Fall 2011
Econ 709

Problem Set #1 Solutions

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1.

Let A and B be events "tossing at least one six in six tosses", "tossing at least two sixes in twelve tosses", respectively. We calculate and compare $P(A)$ and $P(B)$. Then,

$$\begin{aligned}
P(A) &= 1 - P(\text{toss no sixes in six tosses}) \\
&= 1 - \left(\frac{5}{6}\right)^6 \approx 0.665, \\
P(B) &= 1 - P(\text{toss no sixes in twelve tosses}) - P(\text{toss exactly one six in twelve tosses}) \\
&= 1 - \left(\frac{5}{6}\right)^{12} - {}_{12}C_1 \times \frac{1}{6}\left(\frac{5}{6}\right)^{11} \approx 0.619.
\end{aligned}$$

Now we have $P(A) > P(B)$.

2.

Want to show the following two statements:

$$\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C).
\end{aligned}$$

I only prove the first statement. The second one can be proved similarly.
($\subset$) Take any x such that $x \in A \cap (B \cup C)$. Then $x \in A$ *and* $x \in (B \cup C)$. There are two cases: (i) $x \in A$ and $x \in B$, or (ii) $x \in A$ and $x \in C$. For (i), $x \in (A \cap B)$ and thus, $x \in (A \cap B) \cup (A \cap C)$. For (ii), $x \in (A \cap C)$ and thus, $x \in (A \cap B) \cup (A \cap C)$. This proves

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C).$$

($\supset$) Take any x such that $x \in (A \cap B) \cup (A \cap C)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$ then $x \in A \cap (B \cup C)$ because $B \subset (B \cup C)$. If $x \in (A \cap C)$ then $x \in A \cap (B \cup C)$ because $C \subset (B \cup C)$. This proves

$$A \cap (B \cup C) \supset (A \cap B) \cup (A \cap C).$$

3.

(From the definition of σ-field) Want to show:

(i) $\emptyset \in \mathcal{B}$.

(ii) If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$.

(iii) If $A_1, A_2, ... \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$.

(i) is already proved. For (ii), suppose $A \in \mathcal{B}$. Then $A \in \mathcal{E}$ for all $\mathcal{E}$ s.t. $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{E}$ is a σ-field. Since $\mathcal{E}$ is a σ-field, $A^c \in \mathcal{E}$ for all such $\mathcal{E}$. By definition, $A^c \in \mathcal{B}$ and this proves (ii). For (iii), suppose $A_1, A_2, ... \in \mathcal{B}$. Then $A_1, A_2, ... \in \mathcal{E}$ for all $\mathcal{E}$ s.t. $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{E}$ is a σ-field. Since $\mathcal{E}$ is a σ-field, $\cup_{i=1}^{\infty} A_i \in \mathcal{E}$ for all such $\mathcal{E}$. By definition, $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ and this proves (iii). Since $\mathcal{B}$ satisfies (i),(ii), and (iii), $\mathcal{B}$ is a σ-field.

4.

Take any real number $a \in (0,1)$ and consider a sequence of intervals

$$C_n = \left(a - \frac{a}{n}, a + \frac{1-a}{n}\right),$$

for $n \in \mathbf{N}$. Then $C_n \subset (0,1)$ for all n and is a decreasing sequence converging to $\{a\}$. By Theorem 1.3.6,

$$P(\{a\}) = P(\bigcap_{n=1}^{\infty} C_n) = \lim_{n\to\infty} P(C_n) = \lim_{n\to\infty} a + \frac{1-a}{n} - a + \frac{a}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

5.

(a) For each wolf i, define $A_i = \{\text{wolf } i \text{ is white}\}$. Then $P(A_i) = 0.02$ for $i = 1, 2, ..., 6$, and $\{\text{at least one wolf is white}\} = \cup_{i=1}^{6} A_i$. By Boole's inequality,

$$P\left(\bigcup_{i=1}^{6} A_i\right) \leq \sum_{i=1}^{6} P(A_i) = 0.12,$$

so that the upper bound is 0.12. Since $A_i \subset \cup_{i=1}^{6} A_i$,

$$0.02 = P(A_i) \leq P\left(\bigcup_{i=1}^{6} A_i\right),$$

for all $i = 1, 2, .., 6$. Thus, the lower bound is 0.02.
(b) The lower bound is achieved when $A_i = \cup_{i=1}^{6} A_i$ for $i = 1, 2, .., 6$.
(c) The upper bound is achieved when A_i's are mutually exclusive.
(d) Assume wolves randomly form packs, i.e., they are random draws from the population. Then

$$P\left(\bigcup_{i=1}^{6} A_i\right) = 1 - P\left(\bigcap_{i=1}^{6} A_i^c\right) = 1 - (0.98)^6 \approx 0.114.$$

The probability falls within the bounds.

6.

The number of total outcomes is ${}_{52}C_5$ The number of "full house" outcomes is ${}_{13}C_1 \times {}_{12}C_1 \times {}_4C_2 \times {}_4C_3$. The probability of a full house is

$$\frac{{}_{13}C_1 \times_{12} C_1 \times_4 C_2 \times_4 C_3}{{}_{52}C_5} \approx 0.00144$$