Fall 2011
Econ 709
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Problem Set #4 Solutions

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1.

Since $g(\cdot)$ is one-to-one, the inverse $g^{-1}(\cdot)$ is well-defined. Assume $g^{-1}(z)$ is differentiable. Let $W = X$. Then $(f(X), g(Y)) = (W, Z)$ where $f(\cdot)$ is an identity function. Then $f^{-1}(\cdot)$ is also an identity function, $\partial f^{-1}(w)/\partial w = 1$, and $\partial f^{-1}(w)/\partial z = 0$.

$$\begin{aligned}
f_{X|Z}(x|z) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\
&= \frac{f_{W,Y}(w, g^{-1}(z)) \left| \begin{matrix} 1 & 0 \\ 0 & \frac{\partial g^{-1}(z)}{\partial z} \end{matrix} \right|}{f_Y(g^{-1}(z)) \left| \frac{\partial g^{-1}(z)}{\partial z} \right|} \\
&= \frac{f_{W,Y}(w, g^{-1}(z))}{f_Y(g^{-1}(z))} \\
&= \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_{X|Y}(x|y).
\end{aligned}$$

2.

$$\begin{aligned}
P(2X + 3Y < 1) &= P\left(Y < \frac{1-2X}{3}\right) \\
&= P\left(0 < Y < \frac{1-2X}{3}, 0 < X < \frac{1}{2}\right) \\
&= \int_0^{1/2} \int_0^{\frac{1-2x}{3}} f(x,y) dx dy \\
&= \int_0^{1/2} \int_0^{\frac{1-2x}{3}} 6(1-x-y) dx dy = \frac{13}{36}. \\
E[XY + 2X] &= \int_0^1 \int_0^{1-x} (xy + 2x^2) 6(1-x-y) dy dx = \frac{1}{4}.
\end{aligned}$$

3.

The marginal pmf's are

$$p_{X_1}(x_1) = \begin{cases} \frac{4}{18} & \text{when } x_1 = 0 \\ \frac{7}{18} & \text{when } x_1 = 1 \\ \frac{7}{18} & \text{when } x_1 = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$p_{X_2}(x_2) = \begin{cases} \frac{11}{18} & \text{when } x_2 = 0 \\ \frac{7}{18} & \text{when } x_2 = 1 \\ 0 & \text{elsewhere} \end{cases}$$

The conditional means are $E[X_1|X_2 = 0] = 16/11$, $E[X_1|X_2 = 1] = 5/7$, $E[X_2|X_1 = 0] = 3/4$, $E[X_2|X_1 = 1] = 3/7$, and $E[X_2|X_1 = 2] = 1/7$.

4.

By using the definition of the conditional density function, $f_{Y|X}(y|x) = f_{XY}(x,y)/f_X(x) = 1/f_X(x)$ and $f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y) = 1/f_Y(y)$. The marginal pdf $f_X(x)$ is given by

$$f_X(x) = \int_{-x}^{x} 1dy = 2x,$$

and zero elsewhere. The marginal pdf $f_Y(y)$ is given by

$$f_Y(y) = \int_{|y|}^{1} 1dx = 1 - |y|,$$

and zero elsewhere. Now the conditional expectations $E[Y|X = x]$ and $E[X|Y = y]$ are

$$\begin{aligned} E[Y|X = x] &= \int_{-x}^{x} y f_{Y|X}(y|x)dy \\ &= \int_{-x}^{x} y\frac{1}{2x}dy = 0 \text{ for } 0 < x < 1. \\ E[X|Y = y] &= \int_{|y|}^{1} x f_{X|Y}(x|y)dx \\ &= \int_{|y|}^{1} x\frac{1}{1-|y|}dx = \frac{1+|y|}{2} \text{ for } -1 < y < 1. \end{aligned}$$

Therefore, $E[Y|x]$ is a straight line, but $E[X|y]$ is not.

5.

Let X_i be the midpoint of the ith line segment. Since they are independent and uniformly distributed, the marginal pdf's are given by $f_{X_1}(x_1) = 1/14$ for $0 < x_1 < 14$, $f_{X_2}(x_2) = 1/14$ for $6 < x_2 < 20$ and zero elsewhere. The joint pdf is given by $f_{X_1,X_2}(x_1,x_2) = \frac{1}{196}$ for $0 < x_1 < 14$ and

$6 < x_2 < 20$, and zero elsewhere. The two line segments overlap if $|x_1 - x_2| < 2$. The probability of the event is

$$\begin{aligned} P(|X_1 - X_2| < 2) &= P(|X_1 - X_2| < 2, X_1 \le X_2) + P(|X_1 - X_2| < 2, X_1 > X_2) \\ &= P(X_2 - X_1 < 2, X_1 \le X_2) + P(X_1 - X_2 < 2, X_1 > X_2). \end{aligned}$$

The region $X_2 - X_1 < 2, X_1 \le X_2$ and $X_1 - X_2 < 2, X_1 > X_2$ on (x_1, x_2) plane is the sum of a triangle and a parallelogram (check this) and we take the integration of the joint density over this region.

$$P(|X_1 - X_2| < 2) = \int_4^8 \int_6^{x_1+2} \frac{1}{196} dx2dx1 + \int_8^{14} \int_{x_1-2}^{x_1+2} \frac{1}{196} dx2dx1 = \frac{8}{49}.$$

6.

(a) $f_X(x) = 2/3(1+x)$ for $0 < x < 1$, $f_Y(y) = 2/3(1+y)$, $f_Z(z) = 2/3(1+z)$, and zero elsewhere.
(b) $P(0 < X < 1/2, 0 < Y < 1/2, 0 < Z < 1/2) = 1/16$. $P(0 < X < 1/2) = P(0 < Y < 1/2) = P(0 < Z < 1/2) = 5/12$.
(c) Since $f_X(x)f_Y(y)f_Z(z) = (2/3)^3(1+x)(1+y)(1+z) \neq f(x,y,z)$, X, Y, and Z are not (mutually) independent.
(d) $E[X^2YZ] + E[3XY^4Z^2] \approx 0.2657$.
(e) $F_X(x) = \begin{cases} 0 & ,x \le 0 \\ \frac{1}{3}(x^2+2x) & ,0 < x < 1 \\ 1 & ,x \ge 1 \end{cases}$ $F_Y(y)$ and $F_Z(z)$ are the same with $F_X(x)$ by replacing x with y and z, respectively.
(f) Note that $E[X+Y|z] = E[X|z] + E[Y|z]$. Since $E[X|z] = E[Y|z] = (7/12 + z/2)/(z+1)$, $E[X+Y|z] = (7/6+z)/(z+1)$.
(g) The conditional pdf is given by $f_{X|Y,Z}(x|y,z) = \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)} = (x+y+z)(1/2+y+z)^{-1}$ for $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, and zero elsewhere. Then $E[X|Y=y, Z=z] = \frac{2+3y+3z}{3+6y+6z}$.

7.

Let X be wife's height and Y be husband's height. Then $\mu_X = 64, \sigma_X = 1.5, \mu_Y = 70, \sigma_Y = 2, \rho = 0.7$. The conditional distribution of X, given $Y = y$ is $X|y \sim N(\mu_X + \rho\frac{\sigma_x}{\sigma_Y}(y - \mu_Y), \sigma_X^2(1-\rho^2))$. Thus,

$$X|y = 72 \sim N(65.05, 1.1475).$$

The best guess of the height of a woman whose husband's height is 6 feet is $E[X|y = 72] = 65.05$ inches. The 95% prediction interval for her height is $65.05 \pm 1.96 \times \sqrt{1.1475} = (62.95, 67.15)$.

8.

To show $f(x,y)$ is a joint pdf, we show $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$ and $f(x,y) \ge 0$ for $-\infty < x < \infty$ and $-\infty < y < \infty$. First,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)\left[1 + xy\exp\left(-\frac{1}{2}(x^2+y^2-2)\right)\right] dydx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2pi}} \exp\left(-\frac{1}{2}x^2\right) dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2pi}} \exp\left(-\frac{1}{2}y^2\right) dy = 1, \end{aligned}$$

where the first equality follows from the fact that the integration of an odd function over the real line is zero, and the last equality follows from the fact that $1/\sqrt{2pi}\exp(-x^2/2)$ and $1/\sqrt{2pi}\exp(-y^2/2)$ are normal pdfs. Using the inequalities $\exp(x^2/2 - 1/2) \geq |x|$ and $\exp(y^2/2 - 1/2) \geq |y|$, we can show $1 + xy\exp(-(x^2 + y^2 - 2)/2) \geq 0$ for $x, y \in \mathbf{R}$. The calculation above shows that the marginal pdfs are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right), \quad f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right),$$

so that they are normal.