Problem Set 3 (Due 10/17/2011 in class)

October 5, 2011

1. For a random variable X with finite second moment, let a scalar a be a guess of X's value.

   (a) Find the a that minimizes the mean squared error: $MSE(a) = E(X-a)^2$. What is the minimum mean-squared error?

   (b) Assume that X is a continuous random variable, find the a that minimizes the mean absolute deviation: $MAD(a) = E|X-a|$. (Hint: you can assume that X has density $f_X(x)$ and use the definition of expectation.)

2. Prove that the cumulative distribution function of any random variable X is right continuous, i.e., for any $x_0 \in R$,

$$\lim_{x \downarrow x_0} F_X(x) = F_X(x_0).$$

3. The Cantor (ternary) set is created by repeatedly deleting the open middle third subinterval. More precisely, we start with the interval [0,1] and follow these steps of deletion: (1) divide the interval into three equal thirds: [0,1/3], (1/3,2/3), [2/3,1] and delete the open middle third (1/3,2/3); (2) repeat the deletion for the intervals remaining: [0,1/3], [2/3,1], i.e., we divide each of them into equal thirds and delete the open middle third; for example, we divide [0,1/3] into [0,1/9], (1/9,2/9), [2/9,1/3], and delete the open middle third (1/9,2/9); (3) repeat the deletion infinite times. The Cantor set is the collection of all points that are NOT deleted after infinite steps.

   (a) Show that the Cantor set (denoted C) is a Borel set (i.e. $C \in \mathcal{B}(R)$).

   (b) Use the countable additivity of the Lebesgue measure, μ, and the fact that $\mu((a,b)) = b - a \; \forall (a,b) \subset R$ to compute the Lebesgue measure of C.

(c) Show that C is uncountable. (Optional, you can find an answer on Wikipedia or an analysis book.)

4. Consider a random variable X. Suppose that X does not have finite k_0th moment for some positive integer k_0.

   (a) Show that for any $k > k_0$, X does not have finite kth moment. (Hint: prove by contradiction and write X as a sum of two random variables.)

   (b) Show that the moment generating function of X is not well-defined on a neighborhood of zero (*i.e.* $M_X(t) \not< \infty \ \forall t > 0$).

5. 1.10.4

6. 3.1.14

7. 3.3.2

8. 3.3.25

9. 3.4.10