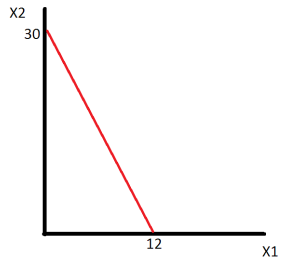


Midterm 1 (A)

You have 70 minutes to complete the exam. The midterm consists of 4 questions (45+15+15+25=100 points) + bonus (just for fun). Make sure you answer the first four questions before working on the bonus one!

**Problem 1 (45p) (Well-behaved preferences)**

a) Show geometrically Freddie's budget set. (5 points)



Find relative price of cheese curds in terms of coke. (2.5 points) **1 unit of cheese curds = 2.5 units of diet coke**

Give economic interpretation of the relative price. (2.5 points)

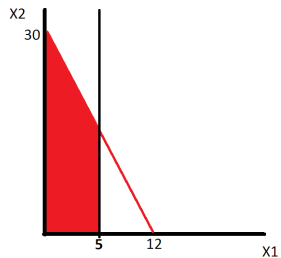
**This is the REAL PRICE of the two goods, meaning that in an exchange economy this would be the rate of exchange between the two goods**

Where can the relative price be seen in the graph? (2.5 points)

**The relative price can be seen (geometrically) as the slope of the budget constraint.**

b) Cheese curds are rationed, show the new budget set on the graph.(5 points)

**The area shaded red is the new budget set.**



c) Find (MRS) as a function of parameters (2.5 points)

$$MRS = -\frac{ax_2}{bx_1}$$

For parameters  $a = 4, b = 2$  and bundle  $(7, 7)$  find value of MRS. (2.5 points)

**MRS(7,7) = -2**

Give economic interpretation of MRS. (1 point)

**The marginal rate of substitution measure the rate at which the consumer is willing to substitute less consumption of good  $x_1$  for more consumption of good  $x_2$  while remaining at the same level of utility.**

Which of the goods is locally more valuable? ( $\frac{1}{2}$  point)

**$x_1$  is locally more valuable at the bundle (7,7)**

What is the geometric interpretation of MRS? (1 point)

**The MRS is the slope of the indifference curve**

Write down two secrets of happiness that determine optimal choice. Explain economic intuition behind the two conditions. (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

The first equation is a budget constraint which requires consumers to spend all of their income on the consumption of two goods,  $x_1$  and  $x_2$ . The second condition is a utility maximization condition that requires the slope of the indifference curve passing through the optimal consumption bundle to be tangent to the price ratio. The second condition will effectively require the marginal utility per dollar of each good to equate at the optimal consumption bundle.

Derive optimal choice of  $x_1$  and  $x_2$  as a function of  $a, b, p_1, p_2$  and  $m$  (show the derivation of magic formulas). Is your solution corner or interior (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

Solve for  $P_1 x_1$  in the second equation and substitute that into the budget constraint and solve for  $X_1$ . To get  $x_2$  simply solve for  $P_2 x_2$  in the second equation and substitute that into the budget constraint.

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (1)$$

$$P_{x_1} x_1 = \frac{a}{b} x_2 P_2 \quad (2)$$

$$M = \frac{a}{b} x_2 P_2 + x_2 P_2 \quad (3)$$

$$M = \left(\frac{a+b}{b}\right) P_2 x_2 \quad (4)$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \quad (5)$$

and

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (6)$$

$$\frac{b}{a} P_{x_1} x_1 = x_2 P_2 \quad (7)$$

$$M = \frac{b}{a} x_1 P_1 + x_1 P_1 \quad (8)$$

$$M = \left(\frac{a+b}{a}\right) P_1 x_1 \quad (9)$$

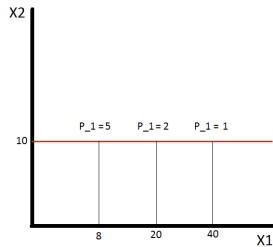
$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1}$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1} \implies x_1 = \frac{2m}{3P_1}$$

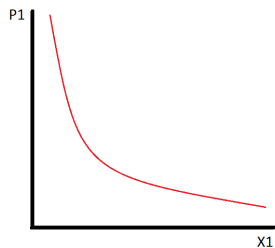
$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \implies x_2 = \frac{m}{3P_2}$$

d) Assume  $a = 4, b = 2$  and  $p_2 = 2, m = 60$ . Find geometrically and determine analytically price offer curve (2.5 points) and demand curve (2.5 points). The price offer curve can be plotted by taking the partial derivative of the demands for  $x_1$  and  $x_2$  with respect to  $P_1$ . This will give us the slope of the price offer curve, with the partial derivative of the demand for  $x_2$  being the "rise" and the partial derivative of the demand for  $x_1$  being the "run". Since the demand for  $x_2$  does not depend on  $P_1$  there is no vertical change in the

price offer curve, making it a horizontal line plotted at 10 units of  $x_2$



The demand curve for  $x_1$  can be plotted by taking the partial derivative of the demand for  $x_1$  with respect to  $P_1$ . The slope of this curve is  $\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{2M}{3P_1^2}$



Is  $x_1$  an ordinary or Giffen good and why? (2 points)

This is an ordinary good because there is a negative relationship between price and the quantity of the good demanded. This can be seen by taking the partial derivative of the demand function with respect to its own price:

$$\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{M}{2P_1^2}$$

e) Show that utility functions  $V(x_1, x_2) = x_1^a x_2^b$  and  $U(x_1, x_2) = a \ln x_1 + b \ln x_2$  represent the same preferences (3 points).

The function  $U()$  appears to be the result of applying the natural logarithm to the function  $V()$ . That means that we can define the function  $F(x) = \ln(x)$  so that  $U(x_1, x_2) = F(V(x_1, x_2))$ . The natural logarithm is a concave function (strictly increasing), meaning that applying the natural logarithm to a function will preserve the ordinality of that function. Thus, the function  $U()$  is merely a monotonic transformation of the function  $V()$ , meaning they represent the same preference order.

**Problem 2 (15p) (Quasilinear Preferences)**

a) Find marginal rate of substitution as a function of  $(x_1, x_2)$  (1 point).

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\frac{2}{x_1}}{1} = \frac{2}{x_1}$$

b) Using two secrets of happiness find optimal consumption choices (4 points)

$$1) 40 = 2x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{2}{4} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) to reveal:  $x_1 = 4, x_2 = 8$**

c) Suppose the price of a computer goes down to  $p_1 = 1$ . Find optimal choice after the price change (1 point).

To find the new optimal choice repeat the above steps with  $P_1 = 1$ :

$$1) 40 = x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{1}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:  $x_1 = 8, x_2 = 8$**

Decompose the change in  $x_1$  into a substitution effect (2 points) and income effect (2 points).

To calculate the substitution and income effects we need to find an 'intermediate point'. This is the optimal consumption choice given the purchasing power necessary to buy the original optimal consumption bundle (4,8) given the new prices  $P_1 = 1, P_2 = 4$ :

$$M' = (1)4 + (4)8 \implies M' = 36$$

To find the 'intermediate' optimal choice repeat the above steps with  $P_1 = 1$  and  $M = 36$ :

$$1) 36 = x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{1}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:  $x_1 = 8, x_2 = 7$**

**The substitution effect is the change in consumption of  $x_1$  due only to the price change (holding purchasing power constant). That means the substitution effect is the difference between the amount of  $x_1$  originally consumed (4 units) and how much is consumed after the price change (8 units). The substitution effect is then 4 units of  $x_1$ , while the income effect is zero (when we increase purchasing power from  $M = 36$  to  $M = 40$  we do not change our level of  $x_1$  consumption, only  $x_2$ ).**

d) Find optimal consumption for  $p_1 = 2, p_2 = 4$  and  $m = 4$  (1 point). 1)  $4 = 2x_1 + 4x_2$

$$2) \frac{2}{x_1} = \frac{2}{4} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) reveals that:  $x_1 = 4, x_2 = -1$ . We cannot consume negative amounts of a consumption good, so this indicates we would prefer to give up 1 unit of  $x_2$  in order to consume more  $x_1$ . This is not an option, however, as the lower bound on  $x_2$  is zero.**

Is your solution interior? (1 point).

**Our solution is  $x_1 = 2, x_2 = 0$ , which is not interior (corner solution). Interior solutions are those at which consumption of each good is strictly greater than zero, which is not the case here.**

Is marginal utility of a dollar equalized? (3 points)

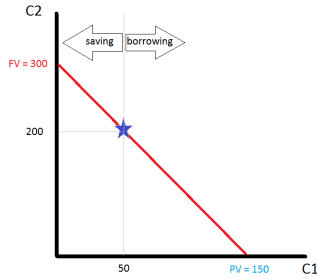
**No, we will very rarely see marginal utility per dollar of each consumption good equalized at a corner solution. In this case we have:**

$$\frac{MU_{x_1}}{P_1} = \frac{2}{x_1} = \frac{1}{2} \neq \frac{1}{4} = \frac{MU_{x_2}}{P_2}$$

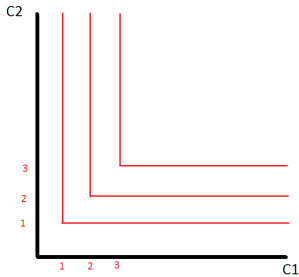
### **Problem 3 (15p) (Perfect complements, Intertemporal choice)**

Casper is a manager in a small startup firm. His income today is relatively small ( $m_1 = 50$ ) but in the future (period two) he expects to become very rich ( $m_2 = 200$ )

a) Depicts Casper's budget set assuming that he can borrow and save at the interest rate  $r = 100\%$  (1 point). Mark consumption plans on the budget line that involve savings and the plans that require borrowing. (1 point) Find Present and Future Value of Casper's income and show the two in the graph. (2 points)



b) In the commodity space plot Casper's indifference curves. (2 points)



c) Find optimal consumption plan  $(C_1, C_2)$  (5 points). Find the level of savings/borrowing in equilibrium (2 points). Is Casper smoothing his consumption over time? (2 points)

$$1) 150 = C_1 + \frac{1}{1+r}C_2$$

$$2) C_1 = C_2$$

**Plug  $C_2$  into the budget constraint for  $C_1$  gives Casper the equation:  $150 = \frac{1+r}{1+r}C_2 + \frac{1}{1+r}C_2$ . Plug in  $r = 1$  and we see that this determines  $C_2 = 100$  and looking back at the second secret of happiness, it also determines  $C_1 = 100$ . We find that  $C_1 = C_2$ , which means that Casper is consumption smoothing. In order to do this, he borrows \$50 from his second period income in the first period (reducing his second period consumption by \$100 and increasing his first period consumption by \$50 in the process.**

**Problem 4 (25p) (Short questions)**

a) Assume utility function  $U(C, R) = C \times R$  and the daily endowment of time equal to  $24h$ . Find optimal choice of consumption  $C$ , relaxation time  $R$  and labor supply  $L$  as a function of real wage rate  $w/p_c$ . (three numbers) use magic formula). Is labor supply elastic or inelastic (one sentence)?

b) Find optimal choice given utility function  $U(x_1, x_2 = 3x_1 + x_2)$ , prices  $p_1 = \$8$  and  $p_2 = \$2$  as well as income  $m = 100$ . Is your solution corner or interior?

$$\frac{MU_{x_1}}{P_1} = \frac{3}{8} < \frac{1}{2} = \frac{MU_{x_2}}{P_2}$$

**The marginal utility per dollar is constant for both goods, but higher for good two, so in this case all income is spent on  $x_2$ . This makes the solution  $(0, 50)$ .**

c) You are going to save \$10,000 when working (age 21-70) and then you are going to live for the next 30 years. Write down equation that determines constant (maximal) level of consumption during retirement age given your savings. Assume annual interest rate  $r = 3\%$ .

$$\frac{\$10,000}{r} \left(1 - \left(\frac{1}{1+r}\right)^{50}\right) = \frac{C}{r} \left(1 - \left(\frac{1}{1+r}\right)^{30}\right) \left(\frac{1}{1+r}\right)^{50}$$

d) Derive Present Value formula for perpetuity.

The formula of a stream of payments that never ends is:

$$\begin{aligned}PV &= \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \dots \\ &= \frac{1}{(1+r)} \left[ x + \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \dots \right] \\ &= \frac{1}{(1+r)} [x + PV]\end{aligned}$$

so we can solve for PV to get a more concise solution:

$$\begin{aligned}\left(1 - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{1+r}{1+r} - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{r}{1+r}\right)PV &= \frac{1}{1+r}x \\ PV &= \frac{x}{r}\end{aligned}$$

**Bonus question (Just for fun)**

a) Prove that for perfect complements  $U(x_1, x_2) = \min(ax_1, bx_2)$ , MRS is equal to zero for all bundles below the optimal proportion line and equal to  $-\infty$  for bundles above it.

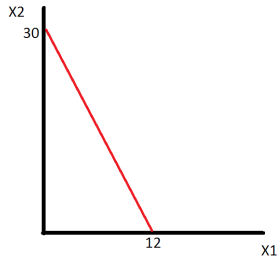
b) Explain in words why the solution to a linear optimization problem such as with perfect substitutes is called a bang bang solution.

### Midterm 1 (B)

You have 70 minutes to complete the exam. The midterm consists of 4 questions (45+15+15+25=100 points) + bonus (just for fun). Make sure you answer the first four questions before working on the bonus one!

#### Problem 1 (45p) (Well-behaved preferences)

a) Show geometrically Freddie's budget set. (5 points)



Find relative price of cheese curds in terms of coke. (2.5 points) **1 unit of cheese curds = 2.5 units of diet coke**

Give economic interpretation of the relative price. (2.5 points)

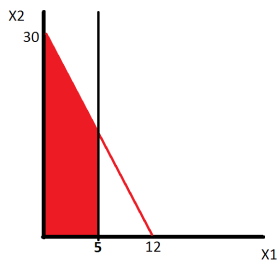
**This is the REAL PRICE of the two goods, meaning that in an exchange economy this would be the rate of exchange between the two goods**

Where can the relative price be seen in the graph? (2.5 points)

**The relative price can be seen (geometrically) as the slope of the budget constraint.**

b) Cheese curds are rationed, show the new budget set on the graph. (5 points)

**The area shaded red is the new budget set.**



c) Find (MRS) as a function of parameters (2.5 points)

$$MRS = -\frac{ax_2}{bx_1}$$

For parameters  $a = 4, b = 2$  and bundle  $(7, 7)$  find value of MRS. (2.5 points)

**MRS(7,7) = -2**

Give economic interpretation of MRS. (1 point)

**The marginal rate of substitution measure the rate at which the consumer is willing to substitute less consumption of good  $x_1$  for more consumption of good  $x_2$  while remaining at the same level of utility.**

Which of the goods is locally more valuable? ( $\frac{1}{2}$  point)

**$x_1$  is locally more valuable at the bundle  $(7, 7)$**

What is the geometric interpretation of MRS? (1 point)

**The MRS is the slope of the indifference curve**

Write down two secrets of happiness that determine optimal choice. Explain economic intuition behind the two conditions. (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

The first equation is a budget constraint which requires consumers to spend all of their income on the consumption of two goods,  $x_1$  and  $x_2$ . The second condition is a utility maximization condition that requires the slope of the indifference curve passing through the optimal consumption bundle to be tangent to the price ratio. The second condition will effectively require the marginal utility per dollar of each good to equate at the optimal consumption bundle.

Derive optimal choice of  $x_1$  and  $x_2$  as a function of  $a, b, p_1, p_2$  and  $m$  (show the derivation of magic formulas). Is your solution corner or interior (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

Solve for  $P_1 x_1$  in the second equation and substitute that into the budget constraint and solve for  $X_1$ . To get  $x_2$  simply solve for  $P_2 x_2$  in the second equation and substitute that into the budget constraint.

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (10)$$

$$P_{x_1} x_1 = \frac{a}{b} x_2 P_2 \quad (11)$$

$$M = \frac{a}{b} x_2 P_2 + x_2 P_2 \quad (12)$$

$$M = \left(\frac{a+b}{b} P_2\right) x_2 \quad (13)$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \quad (14)$$

and

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (15)$$

$$\frac{b}{a} P_{x_1} x_1 = x_2 P_2 \quad (16)$$

$$M = \frac{b}{a} x_1 P_1 + x_1 P_1 \quad (17)$$

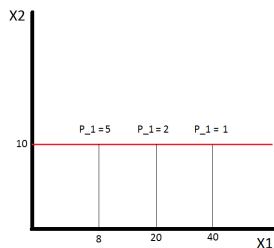
$$M = \left(\frac{a+b}{a} P_1\right) x_1 \quad (18)$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1}$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1} \implies x_1 = \frac{2m}{3P_1}$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \implies x_2 = \frac{m}{3P_2}$$

d) Assume  $a = 4, b = 2$  and  $p_2 = 2, m = 60$ . Find geometrically and determine analytically price offer curve (2.5 points) and demand curve (2.5 points). The price offer curve can be plotted by taking the partial derivative of the demands for  $x_1$  and  $x_2$  with respect to  $P_1$ . This will give us the slope of the price offer curve, with the partial derivative of the demand for  $x_2$  being the "rise" and the partial derivative of the demand for  $x_1$  being the "run". Since the demand for  $x_2$  does not depend on  $P_1$  there is no vertical change in the price offer curve, making it a horizontal line plotted at 10 units of  $x_2$



The demand curve for  $x_1$  can be plotted by taking the partial derivative of the demand for  $x_1$  with respect to  $P_1$ . The slope of this curve is  $\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{2M}{3P_1^2}$



Is  $x_1$  an ordinary or Giffen good and why? (2 points)

This is an ordinary good because there is a negative relationship between price and the quantity of the good demanded. This can be seen by taking the partial derivative of the demand function with respect to its own price:

$$\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{M}{2P_1^2}$$

e) Show that utility functions  $V(x_1, x_2) = x_1^a x_2^b$  and  $U(x_1, x_2) = a \ln x_1 + b \ln x_2$  represent the same preferences (3 points).

The function  $U()$  appears to be the result of applying the natural logarithm to the function  $V()$ . That means that we can define the function  $F(x) = \ln(x)$  so that  $U(x_1, x_2) = F(V(x_1, x_2))$ . The natural logarithm is a concave function (strictly increasing), meaning that applying the natural logarithm to a function will preserve the ordinality of that function. Thus, the function  $U()$  is merely a monotonic transformation of the function  $V()$ , meaning they represent the same preference order.

**Problem 2 (15p) (Quasilinear Preferences)**

a) Find marginal rate of substitution as a function of  $(x_1, x_2)$  (1 point).

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\frac{2}{x_1}}{1} = \frac{2}{x_1}$$

b) Using two secrets of happiness find optimal consumption choices (4 points)

$$1) 40 = 2x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{2}{4} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) to reveal:  $x_1 = 4, x_2 = 8$**

c) Suppose the price of a computer goes down to  $p_1 = 1$ . Find optimal choice after the price change (1 point).

To find the new optimal choice repeat the above steps with  $P_1 = 1$ :

$$1) 40 = x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{1}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:  $x_1 = 8, x_2 = 8$**

Decompose the change in  $x_1$  into a substitution effect (2 points) and income effect (2 points).

To calculate the substitution and income effects we need to find an 'intermediate point'. This is the optimal consumption choice given the purchasing power necessary to buy the original optimal consumption bundle (4,8) given the new prices  $P_1 = 1, P_2 = 4$ :

$$M' = (1)4 + (4)8 \implies M' = 36$$

To find the 'intermediate' optimal choice repeat the above steps with  $P_1 = 1$  and  $M = 36$ :

$$1) 36 = x_1 + 4x_2$$

$$2) \frac{2}{x_1} = \frac{1}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:  $x_1 = 8, x_2 = 7$**

**The substitution effect is the change in consumption of  $x_1$  due only to the price change (holding purchasing power constant). That means the substitution effect is the difference between the amount of  $x_1$  originally consumed (4 units) and how much is consumed after the price change (8 units). The substitution effect is then 4 units of  $x_1$ , while the income effect is zero (when we increase purchasing power from  $M = 36$  to  $M = 40$  we do not change our level of  $x_1$  consumption, only  $x_2$ ).**

d) Find optimal consumption for  $p_1 = 2, p_2 = 4$  and  $m = 4$  (1 point). 1)  $4 = 2x_1 + 4x_2$

$$2) \frac{2}{x_1} = \frac{2}{4} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) reveals that:  $x_1 = 4, x_2 = -1$ . We cannot consumer negative amounts of a consumption good, so this indicates we would prefer to give up 1 unit of  $x_2$  in order to consume more  $x_1$ . This is not an option, however, as the lower bound on  $x_2$  is zero.**

Is your solution interior? (1 point).

**Our solution is  $x_1 = 2, x_2 = 0$ , which is not interior (corner solution). Interior solutions are those at which consumption of each good is strictly greater than zero, which is not the case here.**

Is marginal utility of a dollar equalized? (3 points)

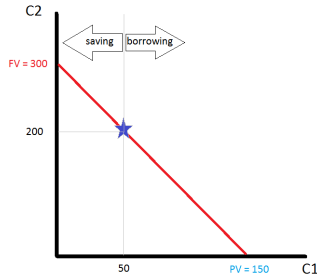
**No, we will very rarely see marginal utility per dollar of each consumption good equalized at a corner solution. In this case we have:**

$$\frac{MU_{x_1}}{P_1} = \frac{\frac{2}{x_1}}{2} = \frac{1}{2} \neq \frac{1}{4} = \frac{MU_{x_2}}{P_2}$$

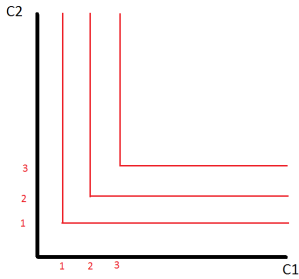
### **Problem 3 (15p) (Perfect complements, Intertemporal choice)**

Casper is a manager in a small startup firm. His income today is relatively small ( $m_1 = 50$ ) but in the future (period two) he expects to become very rich ( $m_2 = 200$ )

a) Depicts Casper's budget set assuming that he can borrow and save at the interest rate  $r = 100\%$  (1 point). Mark consumption plans on the budget line that involve savings and the plans that require borrowing. (1 point) Find Present and Future Value of Casper's income and show the two in the graph. (2 points)



b) In the commodity space plot Casper's indifference curves. (2 points)



c) Find optimal consumption plan  $(C_1, C_2)$  (5 points). Find the level of savings/borrowing in equilibrium (2 points). Is Casper smoothing his consumption over time? (2 points)

$$1) 150 = C_1 + \frac{1}{1+r}C_2$$

$$2) C_1 = C_2$$

**Plug  $C_2$  into the budget constraint for  $C_1$  gives Casper the equation:  $150 = \frac{1+r}{1+r}C_2 + \frac{1}{1+r}C_2$ . Plug in  $r = 1$  and we see that this determines  $C_2 = 100$  and looking back at the second secret of happiness, it also determines  $C_1 = 100$ . We find that  $C_1 = C_2$ , which means that Casper is consumption smoothing. In order to do this, he borrows \$50 from his second period income in the first period (reducing his second period consumption by \$100 and increasing his first period consumption by \$50 in the process.**

#### Problem 4 (25p) (Short questions)

a) Assume utility function  $U(C, R) = C \times R$  and the daily endowment of time equal to  $24h$ . Find optimal choice of consumption  $C$ , relaxation time  $R$  and labor supply  $L$  as a function of real wage rate  $w/p_c$ . (three numbers) use magic formula). Is labor supply elastic or inelastic (one sentence)?

**The two secrets to happiness are:**

$$1) (24 - R)W = P_c C$$

$$2) \frac{MU_{x_1}}{MU_{x_2}} = \frac{P_1}{P_2}$$

$$\implies P_c C = WR$$

$$\implies 24W = P_c C + P_c C$$

$$\implies C = \frac{12W}{P_c} \quad R = 12 \quad L = 12$$

**We find that labor supply is perfectly inelastic because it does not vary with the real wage rate.**

b) Find optimal choice given utility function  $U(x_1, x_2) = 5x_1 + x_2$ , prices  $p_1 = \$10$  and  $p_2 = \$1$  as well as income  $m = 100$ . Is your solution corner or interior?

$$\frac{MU_{x_1}}{P_1} = \frac{5}{10} < \frac{1}{1} = \frac{MU_{x_2}}{P_2}$$

The marginal utility per dollar is constant for both goods, but higher for good two, so in this case all income is spent on  $x_2$ . This makes the solution (0,100).

c) You are going to save \$10,000 when working (age 21-70) and then you are going to live for the next 30 years. Write down equation that determines constant (maximal) level of consumption during retirement age given your savings. Assume annual interest rate  $r=3\%$ .

$$\frac{\$10,000}{r} \left(1 - \left(\frac{1}{1+r}\right)^{50}\right) = \frac{C}{r} \left(1 - \left(\frac{1}{1+r}\right)^{30}\right) \left(\frac{1}{1+r}\right)^{50}$$

d) Derive Present Value formula for perpetuity.

The formula of a stream of payments that never ends is:

$$\begin{aligned} PV &= \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \dots \\ &= \frac{1}{(1+r)} \left[ x + \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \dots \right] \\ &= \frac{1}{(1+r)} [x + PV] \end{aligned}$$

so we can solve for PV to get a more concise solution:

$$\begin{aligned} \left(1 - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{1+r}{1+r} - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{r}{1+r}\right)PV &= \frac{1}{1+r}x \\ PV &= \frac{x}{r} \end{aligned}$$

**Bonus question (Just for fun)**

a) Prove that for perfect complements  $U(x_1, x_2) = \min(ax_1, bx_2)$ , MRS is equal to zero for all bundles below the optimal proportion line and equal to  $-\infty$  for bundles above it.

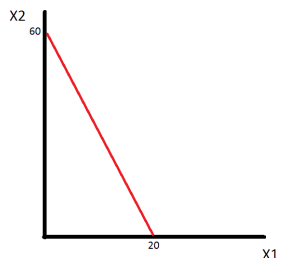
b) Explain in words why the solution to a linear optimization problem such as with perfect substitutes is called a bang bang solution.

### Midterm 1 (C)

You have 70 minutes to complete the exam. The midterm consists of 4 questions (45+15+15+25=100 points) + bonus (just for fun). Make sure you answer the first four questions before working on the bonus one!

#### Problem 1 (45p) (Well-behaved preferences)

a) Show geometrically Freddie's budget set. (5 points)



Find relative price of cheese curds in terms of coke. (2.5 points) **1 unit of cheese curds = 3 units of diet coke**

Give economic interpretation of the relative price. (2.5 points)

**This is the REAL PRICE of the two goods, meaning that in an exchange economy this would be the rate of exchange between the two goods**

Where can the relative price be seen in the graph? (2.5 points)

**The relative price can be seen (geometrically) as the slope of the budget constraint.**

b) Cheese curds are rationed, show the new budget set on the graph. (5 points)

**The area shaded red is the new budget set.**



c) Find (MRS) as a function of parameters (2.5 points)

$$MRS = -\frac{ax_2}{bx_1}$$

For parameters  $a = 4$ ,  $b = 2$  and bundle  $(3, 9)$  find value of MRS. (2.5 points)

**MRS(3,9) = -6**

Give economic interpretation of MRS. (1 point)

**The marginal rate of substitution measure the rate at which the consumer is willing to substitute less consumption of good  $x_1$  for more consumption of good  $x_2$  while remaining at the same level of utility.**

Which of the goods is locally more valuable? ( $\frac{1}{2}$  point)

**$x_1$  is locally more valuable at the bundle (7,7)**

What is the geometric interpretation of MRS? (1 point)

**The MRS is the slope of the indifference curve**

Write down two secrets of happiness that determine optimal choice. Explain economic intuition behind the two conditions. (5 points)

$$M = x_1p_1 + x_2p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

The first equation is a budget constraint which requires consumers to spend all of their income on the consumption of two goods,  $x_1$  and  $x_2$ . The second condition is a utility maximization condition that requires the slope of the indifference curve passing through the optimal consumption bundle to be tangent to the price ratio. The second condition will effectively require the marginal utility per dollar of each good to equate at the optimal consumption bundle.

Derive optimal choice of  $x_1$  and  $x_2$  as a function of  $a, b, p_1, p_2$  and  $m$  (show the derivation of magic formulas). Is your solution corner or interior (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

Solve for  $P_1 x_1$  in the second equation and substitute that into the budget constraint and solve for  $X_1$ . To get  $x_2$  simply solve for  $P_2 x_2$  in the second equation and substitute that into the budget constraint.

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (19)$$

$$P_{x_1} x_1 = \frac{a}{b} x_2 P_2 \quad (20)$$

$$M = \frac{a}{b} x_2 P_2 + x_2 P_2 \quad (21)$$

$$M = \left(\frac{a+b}{b} P_2\right) x_2 \quad (22)$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \quad (23)$$

and

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (24)$$

$$\frac{b}{a} P_{x_1} x_1 = x_2 P_2 \quad (25)$$

$$M = \frac{b}{a} x_1 P_1 + x_1 P_1 \quad (26)$$

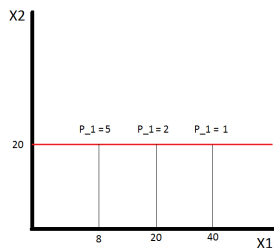
$$M = \left(\frac{a+b}{a} P_1\right) x_1 \quad (27)$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1}$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1} \implies x_1 = \frac{2m}{3P_1}$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \implies x_2 = \frac{m}{3P_2}$$

d) Assume  $a = 4, b = 2$  and  $p_2 = 1, m = 60$ . Find geometrically and determine analytically price offer curve (2.5 points) and demand curve (2.5 points). The price offer curve can be plotted by taking the partial derivative of the demands for  $x_1$  and  $x_2$  with respect to  $P_1$ . This will give us the slope of the price offer curve, with the partial derivative of the demand for  $x_2$  being the "rise" and the partial derivative of the demand for  $x_1$  being the "run". Since the demand for  $x_2$  does not depend on  $P_1$  there is no vertical change in the price offer curve, making it a horizontal line plotted at 20 units of  $x_2$



The demand curve for  $x_1$  can be plotted by taking the partial derivative of the demand for  $x_1$  with respect to  $P_1$ . The slope of this curve is  $\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{2M}{3P_1^2} \implies x_1 = \frac{-40}{P_1}$



Is  $x_1$  an ordinary or Giffen good and why? (2 points)

This is an ordinary good because there is a negative relationship between price and the quantity of the good demanded. This can be seen by taking the partial derivative of the demand function with respect to its own price:

$$\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{M}{2P_1^2}$$

e) Show that utility functions  $V(x_1, x_2) = x_1^a x_2^b$  and  $U(x_1, x_2) = a \ln x_1 + b \ln x_2$  represent the same preferences (3 points).

The function  $U()$  appears to be the result of applying the natural logarithm to the function  $V()$ . That means that we can define the function  $F(x) = \ln(x)$  so that  $U(x_1, x_2) = F(V(x_1, x_2))$ . The natural logarithm is a concave function (strictly increasing), meaning that applying the natural logarithm to a function will preserve the ordinality of that function. Thus, the function  $U()$  is merely a monotonic transformation of the function  $V()$ , meaning they represent the same preference order.

**Problem 2 (15p) (Quasilinear Preferences)**

a) Find marginal rate of substitution as a function of  $(x_1, x_2)$  (1 point).

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\frac{4}{x_1}}{1} = \frac{4}{x_1}$$

b) Using two secrets of happiness find optimal consumption choices (4 points)

$$1) 20 = 4x_1 + 2x_2$$

$$2) \frac{4}{x_1} = \frac{4}{2} \implies x_1 = 2$$

**Plug  $x_1 = 2$  into the first secret (budget constraint) to reveal:  $x_1 = 2, x_2 = 6$**

c) Suppose the price of a computer goes down to  $p_1 = 2$ . Find optimal choice after the price change (1 point).

To find the new optimal choice repeat the above steps with  $P_1 = 1$ :

$$1) 20 = 2x_1 + 2x_2$$

$$2) \frac{4}{x_1} = \frac{2}{2} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) to reveal:**  $x_1 = 4, x_2 = 6$

Decompose the change in  $x_1$  into a substitution effect (2 points) and income effect (2 points).

To calculate the substitution and income effects we need to find an 'intermediate point'. This is the optimal consumption choice given the purchasing power necessary to buy the original optimal consumption bundle (2,6) given the new prices  $P_1 = 2, P_2 = 2$ :

$$M' = (2)2 + (2)6 \implies M' = 116$$

To find the 'intermediate' optimal choice repeat the above steps with  $P_1 = 2$  and  $M = 16$ :

$$1) 16 = 2x_1 + 2x_2$$

$$2) \frac{4}{x_1} = \frac{2}{2} \implies x_1 = 4$$

**Plug  $x_1 = 4$  into the first secret (budget constraint) to reveal:**  $x_1 = 4, x_2 = 4$

**The substitution effect is the change in consumption of  $x_1$  due only to the price change (holding purchasing power constant). That means the substitution effect is the difference between the amount of  $x_1$  originally consumed (2 units) and how much is consumed after the price change (4 units). The substitution effect is then 2 units of  $x_1$ , while the income effect is zero (when we increase purchasing power from  $M = 16$  to  $M = 20$  we do not change our level of  $x_1$  consumption, only  $x_2$ ).**

d) Find optimal consumption for  $p_1 = 4, p_2 = 2$  and  $m = 4$  (1 point). 1)  $4 = 4x_1 + 2x_2$

$$2) \frac{4}{x_1} = \frac{4}{2} \implies x_1 = 2$$

**Plug  $x_1 = 2$  into the first secret (budget constraint) reveals that:**  $x_1 = 2, x_2 = -2$ . **We cannot consumer negative amounts of a consumption good, so this indicates we would prefer to give up 2 unit of  $x_2$  in order to consume more  $x_1$ . This is not an option, however, as the lower bound on  $x_2$  is zero.**

Is your solution interior? (1 point).

**Our solution is  $x_1 = 1, x_2 = 0$ , which is not interior (corner solution). Interior solutions are those at which consumption of each good is strictly greater than zero, which is not the case here.**

Is marginal utility of a dollar equalized? (3 points)

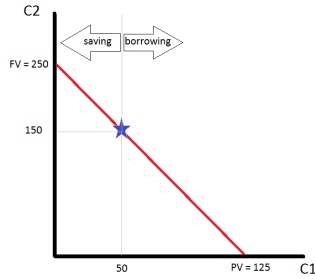
**No, we will very rarely see marginal utility per dollar of each consumption good equalized at a corner solution. In this case we have:**

$$\frac{MU_{x_1}}{P_1} = \frac{\frac{4}{x_1}}{4} = \frac{4}{4} \neq \frac{1}{2} = \frac{MU_{x_2}}{P_2}$$

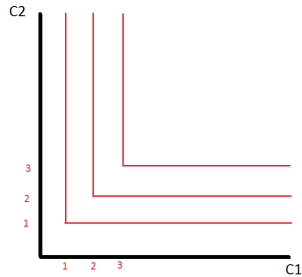
### **Problem 3 (15p) (Perfect complements, Intertemporal choice)**

Casper is a manager in a small startup firm. His income today is relatively small ( $m_1 = 50$ ) but in the future (period two) he expects to become very rich ( $m_2 = 150$ )

a) Depicts Casper's budget set assuming that he can borrow and save at the interest rate  $r = 100\%$  (1 point). Mark consumption plans on the budget line that involve savings and the plans that require borrowing. (1 point) Find Present and Future Value of Casper's income and show the two in the graph. (2 points)



b) In the commodity space plot Casper's indifference curves.(2 points)



c) Find optimal consumption plan  $(C_1, C_2)$  (5 points). Find the level of savings/borrowing in equilibrium (2 points). Is Casper smoothing his consumption over time? (2 points)

$$1) 125 = C_1 + \frac{1}{1+r}C_2$$

$$2) C_1 = C_2$$

**Plug  $C_2$  into the budget constraint for  $C_1$  gives Casper the equation:  $125 = \frac{1+r}{1+r}C_2 + \frac{1}{1+r}C_2$ . Plug in  $r = 1$  and we see that this determines  $C_2 = \frac{250}{3}$  and looking back at the second secret of happiness, it also determines  $C_1 = \frac{250}{3}$ . We find that  $C_1 = C_2$ , which means that Casper is consumption smoothing. In order to do this, he borrows  $\frac{100}{3}$  from his second period income in the first period (reducing his second period consumption by  $\frac{100}{3}$  and increasing his first period consumption by 33.33 dollars in the process.**

**Problem 4 (25p) (Short questions)**

a) Assume utility function  $U(C, R) = C \times R$  and the daily endowment of time equal to  $24h$ . Find optimal choice of consumption  $C$ , relaxation time  $R$  and labor supply  $L$  as a function of real wage rate  $w/p_c$ . (three numbers) use magic formula). Is labor supply elastic or inelastic (one sentence)?

b) Find optimal choice given utility function  $U(x_1, x_2) = 5x_1 + x_2$ , prices  $p_1 = \$8$  and  $p_2 = \$2$  as well as income  $m = 100$ . Is your solution corner or interior?

$$\frac{MU_{x_1}}{P_1} = \frac{5}{8} > \frac{1}{2} = \frac{MU_{x_2}}{P_2}$$

**The marginal utility per dollar is constant for both goods, but higher for good one, so in this case all income is spent on  $x_1$ . This makes the solution (12.5,0).**

c) You are going to save \$50,000 when working (age 21-80) and then you are going to live for the next 20 years. Write down equation that determines constant (maximal) level of consumption during retirement age given your savings. Assume annual interest rate  $r = 2\%$ .

$$\frac{\$50,000}{r} \left(1 - \left(\frac{1}{1+r}\right)^{60}\right) = \frac{C}{r} \left(1 - \left(\frac{1}{1+r}\right)^{20}\right) \left(\frac{1}{1+r}\right)^{60}$$

d) Derive Present Value formula for perpetuity.

The formula of a stream of payments that never ends is:

$$\begin{aligned}PV &= \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \dots \\ &= \frac{1}{(1+r)} \left[ x + \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \dots \right] \\ &= \frac{1}{(1+r)} [x + PV]\end{aligned}$$

so we can solve for PV to get a more concise solution:

$$\begin{aligned}\left(1 - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{1+r}{1+r} - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{r}{1+r}\right)PV &= \frac{1}{1+r}x \\ PV &= \frac{x}{r}\end{aligned}$$

**Bonus question (Just for fun)**

a) Prove that for perfect complements  $U(x_1, x_2) = \min(ax_1, bx_2)$ , MRS is equal to zero for all bundles below the optimal proportion line and equal to  $-\infty$  for bundles above it.

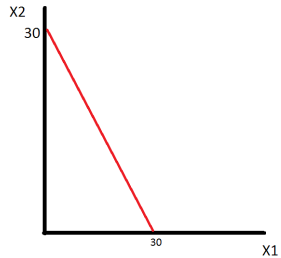
b) Explain in words why the solution to a linear optimization problem such as with perfect substitutes is called a bang bang solution.

### Midterm 1 (D)

You have 70 minutes to complete the exam. The midterm consists of 4 questions (45+15+15+25=100 points) + bonus (just for fun). Make sure you answer the first four questions before working on the bonus one!

#### Problem 1 (45p) (Well-behaved preferences)

a) Show geometrically Freddie's budget set. (5 points)



Find relative price of cheese curds in terms of coke. (2.5 points) **1 unit of cheese curds = 1 unit of diet coke**

Give economic interpretation of the relative price. (2.5 points)

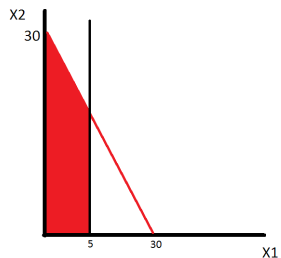
**This is the REAL PRICE of the two goods, meaning that in an exchange economy this would be the rate of exchange between the two goods**

Where can the relative price be seen in the graph? (2.5 points)

**The relative price can be seen (geometrically) as the slope of the budget constraint.**

b) Cheese curds are rationed, show the new budget set on the graph.(5 points)

**The area shaded red is the new budget set.**



c) Find (MRS) as a function of parameters (2.5 points)

$$MRS = -\frac{ax_2}{bx_1}$$

For parameters  $a = 4, b = 2$  and bundle  $(3, 9)$  find value of MRS. (2.5 points)

**MRS(3,9) = -6**

Give economic interpretation of MRS. (1 point)

**The marginal rate of substitution measure the rate at which the consumer is willing to substitute less consumption of good  $x_1$  for more consumption of good  $x_2$  while remaining at the same level of utility.**

Which of the goods is locally more valuable? ( $\frac{1}{2}$  point)

**$x_1$  is locally more valuable at the bundle (7,7)**

What is the geometric interpretation of MRS? (1 point)

**The MRS is the slope of the indifference curve**

Write down two secrets of happiness that determine optimal choice. Explain economic intuition behind the two conditions. (5 points)

$$M = x_1p_1 + x_2p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

The first equation is a budget constraint which requires consumers to spend all of their income on the consumption of two goods,  $x_1$  and  $x_2$ . The second condition is a utility maximization condition that requires the slope of the indifference curve passing through the optimal consumption bundle to be tangent to the price ratio. The second condition will effectively require the marginal utility per dollar of each good to equate at the optimal consumption bundle.

Derive optimal choice of  $x_1$  and  $x_2$  as a function of  $a, b, p_1, p_2$  and  $m$  (show the derivation of magic formulas). Is your solution corner or interior (5 points)

$$M = x_1 p_1 + x_2 p_2$$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{P_{x_1}}{P_{x_2}}$$

Solve for  $P_1 x_1$  in the second equation and substitute that into the budget constraint and solve for  $X_1$ . To get  $x_2$  simply solve for  $P_2 x_2$  in the second equation and substitute that into the budget constraint.

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (28)$$

$$P_{x_1} x_1 = \frac{a}{b} x_2 P_2 \quad (29)$$

$$M = \frac{a}{b} x_2 P_2 + x_2 P_2 \quad (30)$$

$$M = \left(\frac{a+b}{b} P_2\right) x_2 \quad (31)$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \quad (32)$$

and

$$\frac{ax_2}{bx_1} = \frac{P_{x_1}}{P_{x_2}} \quad (33)$$

$$\frac{b}{a} P_{x_1} x_1 = x_2 P_2 \quad (34)$$

$$M = \frac{b}{a} x_1 P_1 + x_1 P_1 \quad (35)$$

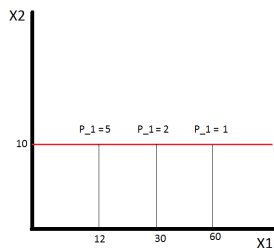
$$M = \left(\frac{a+b}{a} P_1\right) x_1 \quad (36)$$

$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1}$$

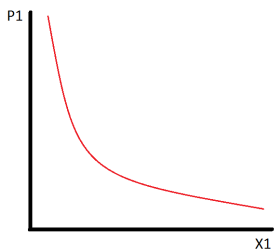
$$x_1 = \frac{a}{a+b} \times \frac{m}{P_1} \implies x_1 = \frac{2m}{3P_1}$$

$$x_2 = \frac{b}{a+b} \times \frac{m}{P_2} \implies x_2 = \frac{m}{3P_2}$$

d) Assume  $a = 4, b = 2$  and  $p_2 = 3, m = 90$ . Find geometrically and determine analytically price offer curve (2.5 points) and demand curve (2.5 points). The price offer curve can be plotted by taking the partial derivative of the demands for  $x_1$  and  $x_2$  with respect to  $P_1$ . This will give us the slope of the price offer curve, with the partial derivative of the demand for  $x_2$  being the "rise" and the partial derivative of the demand for  $x_1$  being the "run". Since the demand for  $x_2$  does not depend on  $P_1$  there is no vertical change in the price offer curve, making it a horizontal line plotted at 10 units of  $x_2$



The demand curve for  $x_1$  can be plotted by taking the partial derivative of the demand for  $x_1$  with respect to  $P_1$ . The slope of this curve is  $\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{2M}{3P_1^2} \implies x_1 = \frac{-60}{P_1}$



Is  $x_1$  an ordinary or Giffen good and why? (2 points)

This is an ordinary good because there is a negative relationship between price and the quantity of the good demanded. This can be seen by taking the partial derivative of the demand function with respect to its own price:

$$\frac{\partial x_1(M, P_1)}{\partial P_1} = -\frac{M}{2P_1^2}$$

e) Show that utility functions  $V(x_1, x_2) = x_1^a x_2^b$  and  $U(x_1, x_2) = a \ln x_1 + b \ln x_2$  represent the same preferences (3 points).

The function  $U()$  appears to be the result of applying the natural logarithm to the function  $V()$ . That means that we can define the function  $F(x) = \ln(x)$  so that  $U(x_1, x_2) = F(V(x_1, x_2))$ . The natural logarithm is a concave function (strictly increasing), meaning that applying the natural logarithm to a function will preserve the ordinality of that function. Thus, the function  $U()$  is merely a monotonic transformation of the function  $V()$ , meaning they represent the same preference order.

**Problem 2 (15p) (Quasilinear Preferences)**

a) Find marginal rate of substitution as a function of  $(x_1, x_2)$  (1 point).

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\frac{4}{x_1}}{1} = \frac{4}{x_1}$$

b) Using two secrets of happiness find optimal consumption choices (4 points)

1)  $80 = 4x_1 + 4x_2$

2)  $\frac{4}{x_1} = \frac{4}{4} \implies x_1 = 4$

**Plug  $x_1 = 4$  into the first secret (budget constraint) to reveal:  $x_1 = 4, x_2 = 16$**

c) Suppose the price of a computer goes down to  $p_1 = 2$ . Find optimal choice after the price change (1 point).

To find the new optimal choice repeat the above steps with  $P_1 = 2$ :

1)  $80 = 2x_1 + 4x_2$

$$2) \frac{4}{x_1} = \frac{2}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:**  $x_1 = 8, x_2 = 16$

Decompose the change in  $x_1$  into a substitution effect (2 points) and income effect (2 points).

To calculate the substitution and income effects we need to find an 'intermediate point'. This is the optimal consumption choice given the purchasing power necessary to buy the original optimal consumption bundle (2,6) given the new prices  $P_1 = 2, P_2 = 2$ :

$$M' = (2)4 + (4)16 \implies M' = 72$$

To find the 'intermediate' optimal choice repeat the above steps with  $P_1 = 2$  and  $M = 72$ :

$$1) 72 = 2x_1 + 4x_2$$

$$2) \frac{4}{x_1} = \frac{2}{4} \implies x_1 = 8$$

**Plug  $x_1 = 8$  into the first secret (budget constraint) to reveal:**  $x_1 = 8, x_2 = 14$

**The substitution effect is the change in consumption of  $x_1$  due only to the price change (holding purchasing power constant). That means the substitution effect is the difference between the amount of  $x_1$  originally consumed (4 units) and how much is consumed after the price change (8 units). The substitution effect is then 4 units of  $x_1$ , while the income effect is zero (when we increase purchasing power from  $M = 72$  to  $M = 80$  we do not change our level of  $x_1$  consumption, only  $x_2$ ).**

d) Find optimal consumption for  $p_1 = 4, p_2 = 4$  and  $m = 4$  (1 point). 1)  $4 = 4x_1 + 4x_2$

$$2) \frac{4}{x_1} = \frac{4}{4} \implies x_1 = 4$$

**Plug  $x_1 = 2$  into the first secret (budget constraint) reveals that:**  $x_1 = 2, x_2 = -3$ . **We cannot consumer negative amounts of a consumption good, so this indicates we would prefer to give up 3 unit of  $x_2$  in order to consume more  $x_1$ . This is not an option, however, as the lower bound on  $x_2$  is zero.**

Is your solution interior? (1 point).

**Our solution is  $x_1 = 1, x_2 = 0$ , which is not interior (corner solution). Interior solutions are those at which consumption of each good is strictly greater than zero, which is not the case here.**

Is marginal utility of a dollar equalized? (3 points)

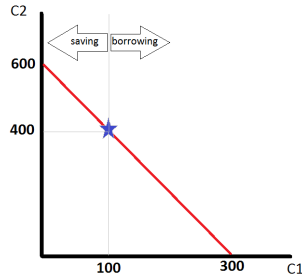
**No, we will very rarely see marginal utility per dollar of each consumption good equalized at a corner solution. In this case we have:**

$$\frac{MU_{x_1}}{P_1} = \frac{\frac{4}{x_1}}{4} = \frac{4}{4} \neq \frac{1}{4} = \frac{MU_{x_2}}{P_2}$$

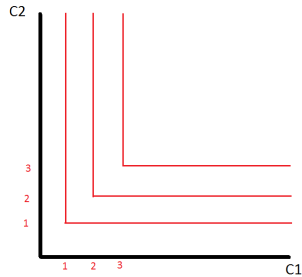
### **Problem 3 (15p) (Perfect complements, Intertemporal choice)**

Casper is a manager in a small startup firm. His income today is relatively small ( $m_1 = 100$ ) but in the future (period two) he expects to become very rich ( $m_2 = 400$ )

a) Depicts Casper's budget set assuming that he can borrow and save at the interest rate  $r = 100\%$  (1 point). Mark consumption plans on the budget line that involve savings and the plans that require borrowing. (1 point) Find Present and Future Value of Casper's income and show the two in the graph. (2 points)



b) In the commodity space plot Casper's indifference curves.(2 points)



c) Find optimal consumption plan  $(C_1, C_2)$  (5 points). Find the level of savings/borrowing in equilibrium (2 points). Is Casper smoothing his consumption over time? (2 points)

$$1) 300 = C_1 + \frac{1}{1+r}C_2$$

$$2) C_1 = C_2$$

**Plug  $C_2$  into the budget constraint for  $C_1$  gives Casper the equation:  $300 = \frac{1+r}{1+r}C_2 + \frac{1}{1+r}C_2$ . Plug in  $r = 1$  and we see that this determines  $C_2 = 200$  and looking back at the second secret of happiness, it also determines  $C_1 = 200$ . We find that  $C_1 = C_2$ , which means that Casper is consumption smoothing. In order to do this, he borrows from his second period income in the first period (reducing his second period consumption by \$200 and increasing his first period consumption by \$100 in the process.**

**Problem 4 (25p) (Short questions)**

a) Assume utility function  $U(C, R) = C \times R$  and the daily endowment of time equal to  $24h$ . Find optimal choice of consumption  $C$ , relaxation time  $R$  and labor supply  $L$  as a function of real wage rate  $w/p_c$ . (three numbers) use magic formula). Is labor supply elastic or inelastic (one sentence)?

b) Find optimal choice given utility function  $U(x_1, x_2 = 5x_1 + x_2)$ , prices  $p_1 = \$8$  and  $p_2 = \$2$  as well as income  $m = 40$ . Is your solution corner or interior?

$$\frac{MU_{x_1}}{P_1} = \frac{5}{8} > \frac{1}{2} = \frac{MU_{x_2}}{P_2}$$

**The marginal utility per dollar is constant for both goods, but higher for good one, so in this case all income is spent on  $x_1$ . This makes the solution (5,0).**

c) You are going to save \$6,000 when working (age 21-80) and then you are going to live for the next 20 years. Write down equation that determines constant (maximal) level of consumption during retirement age given your savings. Assume annual interest rate  $r = 4\%$ .

$$\frac{\$6,000}{r} \left(1 - \left(\frac{1}{1+r}\right)^{60}\right) = \frac{C}{r} \left(1 - \left(\frac{1}{1+r}\right)^{20}\right) \left(\frac{1}{1+r}\right)^{60}$$

d) Derive Present Value formula for perpetuity.

The formula of a stream of payments that never ends is:

$$\begin{aligned}PV &= \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \dots \\ &= \frac{1}{(1+r)} \left[ x + \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \dots \right] \\ &= \frac{1}{(1+r)} [x + PV]\end{aligned}$$

so we can solve for PV to get a more concise solution:

$$\begin{aligned}\left(1 - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{1+r}{1+r} - \frac{1}{1+r}\right)PV &= \frac{1}{1+r}x \\ \left(\frac{r}{1+r}\right)PV &= \frac{1}{1+r}x \\ PV &= \frac{x}{r}\end{aligned}$$

**Bonus question (Just for fun)**

a) Prove that for perfect complements  $U(x_1, x_2) = \min(ax_1, bx_2)$ , MRS is equal to zero for all bundles below the optimal proportion line and equal to  $-\infty$  for bundles above it.

b) Explain in words why the solution to a linear optimization problem such as with perfect substitutes is called a bang bang solution.