Problem Set 5: Solutions

ECON 301: Intermediate Microeconomics
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Problem 1 (Intertemporal Choice)

(a) The present value of Gerald's stream of income over the two periods is

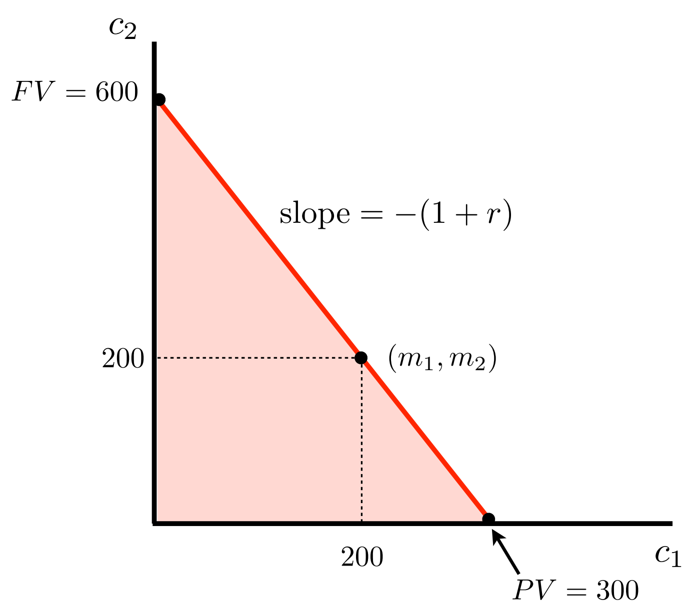
$$PV = m_1 + \frac{m_2}{1+r} \quad \Longrightarrow \quad PV = 200 + \frac{200}{1+1} = 300$$

and in terms of future value the stream of income is worth

$$FV = (1+r)m_1 + m_2 \quad \Longrightarrow \quad FV = (1+1)200 + 200 = 600\,.$$

Note that $PV = \frac{FV}{1+r}$ and $FV = (1+r)PV$.

(b) Gerald's budget set in (c_1,c_2)-space is shown below, with PV and FV points labeled. The PV point coincides with consuming (the present value of) everything in period one and nothing in period two. The FV point represents the point where Gerald consumes nothing in period one and (the future value of) everything in period two.



The slope of the budget line is $-(1+r)$. (The slope of the budget line is $-\frac{p_1}{p_2}$; here $p_1 = 1$ and $p_2 = \frac{1}{1+r}$.)

(c) The two "extreme" consumption points (which are where the budget line intercepts the c_1- and c_2-axes) have the following interpretations:

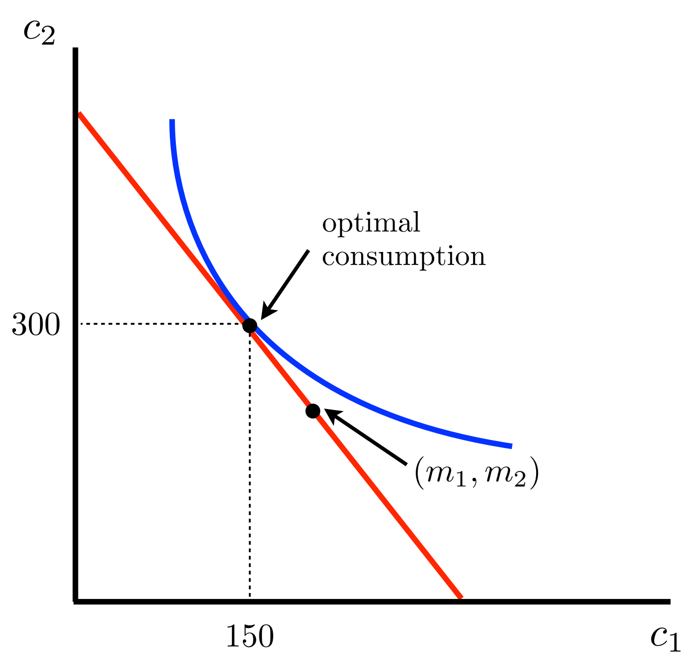
Point $(300, 0)$ on the c_1-axis represents a situation in which Gerald consumes all of his period one income of $m_1 = \$200$ *plus* and additional \$100, the maximum amount he could borrow, which must be payed back in period two at the $r = 100\%$ rate. All of his period two income goes to repaying this and he consumes nothing in the second period ($m_2 = \$200$ and he uses this to repay the $100 \times (1 + 1) = 200$ borrowed in the first period).

Point $(0, 600)$ on the c_2-axis represents the situation where Gerald consumes nothing in the first period and saves all of his m_1 income, earning a return of $r = 100\%$ in the second period. In the second period he consumes all of his $m_2 = \$200$ income and the amount saved with interest from the first period, $m_1(1 + r) = \$200 \times (1 + 1) = \$400$ for a total consumption of \$600 in the second period.

(d) We can always take a monotonic transformation of the utility function to get one that is in Cobb-Douglas form with $a = 1$ and $b = 1$ and use the "magic formulas" to find demand for c_1 and c_2 ($c_1 = \frac{a}{a+b}\frac{m}{p_1}$ and $c_2 = \frac{b}{a+b}\frac{m}{p_1}$). Alternatively, we could derive demand from the utility function given using the two "secrets of happiness" for well-behaved preferences.

In any case we will have (since $p_1 = 1$, $p_2 = \frac{1}{1+r}$, and $m = PV = 300$):

$$c_1 = \frac{1}{2}\frac{300}{1} = 150 \quad \text{and} \quad c_2 = \frac{1}{2}\frac{300}{\left(\frac{1}{1+1}\right)} = 300\,.$$



Here, Gerald is consuming \$150 saving \$50 in period one (consumes \$50 less than what his income in period one is: $S = m_1 - c_1 = \$200 - \$150 = \$50$). The \$50 he saves gives him $\$50 \times (1 + 1) = \$100$ in period two, and he consumes $m_2 + \$100 = \$300$ in period two.

Problem 2 (Intertemporal Choice)

(a) The coefficient δ is the discount rate which signifies how impatient the agent is. The greater δ the more impatient the agent, giving smaller weight to utility from future consumption (high $\delta \implies$ low $\frac{1}{1+\delta}$).

(b) If the income stream for the manager is $m_1 = 0$ and $m_2 = 3,000$, then the present value of the stream is $PV = 0 + \frac{3,000}{1+1} = 1,500 = m$. As in Problem 1, we can use the Cobb-Douglas magic formulas for demand with $a = 1$ and $b = \frac{1}{1+\delta}$ to get

$$c_1 = \frac{1}{1 + \frac{1}{1+\delta}} \cdot \frac{m}{p_1} = \frac{1+\delta}{2+\delta} \cdot 1,500 = 1,000$$

and

$$c_2 = \frac{\frac{1}{1+\delta}}{1 + \frac{1}{1+\delta}} \cdot \frac{m}{p_2} = \frac{1+r}{2+\delta} \cdot 1,500 = 1,000\,.$$

His savings in the first period is $S = m_1 - c_1 = 0 - 1,000 = -1,000$. So he borrows \$1,000 the first period (and repays $(1+r) \times 1,000 = \$2,000$ in period two).

(c) If the income stream for the sportsman is $m_1 = 1,500$ and $m_2 = 0$, we have $PV = 1,500 = m$ and the same formulas we used in part **(b)** give

$$c_1 = \frac{1}{1 + \frac{1}{1+\delta}} \cdot \frac{m}{p_1} = \frac{1+\delta}{2+\delta} \cdot 1,500 = 1,000$$

and

$$c_2 = \frac{\frac{1}{1+\delta}}{1 + \frac{1}{1+\delta}} \cdot \frac{m}{p_2} = \frac{1+r}{2+\delta} \cdot 1,500 = 1,000\,.$$

Savings in the first period is now $S = m_1 - c_1 = 1,500 - 1,000 = 500$.

(d) Both the manager and the sportsman consume such that $c_1 = c_2$ and hence perfectly smooth their consumption over the two periods.

(e) From the second secret of happiness for well-behaved preferences:

$$MRS = -\frac{p_1}{p_2} \implies \frac{1/c_1}{\frac{1}{1+\delta}(1/c_2)} = -\frac{1}{\left(\frac{1}{1+r}\right)}$$

Rearranging this equation to solve for c_1, we have

$$c_2 = \frac{(1+r)}{(1+\delta)} c_1 \,.$$

When $r < \delta$, we have that $\frac{(1+r)}{(1+\delta)} < 1$, which implies that $c_2 < c_1$ (consumption is decreasing over time).

Problem 3 (Annuity and Perpetuity)

(a) A perpetuity gives amount x in each period, and hence its present value is given by

$$PV = \frac{x}{1+r} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \cdots$$

we can rewrite this as

$$PV = \frac{x}{1+r} + \frac{1}{1+r}\left[\frac{x}{1+r} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \cdots\right] .$$

The sum of the elements in the bracket is equal to the present value of the perpetuity and so

$$PV = \frac{x}{1+r} + \frac{1}{1+r}\left[PV\right] .$$

Solving for PV gives

$$PV - \frac{1}{1+r} PV = \frac{x}{1+r}$$

which gives

$$PV^{perp} = \frac{x}{r} \,.$$

(b) The cash flow of an annuity differs from that of a perpetuity in that there are no payments x after terminal period T.

The present value *at time* T of the future payment left in a perpetuity is $PV_T^{perp} = \frac{x}{r}$. These payments will be missing from the perpetuity. The present value in period one of PV_T^{perp} is $PV = \left(\frac{1}{1+r}\right)^T PV_T^{perp} = \left(\frac{1}{1+r}\right)^T \left(\frac{x}{r}\right)$. We subtract this amount off from the value of the perpetuity to get the value of the annuity:

$$PV^{ann} = PV^{perp} - \left(\frac{1}{1+r}\right)^T \left(\frac{x}{r}\right) = \frac{x}{r}\left[1 - \left(\frac{1}{1+r}\right)^T\right] .$$

Problem 4 (Present Value, use a calculator)

(a) To decide whether to buy or rent (forever), we will compare the present value of our perpetual rent payment of \$500 with $r = .01$ to \$600,000. Since $PV^{rent} = \frac{x}{r} = \frac{500}{.001} = 500,000 < 600,000$. You would be better off renting the apartment (which in present value will cost you \$500,000) rather than purchasing the apartment today for \$600,000.

(b) The present value of the payment must coincide with the size of the loan, hence

$$\$4,000 = PV = \frac{x}{.005}\left(1 - \left(\frac{1}{1.005}\right)^{36}\right) \implies x = \$121.69\,.$$

(c) The present value of the bond described is

$$PV = \frac{\$100}{.1}\left(1 - \left(\frac{1}{1.1}\right)^{9}\right) + \frac{\$1,000}{(1.1)^{10}} = \$961.45\,.$$

Because the present value of the bond is greater than the price, it is a good idea to purchase the bond. (Note: If we had assumed that the bond also pays the coupon amount c in the last period, as is often the case in finance, the present value of the bond would be \$1,000. The Varian textbook does not make this assumption.)

(d) We want to save such that $PV(S) = PV(C)$. The present value of the consumption of \$40,000 every year for 20 year beginning 40 years from today is (rounding)

$$PV(C) = \left(\frac{1}{1.05}\right)^{40}\left(\frac{\$40,000}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{20}\right) = \$70,808\,.$$

The present value of saving S for 40 years beginning today is

$$PV(S) = \left(\frac{\$S}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{40}\right) = \$17,159 \times S\,.$$

Equating $PV(S) = PV(C)$ and solving for S gives $S = \$4,127$ must be be the yearly savings amount with $r = 5\%$.

(e) Now we have that the present value of consumption is

$$PV(C) = \left(\frac{1}{1.05}\right)^{40}\left(\frac{\$C}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{20}\right) = \$1,770 \times C\,.$$

The present value of saving \$20,000 for 40 years beginning today is

$$PV(S) = \left(\frac{\$20,000}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{40}\right) = \$343,180\,.$$

Equating $PV(S) = PV(C)$ and solving for C gives $C = \$193,870$ could be consumed every year with $S = \$20,000$ and $r = 5\%$ annually.

Problem 5 (Life-Cycle Problem)

(a) With $r = 5\%$, we want to find the C over 60 years that satisfies $PV(C) = PV(\text{income})$ where income \$200,000 is earned annually for 40 years:

$$PV(C) = \left(\frac{\$C}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{60}\right) = \$18,929 \times C$$

$$\text{and} \quad PV(\text{income}) = \left(\frac{\$200,000}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{40}\right) = \$3,431,800\,.$$

Solving for C from the two equations we get $C = \$181,300$. The level of savings then over the 40 working years is $S_t = m_t - C \approx \$19,000$ for $t = 21, 22, \ldots, 60$ and after retirement $S_t \approx -\$181,000$ for $t = 61, 62, \ldots, 80$.

(b) We still want to find the C that satisfies $PV(C) = PV(\text{income})$. The calculation of $PV(C)$ is unchanged, but now for the present value of income we have

$$PV(\text{income}) = \left(\frac{\$200,000}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{40}\right) + \$1,000,000 = \$4,431,800\,.$$

Solving $PV(C) = PV(\text{income})$ this time we get $C = 234,130$. The level of savings over the 40 working years is $S_t = m_t - C \approx -\$34,000$ for $t = 21, 22, \ldots, 60$ and after retirement $S_t \approx -\$234,000$ for $t = 61, 62, \ldots, 80$.

(c) To solve this, we can either add the present value of the bequest to the present value of consumption or subtract it from the present value of income. If we subtract it from the present value of income, then $PV(C)$ is unchanged and $PV(\text{income})$ becomes:

$$PV(\text{income}) = \left(\frac{\$200,000}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{40}\right) + \$1\text{mil - } \$1\text{mil}\left(\frac{1}{1.05}\right)^{60} = \$4,378,300\,.$$

Solving $PV(C) = PV(\text{income})$ we now have $C = 231,300$ The level of savings over the 40 working years is $S_t = m_t - C \approx -\$31,000$ for $t = 21, 22, \ldots, 60$ and after retirement $S_t \approx -\$231,000$ for $t = 61, 62, \ldots, 80$.