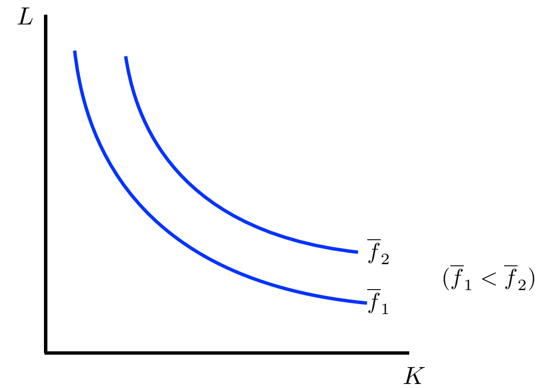
**Problem Set 7: Solutions**

ECON 301: Intermediate Microeconomics
Prof. Marek Weretka

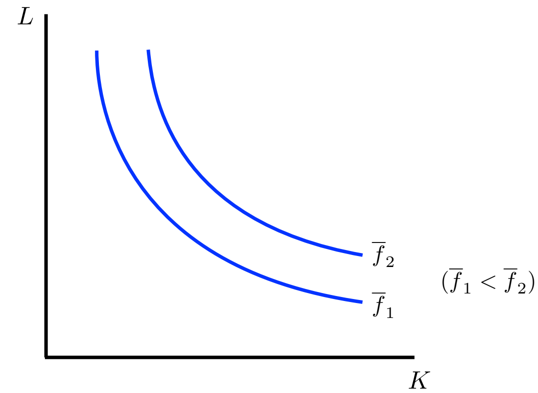
## Problem 1 (Production Functions)

**(a)** The isoquants for each of the three production functions are show below:

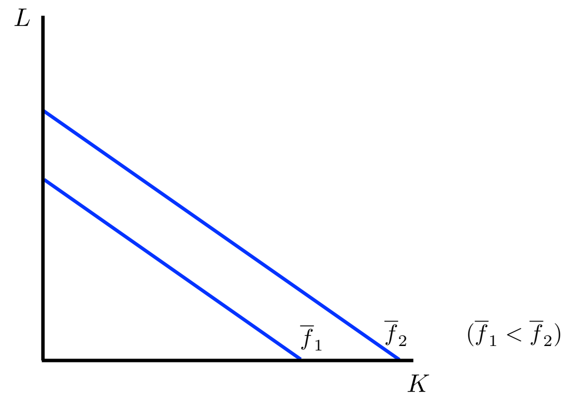
- $f(K,L) = K^2L$




- $f(K,L) = K^{\frac{1}{3}}L^{\frac{1}{3}}$




- $f(K,L) = 2K + L$



**(b)** The marginal productivity of capital, $MP_K$, tells us by how many units output would increase if capital input were increase by one unit (machine). (Mathematically, $MP_K$ is the partial derivative of the production function; the larger the change in capital the further the approximation gets from actual changes in output.)

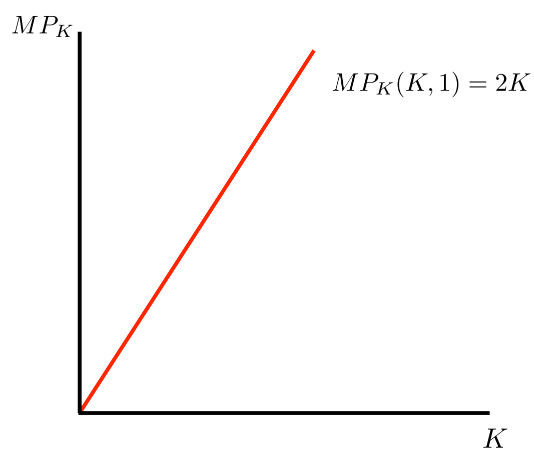
The marginal productivity of labor, $MP_L$, tells us howe much additional output we get from increasing labor input by one unit (worker).

**(c)** Marginal productivity of capital with $\overline{L} = 1$:

- $MP_K$ when $f(K, L) = K^2 L$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = 2KL \quad (MP_K \text{ is increasing in } K)$$

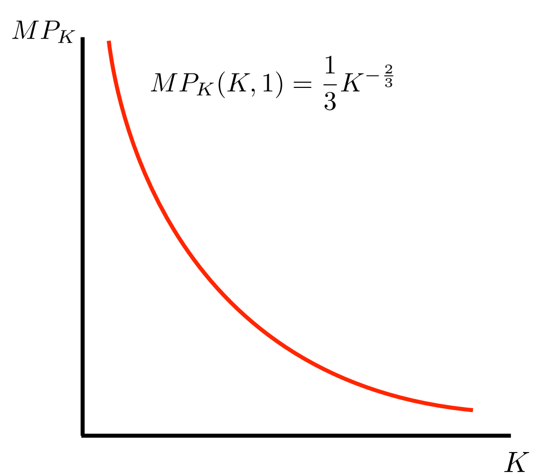
With $\overline{L} = 1$, $MP_K = 2K$:




- $MP_K$ when $f(K, L) = K^{\frac{1}{3}} L^{\frac{1}{3}}$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = \frac{1}{3} K^{-\frac{2}{3}} L^{\frac{1}{3}} \quad (MP_K \text{ is decreasing in } K)$$

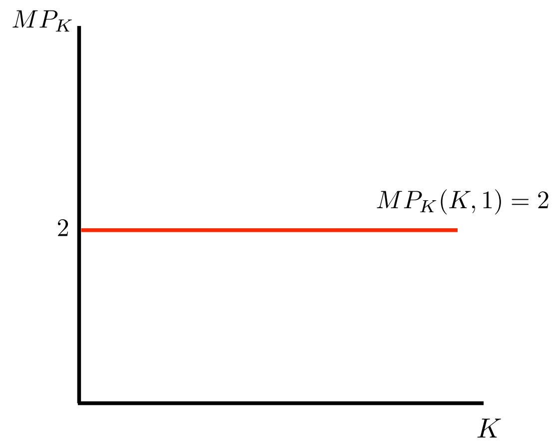
With $\overline{L} = 1$, $MP_K = \frac{1}{3} K^{-\frac{2}{3}}$:



- $MP_K$ when $f(K, L) = 2K + L$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = 2 \quad (MP_K \text{ is constant in } K)$$

With $\overline{L} = 1$, $MP_K = 2$:

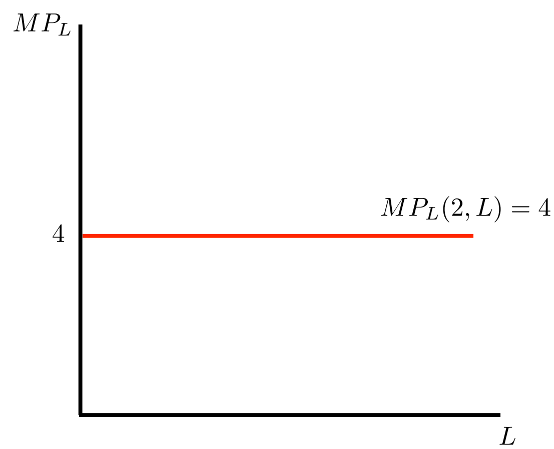



**(d)** Marginal productivity of labor with $\overline{K} = 2$:

- $MP_L$ when $f(K, L) = K^2 L$:

$$MP_L = \frac{\partial f(K, L)}{\partial L} = K^2 \quad (MP_L \text{ is constant in } L)$$

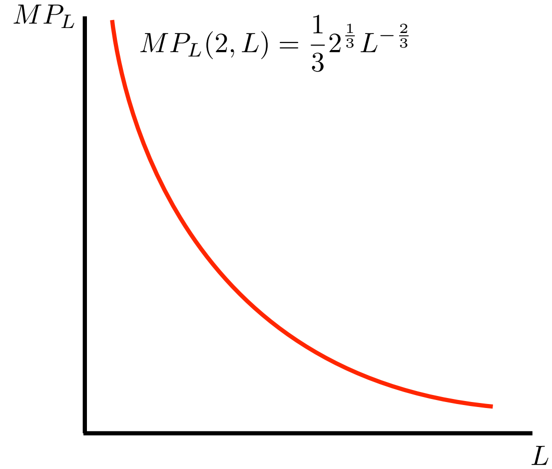
With $\overline{K} = 2$, $MP_L = 4$:



- $MP_L$ when $f(K,L) = K^{\frac{1}{3}}L^{\frac{1}{3}}$:

$$MP_L = \frac{\partial f(K,L)}{\partial L} = \frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}} \quad (MP_L \text{ is decreasing in } L)$$

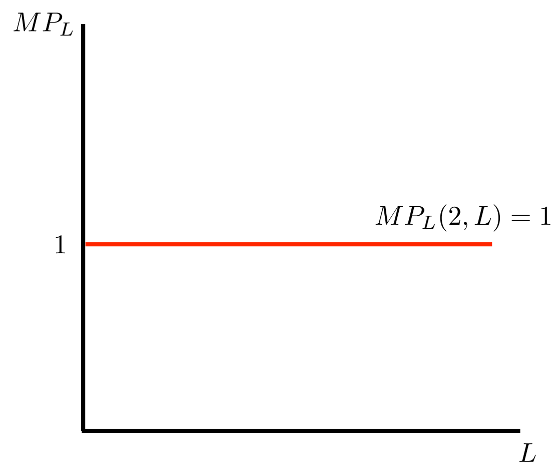
With $\overline{K} = 2$, $MP_L = \frac{1}{3}2^{\frac{1}{3}}L^{-\frac{2}{3}}$:




- $MP_L$ when $f(K,L) = 2K + L$:

$$MP_L = \frac{\partial f(K,L)}{\partial L} = 1 \quad (MP_L \text{ is constant in } L)$$

With $\overline{K} = 2$, $MP_L = 1$:




**(e)** Returns to scale:

- Constant Returns to Scale (CRS), $f(\lambda K, \lambda L) = \lambda f(K,L)$: The means that doubling all inputs leads to a doubling of output (or tripling inputs triples outputs, etc.) An example might be pastry making at a bakery, where twice as much of all inputs ($L$: pastry chefs, $K$: countertops, ovens, and butter, flour, eggs, etc.) leads to twice as much output (the pastries). Also, the Varian textbook mentions data centers: A thousand times as many data centers (inputs) leads to a thousand times as many webpages served (output).

- Decreasing Returns to Scale (DRS), $f(\lambda K, \lambda L) < \lambda f(K, L)$: A doubling of inputs results in *less than* double the output. As the Varian text notes, DRS is usually a short-run phenomenon where in fact there is some other input that *is* held fixed (otherwise a firm could at least replicate a process and achieve CRS). For instance, in farming, a doubling of capital equipment and labor does not lead to a doubling of output so in that case we'd say there is DRS, but this is really because one of the inputs—land—might actually be fixed.

- Increasing Returns to Scale (IRS), $f(\lambda K, \lambda L) > \lambda f(K, L)$: A doubling of inputs results in *more than* a doubling of output. Again, here Varian gives a nice example: An oil pipeline. "If we double the diameter of a pipe, we use twice as much materials, but the cross section of the pipe goes up by a factor of 4. Thus we will likely be able to pump more than twice as much oil through it [up to a certain point]."

**(f)** Let's see whether our three production functions exhibit CRS, DRS, or IRS for $\lambda > 1$:

- $f(K, L) = K^2 L$:

$$f(\lambda K, \lambda L) = (\lambda K)^2 \lambda L = \lambda^3 f(K, L) > \lambda f(K, L) \implies IRS$$

- $f(K, L) = K^{\frac{1}{3}} L^{\frac{1}{3}}$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}} (\lambda L)^{\frac{1}{3}} = \lambda^{\frac{1}{3}+\frac{1}{3}} K^{\frac{1}{3}} L^{\frac{1}{3}} = \lambda^{\frac{2}{3}} f(K, L) < \lambda f(K, L) \implies DRS$$

- $f(K, L) = 2K + L$:

$$f(\lambda K, \lambda L) = 2(\lambda K) + \lambda L = \lambda(2K + L) = \lambda f(K, L) \implies CRS$$

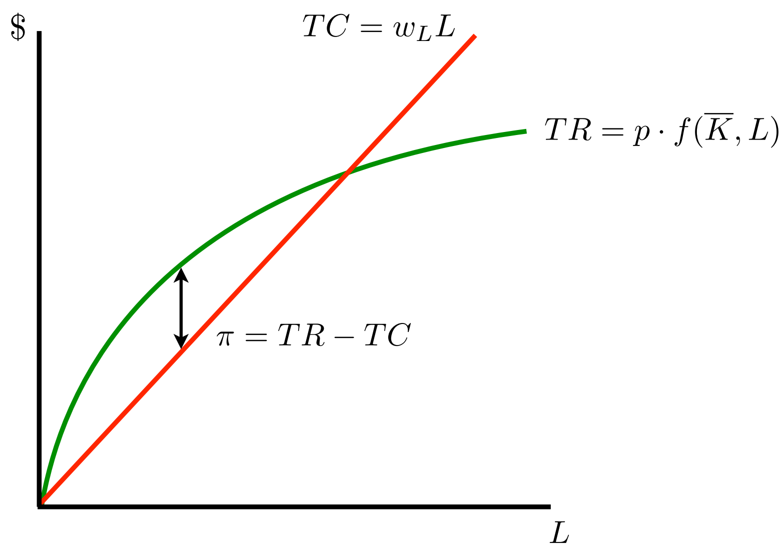
**Problem 2 (Profit Maximization in the Short Run)**

**(a)** The profit of GMC is total revenue ($p \cdot f(\overline{K}, L)$) minus cost ($w_L L$):

$$\pi = p \cdot f(\overline{K}, L) - w_L \cdot L \text{ and } \overline{K} = 16 \implies \pi = 8pL^{\frac{1}{2}} - w_L L\,.$$

Since capital is fixed, we are in the short run and costs include only the variable costs $w_L L$.

**(b)** Total revenue, $p \cdot f(\overline{K}, L)$, and labor cost, $w_L L$, are shown below for $p = 1$ and $w_L = 2$:




**(c)** A well-behaved function $\pi(x)$ is flat at the point at which it attains a local maximum (increasing to the left, flat, then decreasing to the right). Since the derivative is zero when $\pi(x)$ is flat, finding the $x$ at which $\pi'(x) = 0$ tells us where a local maximum is. This is what we call the *first-order condition*. (We can assume for the profit functions we'll be working with that there is only one local maximum and that it is the global maximum.) Warning: A function is also flat where it attains a minimum, therefore we should check whether actually our $x$ is not minimizing the value of the function (this is the *second-order condition*: $\pi''(x) > 0$ means it's a minimum, $\pi''(x) < 0$ means it's a maximum). This won't be an issue in our application to maximization of profit function though.

**(d)** Setting the derivative of the profit function to zero we have

$$\frac{\partial \pi}{\partial L} = 0 \implies p\frac{\partial f(\overline{K}, L)}{\partial L} - w_L = 0 \implies MP_L = \frac{w_L}{p}\,.$$

Alternatively, we can see this using the production function $f(\overline{K}, L) = 8L^{\frac{1}{2}}$ for $\overline{K} = 16$. We then have $\pi = 8pL^{\frac{1}{2}} - w_L L$, so

$$\frac{\partial \pi}{\partial L} = \frac{1}{2}8pL^{-\frac{1}{2}} - w_L$$

and setting this equal to zero (our first order condition), we get

$$\frac{\partial \pi}{\partial L} = 0 \implies \frac{1}{2} 8pL^{-\frac{1}{2}} - w_L = 0 \implies 4L^{-\frac{1}{2}} = \frac{w_L}{p}. \tag{1}$$

Since the marginal product of labor is $MP_L = \frac{\partial f(\overline{K}, L)}{\partial L} = 4L^{-\frac{1}{2}}$, in equation (1) we in fact found the condition that $MP_L = \frac{w_L}{p}$.

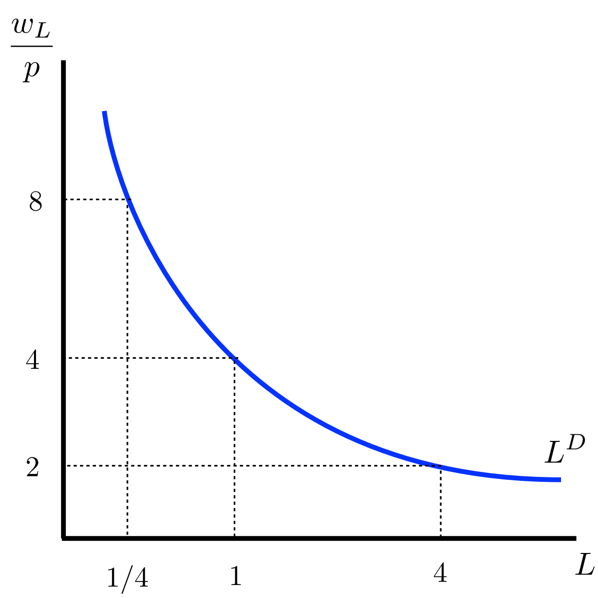
The intuition is that a firm should hire as long as marginal benefits (additions to output $MP_L$) are greater than the marginal costs of doing so (real wage $\frac{w_L}{p}$), up to the point where additional benefits and costs are exactly equal ($MP_L = \frac{w_L}{p}$). Past this point, a firm shouldn't hire any more labor since $MP_L < \frac{w_L}{p}$ (since $MP_L$ is always decreasing).

**(e)** To find the optimal level of labor, we can use the condition we found in part **(d)** in equation (1): $MP_L = \frac{w_L}{p}$ or $4L^{-\frac{1}{2}} = \frac{w_L}{p}$. Solving for $L$ we get the labor demand curve:

$$L^D = \left(\frac{4p}{w_L}\right)^2$$

- For $p = 1, w_L = 8$, we have $L^* = \left(\frac{4 \cdot 1}{8}\right)^2 = \frac{1}{4}$
- For $p = 1, w_L = 4$, we have $L^* = \left(\frac{4 \cdot 1}{4}\right)^2 = 1$
- For $p = 1, w_L = 2$, we have $L^* = \left(\frac{4 \cdot 1}{2}\right)^2 = 4$

These points are shown in the graph below:

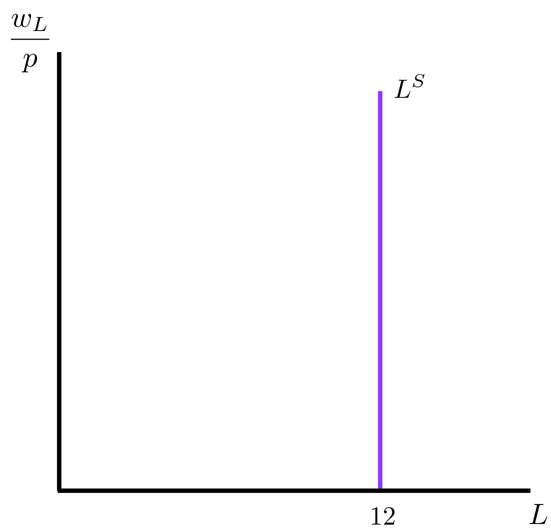


**(f)** We know from part **(a)** what profit is associated with any $p$, $w_L$, and $L^*$: $\pi = 8pL^{\frac{1}{2}} - w_L L$, so we have:

- For $p = 1, w_L = 8,\ L^* = \frac{1}{4}$, we have $\pi = 8(1)(\frac{1}{4})^{\frac{1}{2}} - (8)(\frac{1}{4}) = 2$
- For $p = 1, w_L = 4,\ L^* = 1$, we have $\pi = 8(1)(1)^{\frac{1}{2}} - (4)(1) = 4$
- For $p = 1, w_L = 2,\ L^* = 4$, we have $\pi = 8(1)(4)^{\frac{1}{2}} - (2)(4) = 8$

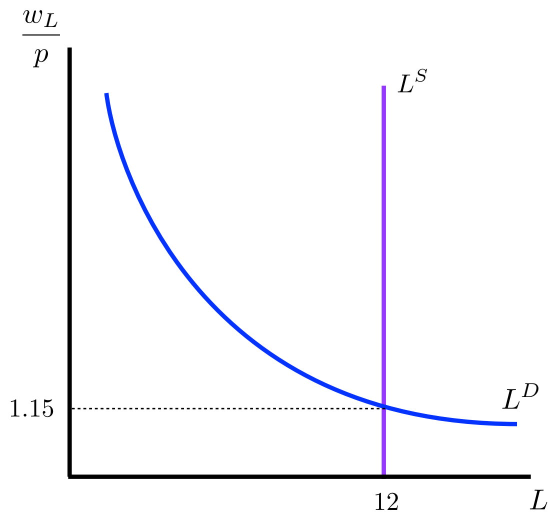
**Problem 3 (Labor Market)**

**(a)** Kate's (perfectly inelastic) labor supply, $L^S = 12$ is shown below:




**(b)** We had that labor demand was given by $L^D = \left(\frac{4p}{w_L}\right)^2$. We get the equilibrium wage rate by equating $L^S = L^D$ and solving for $\frac{w_L}{p}$:

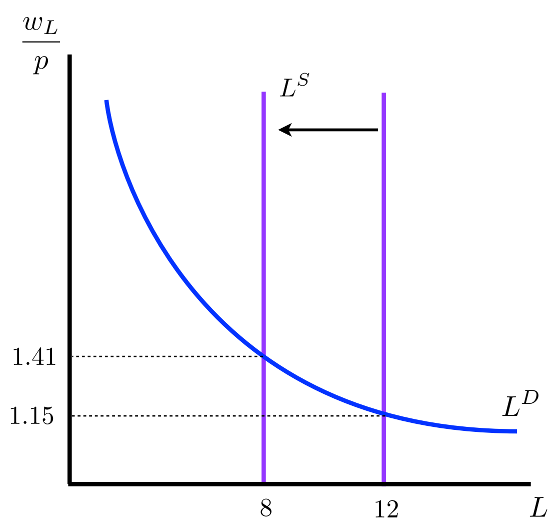
$$L^S = L^D \implies 12 = \left(\frac{4p}{w_L}\right)^2 \implies \frac{w_L}{p} = \left(\frac{4}{3}\right)^{\frac{1}{2}} \approx 1.15$$



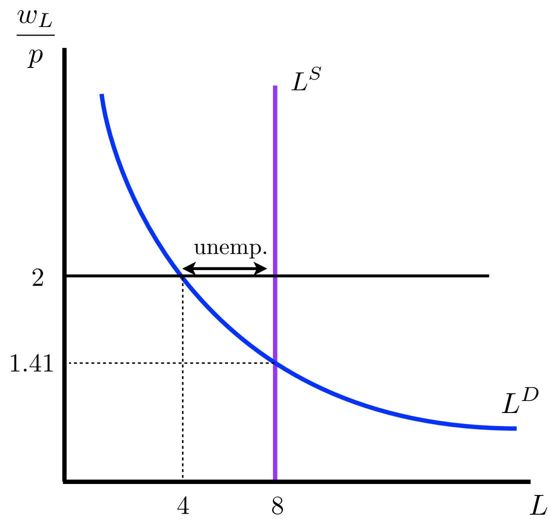

**(c)** At a hypothetical wage above what we found in part **(b)**, the hours of labor demanded is less than the supply at that wage of $L^S = 12$. This excess supply is unemployment. Since there is willingness to work at lower wages, the wage offered would fall, bringing the excess supply (unemployment) to zero. (Same is true at a point below the equilibrium wage we found: There would be excess demand, so to attract more workers the wages would be bid up to the point where there is no excess demand.)

**(d)** Now with $L^S = 12$, equating $L^S = L^D$ and solving for $\frac{w_L}{p}$ we get:

$$L^S = L^D \implies 8 = \left(\frac{4p}{w_L}\right)^2 \implies \frac{w_L}{p} = 2^{\frac{1}{2}} \approx 1.41$$




**(e)** At this price floor of $\frac{w_L}{p} = 2$, we have that $L^S = 8$ (unchanged) but now $L^D = \left(4\frac{p}{w_L}\right)^2 = \left(4\frac{1}{2}\right)^2 = 4$. The unemployment rate is now $\frac{L^S - L^D}{L^S} = \frac{8-4}{8} = .5$ or 50% unemployment. (The unemployment rate was previously zero: $\frac{L^S - L^D}{L^S}$=0 since in the market we have $L^S = L^D = 8$.




**Problem 4 (The Long Run)**

**(a)** To determine the returns to scale, we must compare $f(\lambda K, \lambda L)$ to $\lambda f(K, L)$ for any number $\lambda > 1$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} f(K, L) < \lambda f(K, L) \quad \Longrightarrow \quad DRS$$

So since $f(\lambda K, \lambda L) < \lambda f(K, L)$, this function exhibits decreasing returns to scale.

**(b)** The profit function in terms of $K$ and $L$ is given by:

$$\pi = p \cdot f(K, L) - (w_L \cdot L + w_K \cdot K)\,.$$

With $p = 1$, $w_K = 2$, and $w_L = 1$,

$$\pi = f(K, L) - (L + 2K)\,.$$

**(c)** First, we'll find the optimal combination of inputs $K$ and $L$. From our profit function above, setting the partial derivatives with respect to $K$ and $L$, we get secrets of happiness

$$MP_K = \frac{w_K}{p} \quad \text{and} \quad MP_L = \frac{w_L}{p}$$

and substituting in the marginal productivities of capital and labor as well as prices, this is equivalent to

$$\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{1}{3}} = 2 \quad \text{and} \quad \frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}} = 1\,. \tag{2}$$

Dividing the first equation by the second, we get

$$\frac{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{1}{3}}}{\frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}}} = \frac{2}{1} \quad \Longrightarrow \quad \frac{L}{K} = 2 \quad \Longrightarrow \quad L = 2K\,,$$

so we will be using $K$ and $L$ such that $L = 2K$.

Plugging $L = 2K$ into the first equation in (2), we have

$$\frac{1}{3}K^{-\frac{2}{3}}(2K)^{\frac{1}{3}} = 2 \quad \Longrightarrow \quad K = \left(3 \times 2^{\frac{2}{3}}\right)^{-3} = \frac{1}{108}$$

and so

$$L = 2K \quad \Longrightarrow \quad L = 2 \times \frac{1}{108} = \frac{1}{54}\,.$$

Given these two values, the optimal level of output is

$$y = f(K, L) = \left(\frac{1}{108}\right)^{\frac{1}{3}} \left(\frac{1}{54}\right)^{\frac{1}{3}} = \frac{1}{18}\,.$$

and the profit associated with this level of output and the prices given is

$$\pi = p \cdot f(K, L) - (w_L \cdot L + w_K \cdot K) = 1 \cdot \frac{1}{18} - (2 \cdot \frac{1}{108} + 1 \cdot \frac{1}{54}) = \frac{1}{54}\,.$$

**(d)** The condition for cost minimization is

$$TRS = -\frac{w_K}{w_L}$$

Since for the technical rate of substitution $TRS$ we have

$$TRS = -\frac{MP_K}{MP_L} = -\frac{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{1}{3}}}{\frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}}} = -\frac{L}{K} = -\frac{1/54}{1/108} = -2$$

and

$$-\frac{w_K}{w_L} = -\frac{2}{1} = -2$$

we indeed are satisfied the cost minimization condition $TRS = -\frac{w_K}{w_L}$.