Fall 2011
Econ 709
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Problem Set #3 Solutions

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1.

(a) Since $EX^2 < \infty$, $MSE(a) = E(X-a)^2 = EX^2 - 2aEX + a^2$. The first-order condition (FOC) is

$$\frac{dMSE(a)}{da} = -2EX + 2a = 0,$$

and the second-order sufficient condition (SOSC) is

$$\frac{d^2MSE(a)}{da^2} = 2 > 0.$$

Thus, $a = EX$ minimizes the MSE and the MSE evaluated at $a = EX$ is the variance of X, i.e., $MSE(EX) = E(X-EX)^2 = Var(X)$.

(b) Assume that X has a pdf $f(x)$ and cdf $F(x)$. Then

$$\begin{aligned} MAD(a) &= E|X-a| \\ &= \int_{-\infty}^{\infty} |x-a| f(x)dx \\ &= \int_{-\infty}^{a} (a-x)f(x)dx + \int_{a}^{\infty} (x-a)f(x)dx \\ &= a\int_{-\infty}^{a} f(x)dx - \int_{-\infty}^{a} xf(x)dx + \int_{a}^{\infty} xf(x)dx - a\int_{a}^{\infty} f(x)dx \\ &= aF(a) - a(1-F(a)) - 2\int_{-\infty}^{a} xf(x)dx + \int_{-\infty}^{\infty} xf(x)dx. \end{aligned}$$

Let $G(x) = \int xf(x)dx$. Since the second moment (and thus the first moment) exists, $\lim_{x\to-\infty} G(x)$ and $\lim_{x\to\infty} G(x)$ exist. Therefore,

$$\begin{aligned} \frac{dMAD(a)}{da} &= F(a) + af(a) - (1-F(a)) + af(a) - 2af(a) \\ &= 2F(a) - 1 = 0, \end{aligned}$$

and a is the median of X (SOSC is $2f(a) > 0$).

2.

Fix x_0 and take a decreasing sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$. Let $C_n = \{\omega : X(\omega) \leq x_n\}$. Then C_n is a decreasing sequence of events and we have $\lim_{n\to\infty} C_n = \cap_{n=1}^{\infty} C_n = \{\omega : X(\omega) \leq x_0\}$.

$$\begin{aligned} \lim_{x\downarrow x_0} F_X(x) = \lim_{n\to\infty} F_X(x_n) &= \lim_{n\to\infty} P(C_n) \\ &= P(\lim_{n\to\infty} C_n) \quad (\because \text{HCM Theorem 1.3.6}) \\ &= P\{\omega : X(\omega) \leq x_0\} = F_X(x_0). \end{aligned}$$

3.

We say that a set C is a Borel set if C can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements (Rudin, 1976).
Define $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and so on. Then C_n is the union of 2^n disjoint closed intervals with length 3^{-n}. Now the Cantor set is $C = \cap_{n=1}^{\infty} C_n$.

(a) Since C_n is the union of 2^n disjoint closed intervals (each is the complement of an open interval), C_n is a Borel set. So $C = \cap_{n=1}^{\infty} C_n$ is a Borel set.

(b) Countable additivity: $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_i \cap A_j = 0$ for $i \neq j$.

$$\begin{aligned}
\mu(C) &= 1 - \mu(C^c) \\
&= 1 - \mu(\cup_{n=1}^{\infty} C_n^c) \quad \because \text{De Morgan's law} \\
&= 1 - \sum_{n=1}^{\infty} \mu(C_n^c) \quad \because \text{countable additivity} \\
&= 1 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{2} = 0.
\end{aligned}$$

(c) Omitted.

4.

(a) Suppose that for some positive integer $k > k_0$, X has finite kth moment. Then $E|X^k| < \infty$ and

$$\begin{aligned}
E|X^k| = \int_{\mathbf{R}} |x^k| f(x) dx &= \int_{|x|>1} |x^k| f(x) dx + \int_{|x|\le 1} |x^k| f(x) dx \\
&\ge \int_{|x|>1} |x^{k_0}| f(x) dx - \int_{|x|\le 1} |x^k| f(x) dx \\
&\ge \int_{\mathbf{R}} |x^{k_0}| f(x) dx - \int_{|x|\le 1} |x^{k_0}| f(x) dx - \int_{|x|\le 1} f(x) dx \\
&= \infty,
\end{aligned}$$

which contradicts to the assumption. Thus, X does not have finite kth moment for any $k > k_0$.

(b) Suppose that $M_X(t)$ is well-defined on a neighborhood of 0 and is differentiable with respect to t (differentiation under the integral sign). Since e^{tX} is infinitely differentiable with respect to t, by Taylor's theorem,

$$\begin{aligned}
Ee^{tX} &= M_X(0) + \frac{M_X^{(1)}(0)}{1!} t + \frac{M_X^{(2)}(0)}{2!} t^2 + \cdots + \frac{M_X^{(k_0)}(0)}{k_0!} t + R(t) \\
&= 1 + EX \cdot t + \frac{EX^2}{2!} t^2 + \cdots + \frac{EX^{k_0}}{k_0!} t^{k_0} + R(t),
\end{aligned}$$

where $R(t) = o(|t|^{k_0})$ is a remainder term. Since EX^{k_0} is not well-defined by assumption ($\because$ $E|X^{k_0}| = \infty$), the right-hand side of the above equation is not well-defined. This contradicts to the assumption. Thus, $M_X(t)$ is not well-defined on a neighborhood of 0.

5.

Since $e^{tx} > 0$ for all x, by Markov's inequality,

$$P(e^{tX} \geq e^{at}) \leq e^{-at} M_X(t),$$

for $-h < t < h$. Now note that $e^{tx} \geq e^{at}$ if and only if $x \geq a$ for $0 < t < h$ and $e^{tx} \geq e^{at}$ if and only if $x \leq a$ for $-k < t < 0$. This completes the proof.

6.

Note that $P(X \geq 3) = (2/3)^3$. Then,

$$\begin{aligned} P(X = x | X \geq 3) &= \frac{P(X = x, X \geq 3)}{P(X \geq 3)} \\ &= \frac{1}{3}\left(\frac{2}{3}\right)^{x-3}, \end{aligned}$$

for $x = 3, 4, 5, ...,$ and zero elsewhere.

7.

Since X is a continuous random variable, $P(c < X < d) = P(c < X \leq d) = P(X \leq d) - P(X \leq c)$ and $P(X < c) = P(X \leq c)$. Using the table in the appendix, we find $c = 0.831$ and $d = 12.833$.

8.

(a) Assume $x > 0$ and $y > 0$. $P(X > x) = 1 - P(X \leq x) = 1 - \int_{-\infty}^{x} \lambda e^{-\lambda t} dt = e^{-\lambda t}$ for a positive λ. Then

$$\begin{aligned} P(X > x + y | X > x) &= \frac{P(X > x + y, X > x)}{P(X > x)} \\ &= \frac{P(X > x + y, X > x)}{P(X > x)} \\ &= \frac{P(X > x + y)}{P(X > x)} \quad (\because \{X > x + y\} \subset \{X > x\}) \\ &= \frac{e^{-\lambda x + y}}{e^{-\lambda x}} \\ &= e^{-\lambda y} = P(X > y). \end{aligned}$$

(b) Since property (3.3.7) holds for Y, $P(Y > x + y) = P(Y > x)P(Y > y)$ for $x, y > 0$. Let $g(y) = 1 - F_Y(y)$. Then $g(x + y) = g(x)g(y)$ and $\log g(x + y) = \log g(x) + \log g(y)$. Since Y is a continuous random variable, $g(\cdot)$ is differentiable. By differentiating $\log g(x+y) = \log g(x)+\log g(y)$ with respect to x on both sides,

$$\frac{g'(x + y)}{g(x + y)} = \frac{g'(x)}{g(x)}.$$

By letting $x = 0$,

$$\frac{g'(y)}{g(y)} = \frac{g'(0)}{g(0)} = g'(0),$$

because $g(0) = 1 - F_Y(0) = 1$. Let $g'(0) = -\lambda$ and solve the differential equation

$$\frac{g'(y)}{g(y)} = \frac{d \log g(y)}{dy} = -\lambda.$$

Then $\log g(y) = -\lambda y + C_1$ and $g(y) = C_2 e^{-\lambda y}$ for some constants C_1 and C_2. Since we know $g(0) = 1$, $C_2 = 1$. Therefore, $g(y) = 1 - F_Y(y) = e^{-\lambda y}$ or $F_Y(y) = 1 - e^{-\lambda y}$ for $y > 0$.

9.

The random variable X is $N(3, 4^2)$. Thus,

$$\begin{aligned} P(-1 < X < 9) &= P\left(-1 < \frac{X-3}{4} < 1.5\right) \\ &= P(-1 < Z < 1.5) \\ &= \Phi(1.5) - \Phi(-1) = 0.7745, \end{aligned}$$

where Z is the standard normal random variable and $\Phi(\cdot)$ is its cdf.