Fall 2011
Econ 709
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Problem Set #5 Solutions

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1.

Detailed solution omitted. KS statistic is expected to decrease as the sample size increases. Need to clearly demonstrate the value of the KS statistic for different sample sizes and for different distributions.

2.

We prove the following lemma:

Lemma 1. *Suppose $\{Y_{n,1}\}_{n=1}^{\infty}$ and $\{Y_{n,2}\}_{n=1}^{\infty}$ are two sequences of random vectors such that $Y_{n,1} \in \mathbf{R}^k$ and $Y_{n,2} \in \mathbf{R}^l$ and as $n \to n$, $Y_{n,1} \to_d Y_1$ and $Y_{n,2} \to_p r$ for a random vector $Y_1 \in \mathbf{R}^k$ and a constant vector $r \in \mathbf{R}^l$. Then,*

$$\begin{pmatrix} Y_{n,1} \\ Y_{n,2} \end{pmatrix} \to_d \begin{pmatrix} Y_1 \\ r \end{pmatrix}.$$

Take a constant vector $c = (c_1', c_2') \in \mathbf{R}^k \times \mathbf{R}^l$. Since a linear combination is a continuous function, we have $c_1' Y_{n,1} \to_d c_1' Y_1$ and $c_2' Y_{n,2} \to_d c_2' r$ by CMT. Note that $c_1' Y_{n,1} \in \mathbf{R}$ and $c_2' Y_{n,2} \in \mathbf{R}$, i.e., they are random variables. This implies that we can use the scalar version of the Lemma 1:

$$\begin{pmatrix} c_1' Y_{n,1} \\ c_2' Y_{n,2} \end{pmatrix} \to_d \begin{pmatrix} c_1' Y_1 \\ c_2' r \end{pmatrix}.$$

By CMT, $c_1' Y_{n,1} + c_2' Y_{n,2} \to_d c_1' Y_1 + c_2' r$. We can rewrite it as

$$c' \begin{pmatrix} Y_{n,1} \\ Y_{n,2} \end{pmatrix} = c_1' Y_{n,1} + c_2' Y_{n,2} \to_d c_1' Y_1 + c_2' r = c' \begin{pmatrix} Y_1 \\ r \end{pmatrix}.$$

Since the vector c is arbitrary, by Cramer-Wold device, we conclude

$$\begin{pmatrix} Y_{n,1} \\ Y_{n,2} \end{pmatrix} \to_d \begin{pmatrix} Y_1 \\ r \end{pmatrix}.$$

3.

Without loss of generality, assume $EX_{1i} = EX_{2i} = 0$ throughout the question.
(a) I propose the following estimator for ρ:

$$\begin{aligned} \hat{\rho} &= \frac{n^{-1}\sum_{i=1}^{n}(X_{1i} - \bar{X}_{1n})(X_{2i} - \bar{X}_{2n})}{\left(n^{-1}\sum_{i=1}^{n}(X_{1i} - \bar{X}_{1n})^2\right)^{1/2}\left(n^{-1}\sum_{i=1}^{n}(X_{2i} - \bar{X}_{2n})^2\right)^{1/2}}, \\ &= \frac{n^{-1}\sum_{i=1}^{n} X_{1i}X_{2i} - \bar{X}_{1n}\bar{X}_{2n}}{\left(n^{-1}\sum_{i=1}^{n} X_{1i}^2 - (\bar{X}_{1n})^2\right)^{1/2}\left(n^{-1}\sum_{i=1}^{n} X_{2i}^2 - (\bar{X}_{2n})^2\right)^{1/2}}. \end{aligned}$$

where $\bar{X}_{jn} = n^{-1}\sum_{i=1}^n X_{ji}$ for $j = 1, 2$. Since X_{ji}'s are iid and have finite variance, we can apply the WLLN to have $\bar{X}_{jn} = n^{-1}\sum_{i=1}^n X_{ji} \to_p EX_j = 0$ for $j = 1, 2$. Assume X_{ji}'s have finite fourth moment.[1] Then we can apply the WLLN to have

$$n^{-1}\sum_{i=1}^{n}(X_{ji} - \bar{X}_{jn})^2 \to_p E(X_{ji} - EX_{ji})^2 = \sigma_j^2,$$
$$n^{-1}\sum_{i=1}^{n}(X_{1i} - \bar{X}_{1n})(X_{2i} - \bar{X}_{2n}) \to_p E(X_{1i} - EX_{1i})(X_{2i} - EX_{2i}) = Cov(X_{1i}, X_{2i}) = \rho\sigma_1\sigma_2,$$

for $j = 1, 2$.
Consider a function $g : \mathbf{R}^5 \to \mathbf{R}$ such that $g(a, b, c, e, d) = \frac{e-ab}{(c-a^2)^{1/2}(d-b^2)^{1/2}}$. Then

$$\hat{\rho} = g\left(\bar{X}_{1n}, \bar{X}_{2n}, n^{-1}\sum_{i=1}^{n} X_{1i}^2, n^{-1}\sum_{i=1}^{n} X_{2i}^2, n^{-1}\sum_{i=1}^{n} X_{1i}X_{2i}\right).$$

Since $g(a, b, c, d, e)$ is continuous where $c - a^2 > 0$ and $d - b^2 > 0$, by CMT, we have

$$\hat{\rho} = g\left(\bar{X}_{1n}, \bar{X}_{2n}, n^{-1}\sum_{i=1}^{n} X_{1i}^2, n^{-1}\sum_{i=1}^{n} X_{2i}^2, n^{-1}\sum_{i=1}^{n} X_{1i}X_{2i}\right) \to_p g(EX_{1i}, EX_{2i}, EX_{1i}^2, EX_{2i}^2, EX_{1i}X_{2i}) = \rho.$$

Thus, $\hat{\rho}$ is consistent for ρ.

(b) By multivariate CLT,

$$\sqrt{n}\left(\begin{pmatrix} \bar{X}_{1n} \\ \bar{X}_{2n} \\ n^{-1}\sum_{i=1}^n X_{1i}^2 \\ n^{-1}\sum_{i=1}^n X_{2i}^2 \\ n^{-1}\sum_{i=1}^n X_{1i}X_{2i} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sigma_1^2 \\ \sigma_2^2 \\ \rho\sigma_1\sigma_2 \end{pmatrix}\right) \to_d N(\mathbf{0}, \Sigma),$$

where $\mathbf{0}$ is 5×1 vector of zeros and Σ is 5×5 covariance matrix. Let

$$G(a, b, c, d, e) = \begin{pmatrix} \frac{\partial g(a,b,d,c,e)}{\partial a} & \frac{\partial g(a,b,d,c,e)}{\partial b} & \frac{\partial g(a,b,d,c,e)}{\partial c} & \frac{\partial g(a,b,d,c,e)}{\partial d} & \frac{\partial g(a,b,d,c,e)}{\partial e} \end{pmatrix}.$$

You should check whether $G(0, 0, \sigma_1, \sigma_2, \rho\sigma_1\sigma_2)$ exists and is non-zero to apply the delta method. Write $G(0, 0, \sigma_1, \sigma_2, \rho\sigma_1\sigma_2) \equiv G$. By the delta method,

$$\sqrt{n}(\hat{\rho} - \rho) = \sqrt{n}\left(g\begin{pmatrix} \bar{X}_{1n} \\ \bar{X}_{2n} \\ n^{-1}\sum_{i=1}^n X_{1i}^2 \\ n^{-1}\sum_{i=1}^n X_{2i}^2 \\ n^{-1}\sum_{i=1}^n X_{1i}X_{2i} \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \\ \sigma_1^2 \\ \sigma_2^2 \\ \rho\sigma_1\sigma_2 \end{pmatrix}\right) \to_d N(0, G\Sigma G').$$

Note that $G\Sigma G'$ is a scalar and is the asymptotic variance of $\sqrt{n}(\hat{\rho} - \rho)$.

[1] This is not a necessary condition for the WLLN, though.

4.

If $Y_n \sim b(n,p)$ then $Y_n = \sum_{i=1}^n X_i$ with $X_i \sim Bern(p)$ for $i = 1, ..., n$. Since $EX_i = p$, by the WLLN,

$$\frac{Y_n}{n} = \frac{1}{n}\sum_{i=1}^n X_i \to_p EX_i = p.$$

Let $g(x) = x(1-x)$. Then $g(x)$ is continuous. By CMT (or Slutsky's theorem),

$$\frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) = g\left(\frac{Y_n}{n}\right) \to_p g(p) = p(1-p).$$

5.

Since X_i's are random sample from $\chi^2(50)$, $EX_i = 50$ and $Var(X_i) = 100$. CLT implies that $\sqrt{n}(\bar{X} - EX_i)/10$ is asymptotically standard normal. We use this fact to approximate the probability $P(49 < \bar{X} < 51)$:

$$\begin{aligned} P(49 < \bar{X} < 51) &= P(-1 < \bar{X} - 50 < 1) \\ &= P(\sqrt{100} \times -1 < \sqrt{100}(\bar{X} - 50) < \sqrt{100} \times 1) \\ &= P(-1 < \sqrt{100}(\bar{X} - 50)/10 < 1) \\ &\approx P(-1 < Z < 1) = 2\Phi(1) - 1 = 0.6826. \end{aligned}$$

6.

Let X_i's be random sample. Then

$$P(X_i < 3) = \int_1^3 \frac{1}{x^2}dx = \frac{2}{3}.$$

Each observation is either larger than (or equal to) 3 with probability 1/3 or less than 3 with probability 2/3. Let Y_i be a random variable such that $Y_i = 1\{X_i < 3\}$. Then Y_i is a Bernoulli random variable with $p = 2/3$ and $\sum_{i=1}^{72} Y_i$ is a binomial random variable such that $\sum_{i=1}^{72} Y_i \sim b(72, 2/3)$. Let $Y_n = \sum_{i=1}^{72} Y_i$. The distribution of Y_n can be approximated by Z_n such that $Z_n \sim N(48, 16)$: $P(Y_n > 50) \approx P(Z_n > 50.5)$ with continuity correction. Now

$$\begin{aligned} P(Y_n > 50) &\approx P(Z_n > 50.5) \\ &= P(\frac{Z_n - 48}{4} > \frac{50.5 - 48}{4}) \\ &= P(Z > 0.625) = 0.266, \end{aligned}$$

where Z is standard normal random variable.

7.

First, we show $Y_1 \to_p a$. Take any $\varepsilon > 0$.

$$\begin{aligned} P(|Y_1 - a| < \varepsilon) &= P(a - \varepsilon < Y_1 < a + \varepsilon) \\ &= P(a < Y_1 < a + \varepsilon) \\ &= P(a < \min X_i < a + \varepsilon) \\ &= 1 - P(\min X_i \geq a + \varepsilon) \\ &= 1 - P(X_i \geq a + \varepsilon, i = 1, 2, ..., n) \\ &= 1 - (P(X_i \geq a + \varepsilon))^n \\ &= 1 - \left(\frac{b - a - \varepsilon}{b - a} \right)^n \to 1, \end{aligned}$$

as $n \to \infty$. Similarly, we can show $Y_2 \to_p b$. In class, we learned that if $Y_{1n} \to_d Y_1$ and $Y_{2n} \to_d r$, where r is a constant, then $(Y_{1n}, Y_{2n})' \to_d (Y_1, r)'$. This is Lemma 1. Convergence in probability to a constant is the same as convergence in distribution to a constant. This implies that we have $(Y_1, Y_2)' \to_p (a, b)'$ by Lemma 1.

8.

By Lemma 1, we have

$$\begin{pmatrix} X_n - Y_n \\ X_n \end{pmatrix} \to_d \begin{pmatrix} 0 \\ X \end{pmatrix}.$$

By CMT, $X_n - (X_n - Y_n) = Y_n \to_d X - 0 = X$.