) in Lemma D.7, we have

$$\begin{aligned}
& n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}) \Delta_i(\hat{\alpha}_{k_j}) \\
&= n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}) (\hat{\alpha}_{k_j}(X_{j,i}) - \alpha_{k_j}^*(X_{j,i})) \\
&+ n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}) \Delta_i(\alpha_{k_j}^*) \\
&= E_{Z, F_n} [(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\}) (\hat{\alpha}_{k_j}(X_j) - \alpha_{k_j}^*(X_j))] \\
&+ (\hat{\beta}_{k_j} - \beta_{k_j}^*)' \mu_n [(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\hat{\alpha}_{k_j})\}) P_{k_j}(X_j)] \\
&+ O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}),
\end{aligned} \tag{D.84}$$

where in the last equality, we use

$$\frac{\delta_{0,j,n}^2 k_j^{-r_j}}{\delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j}} = o(1) \text{ and } \frac{\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j}}{\delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j}} = O(1). \quad (\text{D.85})$$

Using (D.11), and (D.29) in Lemma D.3, we get

$$(\widehat{\beta}_{k_j} - \beta_{k_j}^*)' \mu_n [(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\}) P_{k_j}(X_j)] = O_p(\delta_{0,j,n}^2 \delta_{2,j,n}^{1/2}), \quad (\text{D.86})$$

which together with (D.84) implies that

$$\begin{aligned} & n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\widehat{\alpha}_{k_j})\}) \Delta_i(\widehat{\alpha}_{k_j}) \\ &= E_Z \left[(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\widehat{\alpha}_{k_j,n})\}) (\widehat{\alpha}_{k_j,n} - \alpha_{k_j,n}^*) \right] \\ &+ O_p(\delta_{0,j,n}^2 \delta_{2,j,n}^{1/2} + \delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}). \end{aligned} \quad (\text{D.87})$$

Combining the results in (D.83) and (D.87), we get

$$\begin{aligned} & n^{-1} \sum_{i=1}^n (I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\widehat{\alpha}_{k_j})\}) (u_{j,i} - \Delta_i(\widehat{\alpha}_{k_j})) \\ &= E_{Z, F_n} \left[(I\{u_j \leq 0\} - I\{u_j \leq \Delta(\widehat{\alpha}_{k_j})\}) (u_j - (\widehat{\alpha}_{k_j}(X_j) - \alpha_{k_j}^*(X_j))) \right] \\ &+ O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + \delta_{0,j,n} k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}), \end{aligned} \quad (\text{D.88})$$

where we use

$$\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} = \delta_{0,j,n}^2 \delta_{2,j,n}^{1/2} + \delta_{0,j,n} \delta_{2,j,n}^{1/2} k_j^{-r_j} \text{ and } \frac{\delta_{0,j,n} \delta_{1,j,n}^{1/2} k_j^{-r_j}}{\delta_{0,j,n} \delta_{2,j,n}^{1/2} k_j^{-r_j}} = \frac{\delta_{1,j,n}^{1/2}}{\delta_{2,j,n}^{1/2}} < 1. \quad (\text{D.89})$$

Using (D.37) in Lemma D.4, and (D.88), we get

$$\begin{aligned} & n^{-1} \sum_{i=1}^n [I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\widehat{\alpha}_{k_j})\}] [u_{j,i} - \Delta_i(\widehat{\alpha}_{k_j})] \\ &= -\frac{(\widehat{\beta}_{k_j} - \beta_{k_j}^*)' H_{F_n, k_j} (\widehat{\beta}_{k_j} - \beta_{k_j}^*)}{2} + O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}), \end{aligned} \quad (\text{D.90})$$

where we use $\delta_{1,j,n}^2 \delta_{2,j,n} \leq 2\delta_{0,j,n}^2 \delta_{2,j,n} + 2\delta_{2,j,n} k_j^{-2r_j} = O(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2}) + O(k_j^{-2r_j})$ and

$$\frac{\delta_{0,j,n}^{3/2} \xi_{k_j}^{1/4} k_j^{-r_j}}{\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2}} = \frac{\delta_{0,j,n}^{1/2} \xi_{k_j}^{1/4}}{\delta_{2,j,n}^{1/2}} \frac{k_j^{-r_j}}{\delta_{0,j,n} + k_j^{-r_j}} < 1. \quad (\text{D.91})$$

Using (D.65) in Lemma D.5 and (D.90), we get

$$\begin{aligned} & n^{-1} \sum_{i=1}^n [I\{u_{j,i} \leq 0\} - I\{u_{j,i} \leq \Delta_i(\hat{\alpha}_{k_j})\}] [u_{j,i} - \Delta_i(\hat{\alpha}_{k_j})] \\ &= -\frac{\mathbf{U}'_{k_j,n} \mathbf{P}'_{k_j,n} H_{F_n,k_j}^{-1} \mathbf{P}_{k_j,n} \mathbf{U}_{k_j,n}}{2n^2} + O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}), \end{aligned} \quad (\text{D.92})$$

where we use

$$\delta_{0,j,n}^{5/2} \xi_{k_j}^{1/4} < \delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} \quad \text{and} \quad \delta_{0,j,n}^2 k_j^{-r_j} < \delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2}. \quad (\text{D.93})$$

Combining the results in (D.68) of Lemma D.6, (D.82) and (D.92), we get

$$\begin{aligned} & n^{-1} \sum_{i=1}^n [\rho_\tau(u_{j,i}) - \rho_\tau(Y - \hat{\alpha}_{k_j}(X_{j,i}))] \\ &= \frac{\mathbf{U}'_{k_j,n} \mathbf{P}'_{k_j,n} H_{F_n,k_j}^{-1} \mathbf{P}_{k_j,n} \mathbf{U}_{k_j,n}}{2n^2} + O_p(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j}). \end{aligned} \quad (\text{D.94})$$

Using (D.94), and the definitions of $\mathbf{U}_{k_j,n}$, $H_{F_n,k}$ and $\ell_{\alpha,k}(Z; \alpha)$, we immediately get the result in (D.81). ■

Proof of Theorem D.1. We can use Theorem 4.1 to prove the claim. For this purpose, we need to verify Assumptions 4.1 and 4.3. Assumptions 4.1(a) is implied by Assumption C.1(i).

Using (D.14), we can write

$$m(Z; \alpha_j) = (I\{u_j \leq \Delta(\alpha_j)\} - \tau) [u_j - \Delta(\alpha_j)] \quad (\text{D.95})$$

which implies that

$$\begin{aligned} & E_{F_0}[m(Z; \alpha_j) | X_j] \\ &= E_{F_0}[I\{u_j \leq \Delta(\alpha_j)\} [u_j - \Delta(\alpha_j)] | X_j] - \tau E_{F_0}[u_j - \Delta(\alpha_j) | X_j] \\ &= \int_{-\infty}^{\Delta(\alpha_j)} u f_{u_j}(u | X_j) du - \Delta(\alpha_j) [F_{u_j}(\Delta(\alpha_j) | X_j) - \tau] - \tau E_{F_0}[u_j | X_j] \end{aligned} \quad (\text{D.96})$$

where $F_{u_j}(\cdot|X_j)$ denotes the conditional CDF of u_j given X_j . Define

$$g_j(X_j, \alpha_j) = \int_{-\infty}^{\Delta(\alpha_j)} u_j f_{u_j}(u_j|X_j) du_j - \Delta(\alpha_j) [F_{u_j}(\Delta(\alpha_j)|X_j) - \tau]. \quad (\text{D.97})$$

Since $\alpha_j(\cdot) = P_{k_j}(\cdot)' \beta_{k_j}$, it is clear that $g_j(x_j, \alpha_j)$ is continuously differentiable at β_{k_j} for any x_j with

$$\frac{\partial g_j(x_j, P_{k_j}(x_j)' \beta_{k_j})}{\partial \beta_{k_j}} = [\tau - F_{u_j}(\Delta(\alpha_j)|x_j)] P_{k_j}(x_j) \quad (\text{D.98})$$

and uniformly over x_j and β_{k_j} ,

$$\left\| \frac{\partial g_j(x_j, P_{k_j}(x_j)' \beta_{k_j})}{\partial \beta_{k_j}} \right\| \leq \xi_{k_j}^{1/2} < \infty. \quad (\text{D.99})$$

We can use the dominated convergence theorem to show that the

$$\frac{\partial E_{F_0}[m(Z; \alpha_j)]}{\partial \beta_{k_j}} = E_{F_0} \left[\frac{\partial g_j(X_j, \alpha_j)}{\partial \beta_{k_j}} \right]. \quad (\text{D.100})$$

Moreover, $\partial g_j(x_j, \alpha_j)/\partial \beta_{k_j}$ is continuously differentiable at β_{k_j} for any x_j with

$$\frac{\partial^2 g_j(x_j, \alpha_j)}{\partial \beta_{k_j} \partial \beta_{k_j}'} = -f_{u_j}(\Delta(\alpha_j)|X_j) P_{k_j}(X_j) P_{k_j}(X_j)'$$

By Assumptions C.1(iv) and D.1(i), we have uniformly over x_j and β_{k_j} ,

$$\left\| \frac{\partial^2 g_j(x_j, P_{k_j}(x_j)' \beta_{k_j})}{\partial \beta_{k_j} \partial \beta_{k_j}'} \right\| \leq C \xi_{k_j}^2 < \infty. \quad (\text{D.101})$$

Hence we can use the dominated convergence theorem again to show that the

$$\frac{\partial^2 E_{F_0}[m(Z; \alpha_j)]}{\partial \beta_{k_j} \partial \beta_{k_j}'} = E_{F_0} \left[\frac{\partial^2 g_j(X_j, \alpha_j)}{\partial \beta_{k_j} \partial \beta_{k_j}'} \right] \quad (\text{D.102})$$

is well defined. This verifies Assumption 4.1(b). By (D.98) and (D.100), and the first order condition of $\beta_{k_j}^*$,

$$E_{F_0} \left[\frac{\partial g_j(X_j, \alpha_{k_j}^*)}{\partial \beta_{k_j}} \right] = E_{F_0} \left[(\tau - I\{u_j \leq \Delta(\alpha_{k_j}^*)\}) P_{k_j}(x_j) \right] = \mathbf{0}_{k_j \times 1}. \quad (\text{D.103})$$

By the definitions of $\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)$ and (D.103),

$$E_{F_0} [\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)] = E_{F_0} \left[\begin{pmatrix} (I\{u_1 \leq \Delta(\alpha_{k_1}^*)\} - \tau) P_{k_1}(X_1) \\ (\tau - I\{u_2 \leq \Delta(\alpha_{k_2}^*)\}) P_{k_2}(X_2) \end{pmatrix} \right] = \mathbf{0}_{|\mathbf{k}|} \quad (\text{D.104})$$

which verifies Assumption 4.1(c). By Assumption D.1(vi),

$$E_{F_0} [\ell(Z; \alpha_{F_0}^*)^2] = E_{F_0} [(\rho_\tau(u_1) - \rho_\tau(u_2))^2] \leq 2E_{F_0} [u_1^2] + 2E_{F_0} [u_2^2] \leq C. \quad (\text{D.105})$$

Moreover, by Assumptions C.1(ii) and C.1(iii),

$$\begin{aligned} E_{F_0} [\|\ell_{\alpha, \mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)\|^4] &\leq 2E_{F_0} \left[|P'_{k_1}(X_1) P_{k_1}(X_1)|^2 + |P'_{k_2}(X_2) P_{k_2}(X_2)|^2 \right] \\ &\leq C \left[\begin{array}{l} \xi_{k_1} \text{tr} (E_{F_0} [P_{k_1}(X_1)' P_{k_1}(X_1)]) \\ + \xi_{k_2} \text{tr} (E_{F_0} [P_{k_2}(X_2)' P_{k_2}(X_2)]) \end{array} \right] \\ &\leq C(\xi_{k_1} + \xi_{k_2})(k_1 + k_2) = C\xi_{\mathbf{k}} |k|, \end{aligned} \quad (\text{D.106})$$

where $\xi_{\mathbf{k}} = \xi_{k_1} + \xi_{k_2}$. Hence Assumption 4.1(d) is also satisfied. Assumptions 4.1(e) is Assumption D.1(v). Also the bounds on the eigenvalues of $H_{F_0, \mathbf{k}}$ in Assumption 4.1(f) can be easily verified using Assumptions C.1(iv), D.1(i) and D.1(iii). By definition,

$$D_{F_0, \mathbf{k}} = E_{F_0} \left[\begin{pmatrix} u_{k_1}^2 P_{k_1}(X_1) P_{k_1}(X_1)' & -u_{k_1} u_{k_2} P_{k_1}(X_1) P_{k_2}(X_2)' \\ -u_{k_1} u_{k_2} P_{k_2}(X_2) P_{k_1}(X_1)' & u_{k_2}^2 P_{k_2}(X_2) P_{k_2}(X_2)' \end{pmatrix} \right]. \quad (\text{D.107})$$

Under Assumption C.1(iv), we see that

$$\rho_{\max} \left(E_{F_0} \left[u_{k_j}^2 P_{k_j}(X_j) P_{k_j}(X_j)' \right] \right) \leq C \rho_{\max} \left(E_{F_0} [P_{k_j}(X_j) P_{k_j}(X_j)'] \right) \leq C \text{ for } j = 1, 2 \quad (\text{D.108})$$

which together with the form of $D_{F_0, \mathbf{k}}$ in (D.107) and the Aronszajn's Inequality implies that $\rho_{\max}(D_{F_0, \mathbf{k}}) \leq 2C$. This verifies Assumption 4.1(f).

In Lemma D.8, we have derived the second order expansion of $\bar{\ell}_n(\hat{\alpha}_{\mathbf{k}})$, where the remainder term is of the order

$$\delta_{n, \mathbf{k}} = \max_{j=1,2} \left\{ \delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right\}. \quad (\text{D.109})$$

By (D.9), we have

$$\frac{n^{1/2} \delta_{n, \mathbf{k}}}{\sigma_{F_n, n}} = \frac{n^{1/2} \left(\delta_{0,j,n} \delta_{1,j,n} \delta_{2,j,n}^{1/2} + k_j^{-2r_j} + n^{-1/2} k_j^{-r_j} \right)}{\sigma_{F_n, n}}$$

$$= \frac{n^{1/2}\delta_{0,j,n}\delta_{1,j,n}\delta_{2,j,n}^{1/2}}{\sigma_{F_n,n}} + \frac{k_j^{-r_j}n^{1/2}}{k_j^{r_j}\sigma_{F_n,n}} + \frac{1}{k_j^{r_j}\sigma_{F_n,n}} = o(1) \quad (\text{D.110})$$

for $j = 1, 2$. As $\xi_{k_j}k_j \log(n)n^{-1} = o(1)$ by assumption Assumption D.1(iv), we have

$$\begin{aligned} \frac{k_j^{1/2}\xi_{k_j}^{1/2}}{n\sigma_{F_n,n}} \frac{\sigma_{F_n,n}}{n^{1/2}\delta_{0,j,n}\delta_{1,j,n}\delta_{2,j,n}^{1/2}} &= \frac{\xi_{k_j}^{1/2}}{n\delta_{1,j,n}(\delta_{2,j,n}\log(n))^{1/2}} \\ &\leq \frac{\xi_{k_j}^{1/4}}{n^{1/4}k_j^{3/4}(\log(n))^{5/4}} = o(1) \end{aligned} \quad (\text{D.111})$$

which together with (D.9) implies that $k_j\xi_{k_j}(n^2\sigma_{F_n,n}^2)^{-1} = o(1)$. Assumption 4.3(b) is implied by (D.9). So Assumption 4.3 is also satisfied. ■

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left\| \widehat{C}_n - C_n \right\| &\leq \left\| \frac{1}{n} \sum_{i=1}^n \ell(Z_i; \alpha_{k_j}^*) \ell_{\alpha_{k_j}}(Z_i; \alpha_{k_j}^*) - H_{\ell,k_j,n} \right\| \sqrt{\bar{\ell}_{\alpha_j,n}(\alpha_{k_j}^*) H_{F_n,k_j}^{-1} \bar{\ell}_{\alpha_j,n}(\alpha_{k_j}^*)} \\ &= o_p(n^{-\frac{1}{2}}), \end{aligned} \quad (\text{D.112})$$

where the equality is by (B.40) and Assumption B.2(d). Under the i.i.d. assumption,

$$E_{F_n} [\|C_n\|^2] = \text{tr} (n^{-1} H_{\ell,k_j,n} H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1} H_{\ell,k_j,n}) = O(|\mathbf{k}| n^{-1}), \quad (\text{D.113})$$

where the last equality is by Assumptions 4.1(f) and B.2(b). Thus,

$$C_n = O_p(|\mathbf{k}| n^{-1}). \quad (\text{D.114})$$

Under the i.i.d. Assumption,

$$\begin{aligned} E_{F_n} [|A_n B_n|^2] &= \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)' H_{F_n,k_j}^{-1}}{v_{\psi,k_j}^* \sigma_{F_n,n}} E_{F_n} \left[\ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*)' \right] \frac{H_{F_n,\mathbf{k}}^{-1} D_{F_n,\mathbf{k}} H_{F_n,\mathbf{k}}^{-1}}{n} \\ &\quad \times E_{F_n} \left[\ell_{\alpha,\mathbf{k}}(Z; \alpha_{\mathbf{k}}^*) \ell_{\alpha,k_j}(Z; \alpha_{k_j}^*) \right] \frac{H_{F_n,k_j}^{-1} \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^* \sigma_{F_n,n}} \\ &\leq \frac{C \rho_{\max}(D_{F_n,\mathbf{k}})}{\rho_{\min}^4(H_{F_n,\mathbf{k}})} \frac{\psi_{\alpha,k_j}(\alpha_{k_j}^*)' \psi_{\alpha,k_j}(\alpha_{k_j}^*)}{v_{\psi,k_j}^{*2}} \frac{1}{n\sigma_{F_n,n}^2} = o(1) \end{aligned} \quad (\text{D.115})$$

where the inequality is by (B.42), the last equality is by Assumptions 4.1(f), 4.3(b) and B.2(e).

Similarly,

$$\begin{aligned}
E_{F_n} [|A_n C_n|^2] &= \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' H_{F_n, k_j}^{-1}}{v_{\psi, k_j}^* \sigma_{F_n, n}} H_{\ell, k_j, n} H_{F_n, k_j}^{-1} \frac{H_{F_n, \mathbf{k}}^{-1} D_{F_n, \mathbf{k}} H_{F_n, \mathbf{k}}^{-1}}{n} H_{F_n, k_j}^{-1} H_{\ell, k_j, n} \frac{H_{F_n, k_j}^{-1} \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^* \sigma_{F_n, n}} \\
&\leq \frac{\rho_{\max}((H_{\ell, k_j, n})^2) \rho_{\max}(D_{F_n, \mathbf{k}})}{\rho_{\min}^2((H_{F_n, \mathbf{k}})^2)} \frac{\psi_{\alpha, k_j}(\alpha_{k_j}^*)' \psi_{\alpha, k_j}(\alpha_{k_j}^*)}{v_{\psi, k_j}^{*2}} \frac{1}{n \sigma_{F_n, n}^2} = o(1), \tag{D.116}
\end{aligned}$$

where the last equality is by Assumptions 4.1(f), 4.3(b), B.2(b) and B.2(e). Therefore

$$\begin{aligned}
&\widehat{A}_n(\widehat{B}_n + \widehat{C}_n + o_p(\sigma_{F_n, n})) \\
&= \widehat{A}_n(\widehat{B}_n + \widehat{C}_n) + o_p(1) \\
&= \widehat{A}_n(\widehat{B}_n - B_n + \widehat{C}_n - C_n) + A_n B_n + A_n C_n + (\widehat{A}_n - A_n)(B_n + C_n) + o_p(1) \\
&= O_p(\sigma_{F_n, n}^{-1}) o_p(n^{-1/2}) + o_p(1) + o_p(|\mathbf{k}|^{-1/2} \sigma_{F_n, n}^{-1}) o_p(|\mathbf{k}|^{1/2} n^{-1/2}) + o_p(1) \\
&= o_p(1). \tag{D.117}
\end{aligned}$$

Thus, the last summand in (B.33) is $o_p(1)$. This completes the proof of (B.32) and hence claim (c) of the lemma. ■

E Auxiliary Lemmas

Lemma E.1 *Let X be a scalar random variable with variance σ_X^2 and $Y = (Y_1, \dots, Y_d)'$ be a d -dimensional random vector with variance-covariance matrix D_Y . Let ρ be the vector correlation coefficient of X and Y . That is, $\rho = \sigma_X^+(D_Y^{1/2})^+ \text{Cov}(X, Y)$, where $A^{1/2}$ is the unique symmetric matrix square root of the positive semi-definite matrix A and A^+ is the Moore-Penrose inverse of A . Then (a) $\rho' \rho \leq 1$; (b) for any positive semi-definite matrix A , $\rho' A \rho \leq \lambda_{\max}(A)$, where $\lambda_{\max}(A)$ is the maximum eigenvalue of A ; and (c) $((D_Y^{1/2})^+ \text{Cov}(X, Y))' D_Y^{1/2} = \text{Cov}(X, Y)'$.*

Proof of Lemma E.1. First we show part (a). If $\sigma_X = 0$, then $\rho = 0$ by definition and part (a) holds. Thus below, we consider the nontrivial case of $\sigma_X > 0$. Without loss of generality, normalize $\sigma_X = 1$.

Consider the simple case that D_Y is invertible first. Then,

$$\begin{pmatrix} 1 & \rho' \\ \rho & I_d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D_Y^{-1/2} \end{pmatrix} \Sigma_{(X, Y)'} \begin{pmatrix} 1 & 0 \\ 0 & D_Y^{-1/2} \end{pmatrix}, \tag{E.1}$$

where $\Sigma_{(X, Y)'}$ is the variance covariance matrix of the random vector $(X, Y)'$. Due to the positive

(semi-)definiteness of D_Y and $\Sigma_{(X,Y)'}$, we have $\begin{pmatrix} 1 & \rho' \\ \rho & I_d \end{pmatrix}$ is positive semi-definite. That implies that

$$\left| \begin{pmatrix} 1 & \rho' \\ \rho & I_d \end{pmatrix} \right| \geq 0. \quad (\text{E.2})$$

Direct calculation shows that $\left| \begin{pmatrix} 1 & \rho' \\ \rho & I_d \end{pmatrix} \right| = 1 - \rho'\rho$. Thus, $\rho'\rho \leq 1$.

Now consider the simple noninvertible case: D_Y is a singular diagonal matrix. Without loss of generality, suppose that the first J elements of D_Y are zeros for a positive integer $J \leq d$ and the rest are strictly positive. If $J = d$, then $\rho = 0$ and part (a) holds trivially. If $J < d$, we have $\rho_j = 0$ for $j = 1, \dots, J$, and

$$\rho_{(J+1):d} = D_{Y,(J+1):d}^{-1/2} \text{Cov}(X, Y_{(J+1):d}), \quad (\text{E.3})$$

where $\rho_{(J+1):d} = (\rho_{J+1}, \dots, \rho_d)$, $Y_{(J+1):d} = (Y_{J+1}, \dots, Y_d)$ and $D_{Y,(J+1):d}$ is D_Y with the first J rows and columns removed. By the arguments for the invertible D_Y case, we have $\rho'_{(J+1):d} \rho_{(J+1):d} \leq 1$. Thus $\rho'\rho = \rho'_{(J+1):d} \rho_{(J+1):d} \leq 1$.

Finally, consider the case where D_Y is singular but not diagonal. Because D_Y is a variance-covariance matrix and thus is positive semi-definite, D_Y have the following eigenvalue decomposition:

$$D_Y = Q_Y \Lambda_Y Q_Y', \quad (\text{E.4})$$

where Q_Y is an orthonormal matrix and Λ_Y is a diagonal matrix whose diagonal elements are eigenvalues of D_Y . Using this decomposition, we have

$$\rho = Q_Y (\Lambda^{1/2})^+ Q_Y' \text{Cov}(X, Y). \quad (\text{E.5})$$

Thus,

$$Q_Y' \rho = (\Lambda^{1/2})^+ \text{Cov}(X, Q_Y' Y). \quad (\text{E.6})$$

Then by the arguments for the singular diagonal D_Y , we have $\rho' Q_Y Q_Y' \rho \leq 1$. But because $Q_Y Q_Y' = I_d$, we have $\rho'\rho = \rho' Q_Y Q_Y' \rho \leq 1$. This concludes the proof of part (a).

Now we show part (b). The matrix A has the following decomposition:

$$A = Q_A \Lambda_A Q_A', \quad (\text{E.7})$$

where Q_A is a orthonormal matrix, and Λ_A is a diagonal matrix of eigenvalues of A . Thus,

$$\begin{aligned}\rho' A \rho &= \rho' Q_A \Lambda_A Q_A' \rho = \rho' Q_A \lambda_{\max}(A) I_d Q_A' \rho + \rho' Q_A (\lambda_{\max}(A) I_d - \Lambda_A) Q_A' \rho \\ &\leq \rho' Q_A \lambda_{\max}(A) I_d Q_A' \rho = \lambda_{\max} \rho' \rho \leq \lambda_{\max}(A).\end{aligned}\tag{E.8}$$

This shows part (b).

Finally we show part (c). Without loss of generality, suppose that D_Y has J many zero eigenvalues which corresponds to the first J diagonal elements in Λ . The claim of the lemma holds trivially when $J = 0$ or d . Hence we only need to consider the case that $0 < J < d$. By (E.4), $D_Y^{1/2} = Q_Y \Lambda_Y^{1/2} Q_Y'$ and $(D_Y^{1/2})^+ = Q_Y (\Lambda_Y^{1/2})^+ Q_Y'$ where $(\Lambda_Y^{1/2})^+$ is a symmetric matrix. Let $Y^* = Q_Y' Y$, then the first J element of Y^* is zero almost surely. Let Y_{d-J}^* denotes the last $d - J$ elements of Y^* . Then

$$\begin{aligned}((D_Y^{1/2})^+ \text{Cov}(X, Y))' D_Y^{1/2} Q_Y &= \text{Cov}(X, Y)' (D_Y^{1/2})^+ D_Y^{1/2} Q_Y \\ &= \text{Cov}(X, Y^*)' \begin{pmatrix} 0_{J \times J} & 0_{J \times (d-J)} \\ 0_{(d-J) \times J} & I_{d-J} \end{pmatrix} \\ &= \begin{pmatrix} 0_{1 \times J} & \text{Cov}(X, Y_{d-J}^*)' \end{pmatrix}.\end{aligned}\tag{E.9}$$

By definition $\text{Cov}(X, Y)' Q_Y = \text{Cov}(X, Y^*)' = \begin{pmatrix} 0_{1 \times J} & \text{Cov}(X, Y_{d-J}^*)' \end{pmatrix}$ which together with the above equation implies that

$$((D_Y^{1/2})^+ \text{Cov}(X, Y))' D_Y^{1/2} Q_Y = \text{Cov}(X, Y)' Q_Y.\tag{E.10}$$

As Q_Y is a non-singular matrix, the claim of the lemma follows immediately by (E.10). ■

Lemma E.2 Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{12} = A_{21}'$, A_{11} and A_{22} are $k_1 \times k_1$ and $k_2 \times k_2$ symmetric matrices respectively. Then $\rho_{\max}(A) \leq C$ implies that:

- (a) $\rho_{\max}(A_{11}^2 + A_{12} A_{21}) \leq C^2$ and $\rho_{\max}(A_{22}^2 + A_{21} A_{12}) \leq C^2$;
- (b) $\rho_{\max}(A_{12} A_{21}) \leq C^2$ and $\rho_{\max}(A_{21} A_{12}) \leq C^2$.

Proof of Lemma E.2. By definition,

$$A^2 = \begin{pmatrix} A_{11}^2 + A_{12} A_{21} & A_{11} A_{12} + A_{12} A_{22} \\ A_{21} A_{11} + A_{22} A_{21} & A_{22}^2 + A_{21} A_{12} \end{pmatrix}.$$

Then it is clear that

$$\rho_{\max}(A_{11}^2 + A_{12} A_{21}) \leq \rho_{\max}(A^2) = \rho_{\max}^2(A),$$

which proves the first claim in (a). The second claim in (a) can be proved by the same way. Let $\gamma_{k_1} \in R^{k_1}$ be the eigenvector of the largest eigenvalue of $A_{12}A_{21}$. Then the above inequality implies that

$$\gamma'_{k_1} A_{11}^2 \gamma_{k_1} + \rho_{\max}(A_{12}A_{21}) \leq \rho_{\max}^2(A)$$

which immediately shows that $\rho_{\max}(A_{12}A_{21}) \leq \rho_{\max}^2(A) \leq C^2$. The second result in (b) can be proved similarly. ■

F Extra Simulation Studies

In this section we report some simulation results on the finite sample performance of our model selection test when applied on the comparison of two parametric models. We also compare the performance of our test with that proposed in Shi (2015b).

F.1 Simulation design 1

Consider two linear regression models

$$\begin{aligned} \mathcal{M}_1 & : Y = \beta_{1,0} + X_1\beta_{1,1} + X_2\beta_2 + u_1, \\ \mathcal{M}_2 & : Y = \beta_{2,0} + X_1\beta_{2,1} + \sum_{k=1}^K X_{3,k}\beta_{3,k} + u_2. \end{aligned}$$

The latent DGP (denoted as S1) is

$$Y = 0.5 + 0.5X_1 + X_2a + \sum_{k=1}^K X_{3,k}b + \varepsilon \quad (\text{F.1})$$

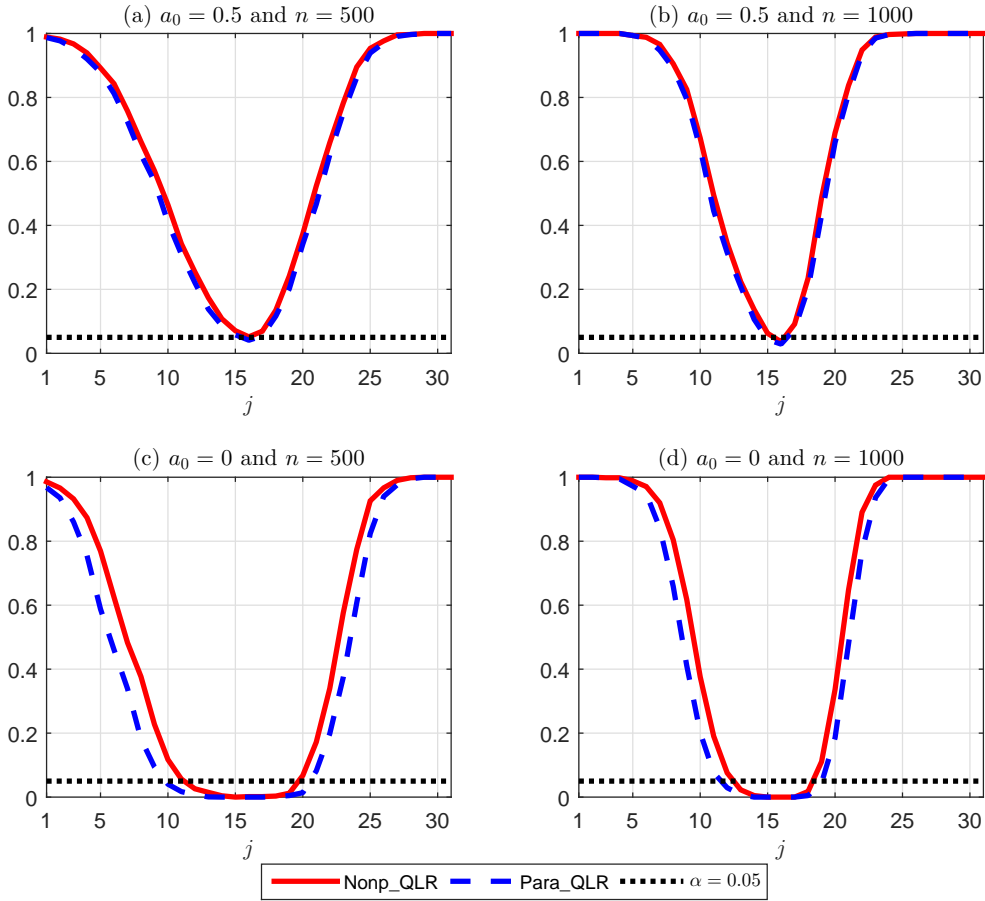
where $(X_1, X_2, X_{3,1}, \dots, X_{3,K}, \varepsilon)$ is a standard normal random vector. Thus,

$$\begin{aligned} -2f(\mathcal{M}_1, F_0) &= E_{F_0}[u_1^2] = Kb^2 + 1; \\ -2f(\mathcal{M}_2, F_0) &= E_{F_0}[u_2^2] = a^2 + 1. \end{aligned} \quad (\text{F.2})$$

Therefore, the null hypothesis holds if and only if $a^2 = Kb^2$, and when $a^2 > Kb^2$, $f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0)$. When $a^2 = Kb^2 = 0$, $u_1 = u_2$ and hence $\omega_{F_0,*}^2 = 0$. Otherwise, $\omega_{F_0,*}^2 > 0$. There are 31 DGPs considered in this design, which are determined by different combinations of a and b :

$$\left(\begin{array}{c} a \\ b \end{array} \right)_{j=1,\dots,31} = \left\{ \left(\begin{array}{c} a_0 + \frac{16-s}{50} \\ b_0 \end{array} \right)_{s=1,\dots,15}, \left(\begin{array}{c} a_0 \\ b_0 \end{array} \right), \left(\begin{array}{c} a_0 \\ b_0 + \frac{s}{50} \end{array} \right)_{s=1,\dots,15} \right\} \quad (\text{F.3})$$

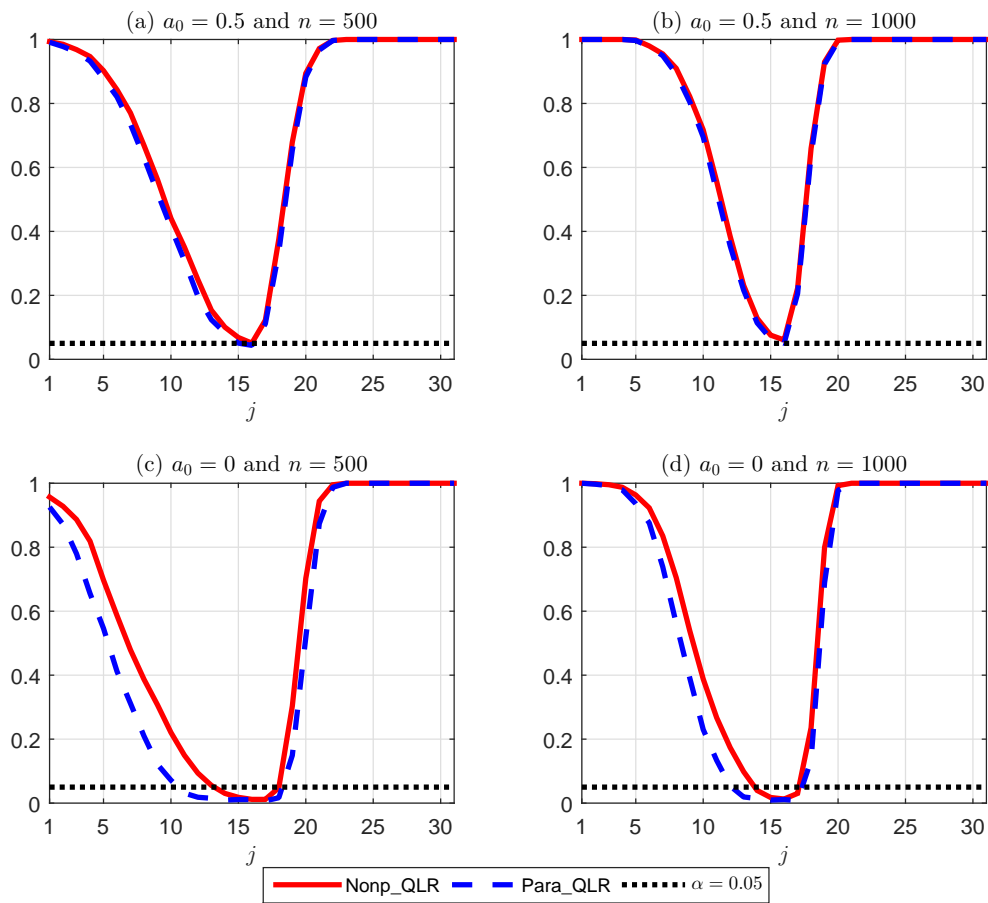
Figure 6: Finite Sample Rejection Rates of the Tests ($K = 2$)



where $a_0^2 = Kb_0^2$ and we consider two possible values for a_0 in this design: $a_0 = 0$ or 0.5 . The null hypothesis H_0 holds when $(a, b) = (a_0, b_0)$. Model \mathcal{M}_1 is better than model \mathcal{M}_2 under the first 15 DGPs in (F.3), while model the model \mathcal{M}_2 is better under the last 15 DGPs. The finite sample rejection rates of the tests are calculated using 10000 simulated samples.

Figure 6 presents the finite sample rejection rates of the tests when $K = 2$. When $a_0 = 0.5$, $\omega_{F_0,*}^2 > 0$ and our nondegenerate test statistic has asymptotic standard normal distribution regardless the value of K . From graphs (a) and (b), we can see that the finite sample rejection rates of our test (the red solid line) and the parametric test proposed in Shi (2015b) (the blue dashed line) are very close to the nominal level 5% (H_0 holds at $j = 16$). When the sample size is 500, both tests have good power which is further improved with larger sample size 1000. When $a_0 = 0$, $\omega_{F_0,*}^2 = 0$ and the standard normal distribution may not be a good approximation of the finite sample distribution of our test statistic since K is small here. In this case (graphs (c) and (d)), both our test and the parametric test proposed in Shi (2015b) under-reject. The finite sample rejection rates of both tests are close to zero under the null. The power of the tests becomes worse

Figure 7: Finite Sample Rejection Rates of the Tests ($K = 8$)



for both tests when $a_0 = 0$, although our nonparametric test has better power. Increasing the sample size from 500 to 1000 improves the power of both tests.

The finite sample rejection rates of the tests when $K = 8$ are included in Figure 7. In graphs (a) and (b), we observe similar properties of both tests to the case with small K ($K = 2$), which is expected since in both cases $\omega_{F_0,*}^2 > 0$ and our nondegenerate test statistic has asymptotic standard normal distribution. When $a_0 = 0$, $\omega_{F_0,*}^2 = 0$ and both tests are still under-rejection under the null. However the null rejection rates of both tests are closer to the nominal level with larger K . In graphs (c) and (d), we see that our nondegenerate test has larger power particularly when model \mathcal{M}_1 is better than model \mathcal{M}_2 (i.e., $j = 1, \dots, 15$). The power of these tests are similar when model \mathcal{M}_2 is better than model \mathcal{M}_1 (i.e., $j = 17, \dots, 31$). Increasing the sample size from 500 to 1000 improves the power of both tests.

F.2 Simulation design 2

In the second simulation design, the latent DGP is

$$Y = 0.5 + 0.25 \sum_{k=1}^K X_{1,k} + X_2 a + X_3 b + \varepsilon, \quad (\text{F.4})$$

where $(X_{1,1}, \dots, X_{1,K}, X_2, X_3, \varepsilon)$ is a standard normal random vector. There are two linear regression models

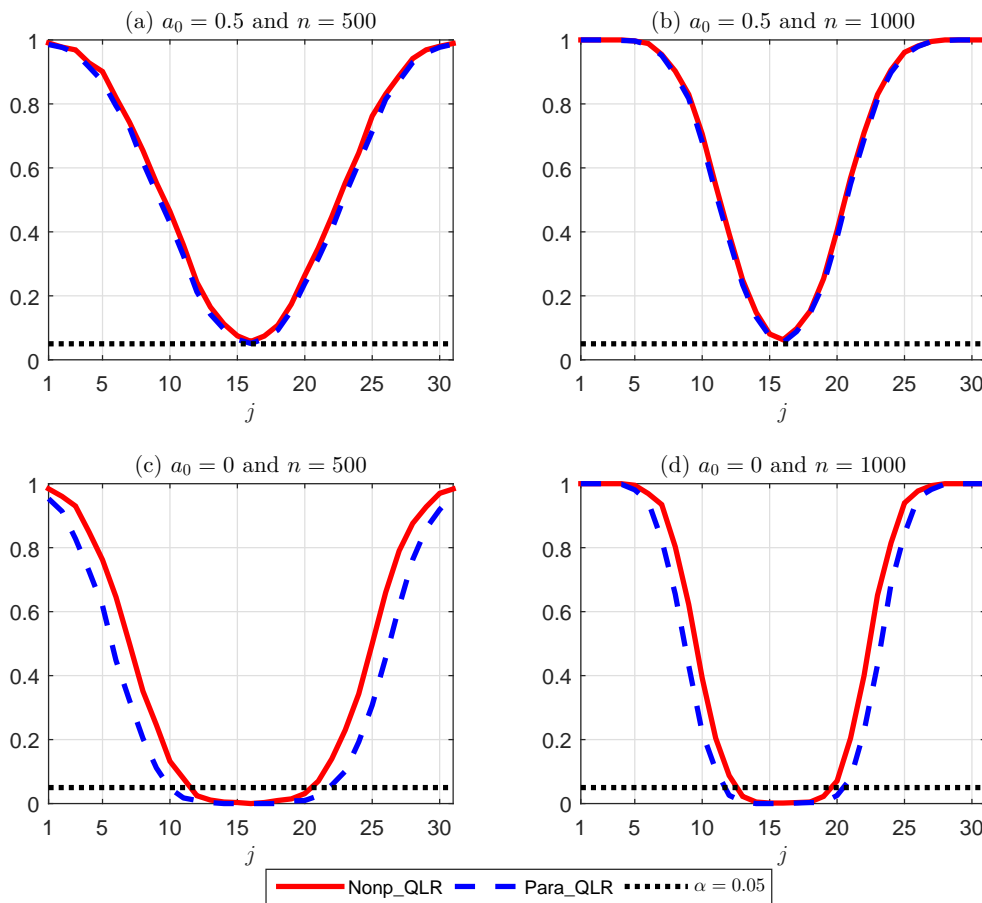
$$\begin{aligned} \mathcal{M}_1 & : Y = \beta_{1,0} + \sum_{k=1}^K X_1 \beta_{1,1,k} + X_2 \beta_2 + u_1, \\ \mathcal{M}_2 & : Y = \beta_{2,0} + \sum_{k=1}^K X_1 \beta_{2,1,k} + X_3 \beta_3 + u_2. \end{aligned}$$

Thus

$$\begin{aligned} -2f(\mathcal{M}_1, F_0) &= E_{F_0}[u_1^2] = b^2 + 1; \\ -2f(\mathcal{M}_2, F_0) &= E_{F_0}[u_2^2] = a^2 + 1. \end{aligned} \quad (\text{F.5})$$

Therefore, the null hypothesis holds if and only if $a^2 = b^2$, and when $a^2 > b^2$, $f(\mathcal{M}_1, F_0) > f(\mathcal{M}_2, F_0)$. When $a^2 = b^2 = 0$, $u_1 = u_2$ and hence $\omega_{F_0,*}^2 = 0$. Otherwise, $\omega_{F_0,*}^2 > 0$. There are 31

Figure 8: Finite Sample Rejection Rates of the Tests



DGPs considered in this design, which are determined by different combinations of a and b :

$$\begin{pmatrix} a \\ b \end{pmatrix}_{j=1,\dots,31} = \left\{ \begin{pmatrix} a_0 + \frac{16-s}{50} \\ b_0 \end{pmatrix}_{s=1,\dots,15}, \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \begin{pmatrix} a_0 \\ b_0 + \frac{s}{50} \end{pmatrix}_{s=1,\dots,15} \right\} \quad (\text{F.6})$$

where $a_0^2 = b_0^2$ and we consider two possible values for a_0 in this design: $a_0 = 0$ or 0.5 . The null hypothesis H_0 holds when $(a, b) = (a_0, b_0)$. The model \mathcal{M}_1 is better than the model \mathcal{M}_2 under the first 15 DGPs in (F.6), while model the model \mathcal{M}_2 is better under the last 15 DGPs. The finite sample rejection rates of the tests are calculated using 10000 simulated samples.

Figure 8 presents the finite sample rejection rates of the tests when $K = 10$. When $a_0 = 0.5$, $\omega_{F_0,*}^2 > 0$ and the null rejection rates of both tests are very close to the nominal level 5% (H_0 holds at $j = 16$) and their rejection rates are almost the same under the alternative, which is similar to what we have observed in the first simulation design. When $a_0 = 0$, $\omega_{F_0,*}^2 = 0$ and we see that both tests under-reject under the null. Since the two models \mathcal{M}_1 and \mathcal{M}_2 have the same

dimensions, the rejections rates of both tests are symmetric around $j = 16$. Our nonparametric test has better power than the parametric test proposed in Shi (2015b) in both sample sizes 500 and 1000 (graphs (c) and (d)).

G Additional Empirical Results

The test result based on NLSY 97 is reported in Section 8 of the main text. Here we report the test result based on NLSY 79. The NLSY 79 sample is obtained in the same way as NLSY 97 and contains 2554 observations. The sample may not be exactly the same as that used in Cameron and Heckman (1998), since we are not able to obtain the exact sample used in their paper.

Table 2: Model Selection Tests Based on NLSY 79

	Test Statistic	p-value
Robust Test	4.684	.000
Vuong (1989) Test	5.472	.000

Table 2 reports the test results. As opposed to the result based on NLSY 97, here we see that the null hypothesis that the logit transition model and the ordered logit model are equally close to the data distribution is strongly rejected even when our robust test is used. This may indicate that the schooling choice decision was made differently by the NLSY 97 cohort than the NLSY 79 cohort.