Online Appendix to "Stable Games and their Dynamics"

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O.1 Analysis of the War of Attrition

In this section, we prove that random matching of a single population to play a war of attrition generates a stable game. Recalling the description in Example 2.4, we see that the payoff matrix for the war of attrition is

$$A = \begin{pmatrix} \frac{v}{2} - c_1 & -c_1 & \cdots & -c_1 \\ v - c_1 & \frac{v}{2} - c_2 & \cdots & -c_2 \\ \vdots & \vdots & \ddots & \vdots \\ v - c_1 & v - c_2 & \cdots & \frac{v}{2} - c_n \end{pmatrix}.$$

Reasoning as in Example 2.3, we consider the symmetric matrix

$$\hat{A} = A + A' = v\mathbf{1}\mathbf{1}' - 2\begin{pmatrix} c_1 & c_1 & \cdots & c_1 \\ c_1 & c_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_n \end{pmatrix} = v\mathbf{1}\mathbf{1}' - 2C,$$

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where the matrix C can be decomposed as

$$C = \begin{pmatrix} c_1 & c_1 & \cdots & c_1 \\ c_1 & c_1 & \cdots & c_1 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_1 & \cdots & c_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & c_2 - c_1 & \cdots & c_2 - c_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_2 - c_1 & \cdots & c_2 - c_1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n - c_{n-1} \end{pmatrix}.$$

Thus, if $z \in TX$, then

$$\begin{aligned} z'\hat{A}z &= vz'\mathbf{1}\mathbf{1}'z - 2z'Cz \\ &= (v - 2c_1)z'\mathbf{1}\mathbf{1}'z - 2\sum_{k=2}^{n}\sum_{i=k}^{n}\sum_{j=k}^{n}(c_k - c_{k-1})z_i z_j \\ &= -2\sum_{k=2}^{n}(c_k - c_{k-1})\left(\sum_{i=k}^{n} z_i\right)^2 \\ &\leq 0, \end{aligned}$$

so $F(x) = Ax$ is a stable game.

O.2 Cycling in Stable Games

Proposition O.2.1. *Consider the EPT dynamic* (E) *generated by revision protocol* (9) *in standard Rock-Paper-Scissors.*

(*i*) *When $\varepsilon < .1094$, there are initial conditions from which solutions to* (E) *converge to periodic orbits.*

(*ii*) *Fix $\delta > 0$. When ε is sufficiently small, solutions to* (E) *from all initial conditions that are not within δ of the equilibrium x^* converge to periodic orbits.*

For intuition, consider Figure 1, which presents a portion of a solution to the dynamic (E) generated by (9) in standard RPS when $\varepsilon = \frac{1}{10}$. Scissors earns a positive payoff as soon as this trajectory crosses segment ax^*, and becomes the sole strategy that does so once segment $e_P x^*$ is reached. However, protocol (9) puts very little probability on Scissors until Paper, the strategy it beats, yields a payoff close to zero. As a result, the solution heads almost directly towards state e_P until Scissors becomes the sole strategy earning a payoff of ε. This extends the phase during which the solution approaches the vertex e_P before turning towards e_S. By symmetry, the same phenomenon occurs near the other two vertices, and as a result, the solution never strays far from the boundary of the simplex.

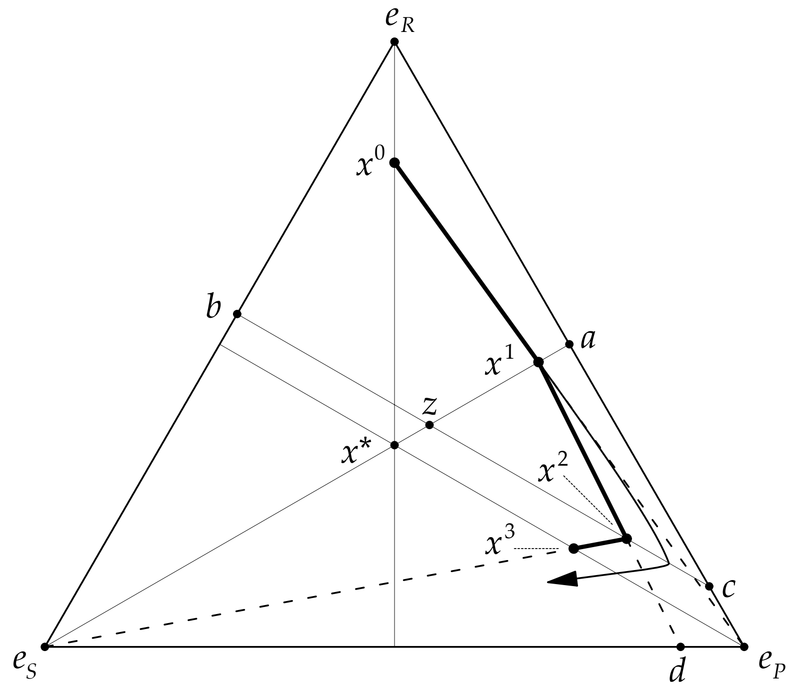


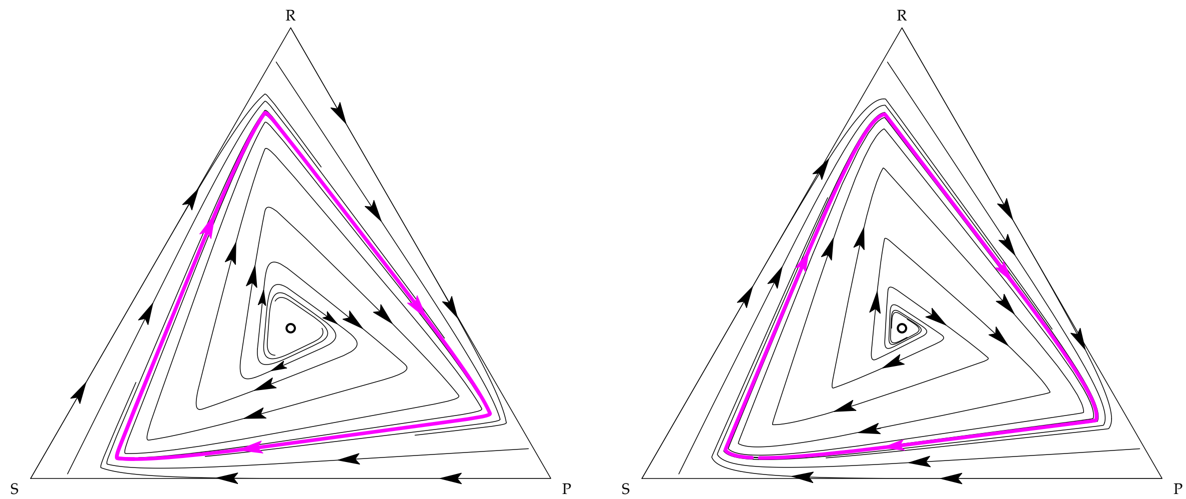
Figure 1: The proof of Proposition O.2.1.

Considering a zero-sum game simplifies the proof of the existence of cycles, but is not necessary for the result to hold: cycles occur under this dynamic even in strictly stable games. In Figure 2, we present solutions to the dynamic generated by protocol (9) with $\varepsilon = \frac{1}{10}$ in both standard RPS ($w = 1$, $l = 1$) and good RPS ($w = 3$, $l = 2$). In each case, convergence to a periodic orbit occurs from most initial conditions.

Proof of Proposition O.2.1. Since standard RPS is zero-sum, we have that $\hat{F}(x) = F(x) - \mathbf{1}x'F(x) = F(x)$: excess payoffs and original payoffs are always the same. This fact simplifies the analysis below.

Consider the trajectory that starts from some initial state $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ that lies on segment $e_R x^*$ and satisfies $\alpha > \underline{\alpha} = \frac{1+\varepsilon}{3-3\varepsilon}$ (see Figure 1). This trajectory travels clockwise around the simplex. Our main task is to obtain an lower bound on the distance of this solution from state x^* when the solution crosses segment $e_P x^*$. Doing so enables us to bound the action of the Poincaré map of the dynamic on $e_R x^*$, which in turn lets us use the Poincaré-Bendixson Theorem to demonstrate the existence of a periodic orbit.

When the current state lies in the triangle with vertices e_R, x^*, and $a = (0, \frac{1}{2}, \frac{1}{2})$, as it does at x^0, only strategy P has a positive payoff, so the target state under dynamic V is $\tau(F(x)) = e_P$. Therefore, the trajectory from x^0 leaves triangle $e_R x^* a$ at state $x^1 = (\frac{2\alpha}{1+3\alpha}, \frac{1-\alpha}{1+3\alpha},$



(i) standard RPS ($w = 1$, $l = 1$)　　(ii) good RPS ($w = 3$, $l = 2$)

Figure 2: Cycling in standard and good Rock-Paper-Scissors games.

$\frac{2\alpha}{1+3\alpha}$). Since $\alpha > \underline{\alpha} = \frac{1+\varepsilon}{3-3\varepsilon}$, x^1 lies on the interior of segment az, where $z = (\frac{1+\varepsilon}{3}, \frac{1-2\varepsilon}{3}, \frac{1+\varepsilon}{3})$. For future reference, we observe that z is the intersection of segments $ax^*$and bc, where $b = (\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}, 0)$ and $c = (\varepsilon, 0, 1-\varepsilon)$.

In triangle $e_P x^* a$, only strategies P and S earn positive payoffs. By construction, $\tau_S(F(x)) = \varepsilon^2[F_S(x)]_+$ as long as the payoff to P is at least ε, which is the case in triangle $e_R bc$. The intersection of these two triangles is the triangle azc. When the current state x is in this region, the target state is always a point $(0, \tau_P(F(x)), \tau_R(F(x)))$ at which

$$\begin{aligned}\tau_S(F(x)) &= \frac{\tau_S(F(x))}{\tau_S(F(x)) + \tau_P(F(x))} \\ &= \frac{[F_S(x)]_+ g^\varepsilon(F_P(x))}{[F_S(x)]_+ g^\varepsilon(F_P(x)) + [F_P(x)]_+ g^\varepsilon(F_R(x))} \\ &\leq \frac{1 \times \varepsilon^2}{(1 \times \varepsilon^2) + (\varepsilon \times 1)} \\ &= \frac{\varepsilon}{\varepsilon + 1}.\end{aligned}$$

Now the ray from point x^1 through point $d = (0, \frac{\varepsilon}{1+\varepsilon}, \frac{1}{1+\varepsilon})$ intersects segment bc at $x^2 = (\frac{2\alpha\varepsilon(2+\varepsilon)}{3\alpha(1+2\varepsilon)-1}, \frac{\varepsilon(1+\alpha-4\alpha\varepsilon)}{3\alpha(1+2\varepsilon)-1}, \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{3\alpha(1+2\varepsilon)-1})$. Hence, the inequality above implies that the solution trajectory from x^1 (and hence the one from x^0) hits segment zc at a point between x^2 and c.

Finally, consider the behavior of solution trajectories passing through the polygon $ce_P x^* z$. In this region, the target point is always on segment $e_S e_P$. In fact, once the solution hits segment $e_P x^*$, strategy S becomes the sole strategy earning a positive payoff, so the target point must be e_S. Thus, the solution starting from x^2 must hit $e_P x^*$ no closer to x^* than $x^3 = (\frac{2\alpha\varepsilon(2+\varepsilon)}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}, \frac{2\alpha\varepsilon(2+\varepsilon)}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}, \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1})$, the point where a ray from x^2 through e_S crosses segment $e_P x^*$. Since the solution starting from x^0 hits segment zc to the right of x^2, it too must hit $e_P x^*$ to the right of x^3. We have thus established a lower bound of $\beta(\alpha) = \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}$ on the value of x_P at the point where the solution starting from $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ intersects segment $e_P x$.

The function β is an increasing hyperbola whose asymptotes lie at $\alpha = \frac{1}{3+9\varepsilon+6\varepsilon^2}$ and $\beta = \frac{3+\varepsilon+2\varepsilon^2}{3+9\varepsilon+6\varepsilon^2}$. It intersects the 45° line at

$$\alpha_\pm = \frac{2+\varepsilon+\varepsilon^2 \pm \sqrt{1-8\varepsilon-10\varepsilon^2-4\varepsilon^3+\varepsilon^4}}{3+9\varepsilon+6\varepsilon^2}.$$

whenever the expression under the square root is positive. This is true whenever $\varepsilon < .1094$. In this case, $(\alpha_-, \alpha_+) \subset (\frac{1}{3}, 1)$, and β is above the 45° line on the former interval. Hence, any solution that begins at a point $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ with $\alpha > \max\{\underline{\alpha}, \alpha_-\}$ will hit segment $e_P x^*$ at some point y with $y_P > \beta(\alpha) \in (\alpha, \alpha_+)$. It then follows from the symmetry of the game and of the choice rule that that the region bounded on the inside by the solution from x^0 to y, its 120° and 240° rotations about x^*, and the pieces of $e_P x^*$, $e_S x^*$, and $e_R x^*$ that connect the three solutions, and on the outside by the boundary of X is a trapping region for the dynamic V. By Proposition 4.1, the only rest point of the dynamic is the Nash equilibrium x^*, which lies outside of this region. Therefore, the Poincaré-Bendixson Theorem (Hirsch and Smale (1974, Theorem 11.4)) implies that every solution with an initial condition in the region converges to a periodic orbit. If we take ε to zero, $\underline{\alpha}$ and α_- approach $\frac{1}{3}$, which implies that the radius of the ball around x^* from which convergence to a periodic orbit is not guaranteed vanishes. This completes the proof of the proposition. ■

References

Hirsch, M. W. and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, San Diego.