Econ 715 - Problem Set

Econometrics Methods

Instructor: Xiaoxia Shi

E-mail: xshi@ssc.wisc.edu

Class Webpage: http://www.ssc.wisc.edu/˜xshi/teaching/econ715

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This is a "set of problems", also called "problem set". The set is constantly updating. Please try the problems on your own schedule and discuss with me during office hours.

1. Consider the linear regression:

$$Y = X\beta + \varepsilon \ . \tag{1}$$

Suppose one would like to test the null hypothesis:

$$H_0 : R\beta = r \ , \tag{2}$$

where R is a $m \times k$ matrix and r is a m-vector. A Wald test uses the following Wald statistic:

$$W_n = \hat{\sigma}_n^{-2} \left(R\hat{\beta}_n - r \right)' \left[R \left(X'X \right)^{-1} R' \right]^{-1} \left(R\hat{\beta}_n - r \right) \ , \tag{3}$$

where n is the sample size, $\hat{\sigma}_n^2 = \frac{1}{n-k-1} \sum_{i=1}^n \hat{\varepsilon}_i^2$, $\hat{\varepsilon}_i$ is the OLS residual, $\hat{\beta}_n$ is the OLS estimator. The Wald test of significance level α rejects H_0 if $W_n > \chi_m^2 (1 - \alpha)$, where $\chi_m^2 (1 - \alpha)$ is the $1 - \alpha$ quantile of χ_m^2 distribution.

A F-test uses the F-statistic:

$$F_n = \frac{\left(SSR_r - SSR_{ur} \right) / m}{SSR_{ur} / \left(n - k - 1 \right)} \ , \tag{4}$$

where SSR_r is the residual sum of squared from the regression with $R\beta = r$ imposed, SSR_{ur} is the residual sum of squared from the unrestricted regression.

Question: prove that $W_n / m = F_n$ when $r = 0$.

2. Consider the following linear structural model

$$\begin{aligned} Y_1 &= b_1 + a_1 X_1 - a_2 X_2 + \varepsilon_1 \\ Y_2 &= b_2 + a_2 X_2 - a_3 X_3 + \varepsilon_2 \\ &\dots \\ Y_J &= b_J + a_J X_J - a_{J+1} X_{J+1} + \varepsilon_J, \end{aligned}$$

where $E[\varepsilon_1, ..., \varepsilon_J | Z] = 0$ for a d_z-dimensional random vector Z. Suppose we have i.i.d. data set of size n on all the variables $(Y_1, ..., Y_J,\ X_1, ..., X_{J+1}, Z)$. Write down the 2 stage least square (2SLS) estimator for all the parameters. Under what conditions is the estimator consistent?

3. Prove Equation 6 in Lecture 2.

4. Prove Lemma 3.1 in Lecture 3.

5. Prove Lemma 4.3 in Lecture 4

6. Propose a test for $H_0 : \beta = 1$ in the nonlinear regression model $Y = \alpha X^\beta + u$, where $E(u|X) = 0$, $Var(X),\ Var(u) < \infty$ and $Var(X) > 0$. The test should be easy enough to be performed by someone who can only run linear regression.

7. Prove that the function $S(m, \Sigma) = \inf_{t \leq 0}(m - t)'\Sigma(m - t)$ is continuous on $[-\infty, \infty)^k \times \Psi$, where Ψ is a closed set of positive definite matrices of dimension $k \times k$. (Andrews and Guggenberger (2010)'s proof of their Lemma 1 states that this is obvious, but it may not be that obvious due to the infinity in the domain of m.)

8. Suppose the density of X belongs to the parametric family: $\{f(x, \theta) : \theta \in \Theta\}$, i.e., there exists $\theta_0 \in \Theta$ such that $f_X(x) = f(x, \theta_0)$. Unlike in a standard maximum likelihood model, assume that there are more than one $\theta \in \Theta$ such that $f_X(x) = f(x, \theta)$. Assume that all such θ's lie in the interior of Θ. Propose a confidence set for θ_0. List your assumptions for the confidence set to be asymptotically valid.

9. Let $\{X_1, X_2, ...\}$ be an iid sequence with support $\mathcal{X} \subset R^{d_x}$. Let $g : \mathcal{X} \times \Theta \to R^k$ be a function.

   (a) Describe one set of sufficient conditions under which

$$\sup_{\theta \in \Theta} \left\| n^{-1} \sum_{i=1}^{n} g(X_i, \theta) - Eg(X_i, \theta) \right\| \to_p 0.$$

   (b) Sketch a proof of the convergence under your conditions.

   (c) Describe one set of primitive conditions under which the quantile regression criterion function converges uniformly in probability:

$$\hat{Q}_n(\beta) = n^{-1} \sum_{i=1}^{n} |Y_i - X_i \beta|$$

10. (a) Describe one set of sufficient conditions under which the extremum estimators are consistent.

   (b) Sketch a proof of consistency under your conditions.

   (c) Describe how violations to those conditions could give you trouble.

11. Suppose that we have the following moment condition model:

$$E[(Y - X\beta)Z] = 0, \tag{5}$$

where X denotes the d_x-dimensional possibly endogenous regressors and Z denotes the d_z-dimensional exogenous instruments, and Y is a scalar.

(a) Suppose we observe an iid sample $\{Y_i, X_i, Z_i\}$. Write down the GMM criterion function used to obtain an estimator of β.

(b) Describe a set of sufficient conditions under which the criterion function converges uniformly in probability.

(c) Describe one set of primitive conditions under which the global identification condition (Assumption ID) holds.

12. (a) Describe one set of conditions under which an extremum estimator is asymptotically normal.

    (b) sketch a proof of the asymptotic normality under your assumptions.

    (c) Verify those conditions in a nonlinear regression setup:

$$\hat{Q}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \beta X_i^{\lambda}\right)^2$$

13. Consider the null hypothesis:

$$H_0 : h(\theta_0) = 0$$

You have an estimator $\hat{\theta}_n$ such that $\hat{\theta}_n \to_p \theta_0$ and $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \to_d N(0, V)$, V_n is a consistent estimator of V.

(a) Write down the Wald statistic.

(b) What is the asymptotic distribution of the Wald statistic under H_0?

(c) How do you test H_0?

14. Consider the moment condition model:

$$Em(X_i,\theta)\begin{cases} =0 & if \quad \theta=\theta_0 \\ \neq 0 & if \quad \theta\in\Theta/\{\theta_0\}\end{cases}$$

The GMM estimator $\hat{\theta}_n$ is defined as

$$\hat{\theta}_n = \arg\min_{\theta\in\Theta} \bar{m}_n(\theta)' W_n \bar{m}_n(\theta)$$

(a) What is the optimal weight matrix? In what sense do we say that it is optimal?

(b) How do we obtain the optimal GMM estimator?

(c) What is the optimal weight matrix when the moment condition is $E[(Y-X\beta)Z]=0$?

15. Consider the null hypothesis: $\left(\theta_0=(\theta_{10},\theta_{20})'\right)$

$$H_0:\theta_{10}\leq 0, \theta_{20}\leq 0$$

You have an estimator $\hat{\theta}_n=\left(\hat{\theta}_{1,n},\hat{\theta}_{2,n}\right)$ such that $\hat{\theta}_n\to_p\theta_0$ and $\sqrt{n}\left(\hat{\theta}_n-\theta_0\right)\to_d N(0,I)$

(a) Write down a Wald statistic that can be used to test this hypothesis.

(b) What is the asymptotic distribution of the Wald statistic under H_0?

(c) How do we test H_0 based on the Wald statistic and its asymptotic distribution?

16. Consider an iid sample $\{X_1, ..., X_n\}$. We would like to test

$$H_0 : EX_1 = 0$$

    (a) If $X_1 \sim N(0, 1)$, how would you test the H_0? What is the rejection probability under H_0? What is the rejection probability under the alternative hypothesis such that $EX_1 = 0.1$?

    (b) If we do not know the distribution of X, but only know $E|X_1|^3 < \infty$, answer the same questions above.

17. (a) Derive the asymptotic distribution of the GMM estimator with the identity weight matrix. Make assumptions as needed as you go along.

    (b) What if the model is misspecified?

18. Consider the moment condition model:

$$Em(X_i, \theta) \begin{cases} = 0 & if \quad \theta = \theta_0 \\ \neq 0 & if \quad \theta \in \Theta / \{\theta_0\} \end{cases}$$

    (a) Assume correct identification, derive the GMM estimator with weight matrix W.

    (b) Derive the optimal weight matrix.