Fall 2011
Econ 709
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Problem Set #2 Solutions

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1.

(simulation exercise) Omitted.

2.

(a) Let X and Y be random variables for the number of sixes Player 1 and 2 gets, respectively. Let T be a random variable for total number of sixes. Then $T = X + Y$. The probability of having total t sixes given P1's x sixes is given by

$$\begin{aligned}
P(T = t|X = x) &= \frac{P(T = t \cap X = x)}{P(X = x)} \\
&= \frac{P(Y = t - x \cap X = x)}{P(X = x)} \\
&= \frac{P(Y = t - x)P(X = x)}{P(X = x)} \\
&= P(Y = t - x).
\end{aligned}$$

The third equality holds because X and Y are independent. Since $t-x = 0, 1, 2, 3$, we find $P(Y = y)$ for $y = 0, 1, 2, 3$:

$$\begin{aligned}
P(Y = 0) &= \left(\frac{5}{6}\right)^3 = \frac{125}{216} \approx 0.58 \\
P(Y = 1) &= {}_3C_1 \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) = \frac{75}{216} \approx 0.35 \\
P(Y = 2) &= {}_3C_2 \left(\frac{5}{6}\right) \left(\frac{1}{6}\right)^2 = \frac{15}{216} \approx 0.07 \\
P(Y = 3) &= \left(\frac{1}{6}\right)^3 = \frac{1}{216} \approx 0.005.
\end{aligned}$$

This means that $P(T = t|X = x) = P(Y = t - x)$ is maximized when $t - x = 0$, i.e, $t = x$. Therefore, P1's best guess is $X = x$, the number of sixes she has, for $x = 0, 1, 2, 3$.

(b) The probability of having total t sixes given P2's $Y = y$ and P1's $X = x$ is given by

$$\begin{aligned}
P(T = t|X = x, Y = y) &= \frac{P(T = t \cap X = x, Y = y)}{P(X = x, Y = y)} \\
&= \frac{P(X = x, Y = y)}{P(X = x, Y = y)} = 1.
\end{aligned}$$

The second equality holds because knowing X and Y is equivalent to knowing T (recall that $T = X + Y$). This means that P2 knows the total number of sixes on the table with certainty. Her best guess is $X + Y$.

(c) Each player has two actions: Challenge (C) or Make a bigger guess (B). Let P2 be the second mover. First suppose $y > 0$. Given $X = x$, P2 makes a bigger guess, $x + y$. Since P1's best guess is smaller than P2's guess, P1 challenges and loses \$1. This implies that the second mover always wins if $y > 0$. Now suppose $y = 0$. Given $X = x$, P2's best guess is also x, so she challenges with prob. 0.5 (and lose \$1) or makes a bigger guess $x+1$ with prob. 0.5. For the latter case, P1 always challenges and wins \$1 because P1's guess (x) is smaller than P2's guess ($x+1$). Thus, the second mover always loses when $y = 0$. But we already calculated that $P(Y = 0) \approx 0.58$ is higher than the sum of the other events, which in turn implies the second mover has a disadvantage. Therefore, the first mover has an advantage.

(d) (simulation exercise) omitted.

3.

Let P=positive, N=negative, S_0=without disease, S_1=early stage, S_2=intermeditate stage, S_3=late stage.

(a) The probability that a person actually is healthy given that the test returns negative is

$$\begin{aligned} P(S_0|N) &= \frac{P(S_0 \cap N)}{P(N)} \\ &= \frac{P(N|S_0)P(S_0)}{\sum_{i=0}^{3} P(N|S_i)P(S_i)} \\ &= \frac{(1-0.05)\times 0.6}{0.95\times 0.6 + 0.5\times 0.3 + 0.05\times 0.099 + 0\times 0.001} \approx 0.786 \end{aligned}$$

(b) We want $P(S_0|N) = 0.95$ by considering a prior distribution $P(S_i)$ for $i = 0, 1, 2, 3$.

$$\begin{aligned} P(S_0|N) &= \frac{P(N|S_0)P(S_0)}{\sum_{i=0}^{3} P(N|S_i)P(S_i)} \\ &= \frac{0.95\times P(S_0)}{0.95\times P(S_0) + 0.5\times P(S_1) + 0.05\times P(S_2) + 0\times P(S_3)} = 0.95. \end{aligned}$$

Solving this equation, we get $P(S_0) = 10P(S_1) + P(S_2)$. To be a valid probability, we need $\sum_{i=0}^{3} P(S_i) = 1$. Another reasonable constraint would be $P(S_1) \geq P(S_2) \geq P(S_3)$. One prior that satisfies the constraints would be $P(S_0) = 0.905$, $P(S_1) = 0.09$, $P(S_2) = 0.005$, and $P(S_3) = 0$.

4.

Suppose that X and Y are independent random variables.

WTS1: $g(X)$ and $f(Y)$ are random variables.

Note $g(X) : (\Omega, \mathcal{F}) \to (\Omega_g, \mathcal{B}(\Omega_g))$. $\forall A \in \mathcal{B}(\Omega_g)$, $E = g^{-1}(A) \in \mathcal{B}(\Omega_X)$ since g is measurable. Then $X^{-1}(E) = X^{-1}(g^{-1}(A)) = (g \circ X)^{-1}(A) \in \mathcal{F}$ since X is measurable (i.e., random variable). The proof for $f(Y)$ is similar.

WTS2: $g(X)$ and $f(Y)$ are independent.
$\forall A \in \mathcal{B}(\Omega_g), \forall B \in \mathcal{B}(\Omega_f)$, we have $g^{-1}(A) \in \mathcal{B}(\Omega_X)$ and $f^{-1}(B) \in \mathcal{B}(\Omega_Y)$ as we just did in the proof of WTS1. Now

$$\begin{aligned} P((g \circ X)^{-1}(A) \cap (f \circ Y)^{-1}(B)) &= P(X^{-1}(g^{-1}(A)) \cap Y^{-1}(f^{-1}(B))) \\ &= P(X^{-1}(g^{-1}(A)))P(Y^{-1}(f^{-1}(B))) \quad (\because X \perp Y) \\ &= P((g \circ X)^{-1}(A))P((f \circ Y)^{-1}(B)). \end{aligned}$$

Thus, $g(X)$ and $f(Y)$ are independent random variables.

5.

The induced probability P_X on D is defined by $P_X(A) = P(\{c : X(c) \in A\}), \forall A \in \mathcal{F}_D$, where $\mathcal{F}_D$ is a sigma-field on D.

(i) $P_X(\emptyset) = 0$.

(ii) $P_X(\{1\}) = P_X(\{2\}) = P_X(\{3\}) = P_X(\{4\}) = 1/13$ and $P_X(\{0\}) = 9/13$.

(iii) For any set $A \in \mathcal{F}_D$, $P_X(A) = \sum_{i=0}^{4} P_X(\{i\})\mathbf{1}(i \in A)$, where $\mathbf{1}(\cdot)$ is the indicator function.

6.

(i)$P(X = 1 \text{ or } 2) = p_X(1) + p_X(2) = 1/5$.
(ii)$P(1/2 < X < 5/2) = p_X(1) + p_X(2) = 1/5$.
(iii)$P(1 \leq X \leq 2) = p_X(1) + p_X(2) = 1/5$.

7.

The graph is omitted.
(a)$P(-1/2 < X \leq 1/2) = P(X \leq 1/2) - P(X \leq -1/2) = F(1/2) - F(-1/2) = 1/4$.
(b)$P(X = 0) = P(X \leq 0) - P(X < 0) = 1/2 - 1/2 = 0$.
(c)$P(X = 1) = P(X \leq 1) - P(X < 1) = 1/4$.
(d)$P(2 < X \leq 3) = P(X \leq 3) - P(X \leq 2) = 0$.