Deflationary Equilibrium with Uncertainty*

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This Draft: January 2025

Abstract

We analyze the so-called deflationary equilibrium of the New Keynesian model with an interest rate lower bound when the future course of the economy is uncertain. In the deflationary equilibrium, we find that the rate of inflation is higher at the risky steady state—which takes uncertainty into account—than at the deterministic steady state—which abstracts away from uncertainty. The rate of inflation at the risky steady state can be positive if the target rate set by the central bank is positive. Our theory is consistent with the Japanese experience in the 2010s when the rate of inflation was on average positive while the interest rate lower bound was binding.

JEL: E32, E52, E61, E62, E63

Keywords: Effective Lower Bound, Deflationary Equilibrium, Liquidity Trap, Risky Steady State, Uncertainty.

[*]The views expressed in this paper, and all errors and omissions, should be regarded as those of the authors, and are not necessarily those of the European Central Bank. We thank Young In Lee for her excellent research assistance and seminar participants at the Bank of Japan and the Summer Workshop on Economic Theory for useful comments. Part of the analysis was conducted while Taisuke was a visiting scholar at the Institute for Monetary and Economic Studies (IMES) of the Bank of Japan. Taisuke thanks IMES for their hospitality.

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1 Introduction

In this paper, we theoretically analyze the so-called deflationary equilibrium of the New Keynesian model with an effective lower bound (ELB) constraint on nominal interest rates. In particular, we explore how uncertainty about the future course of the economy affects allocations under the deflationary equilibrium, paying particular attention to the steady-state allocation. The steady state of a model without uncertainty is often referred to as the deterministic steady state, whereas the steady state of a model with uncertainty is often referred to as the risky steady state (Coeurdacier, Rey, and Winant (2011)). It is well known that the deterministic steady state of the deflationary equilibrium features the binding ELB constraint and a below-target rate of inflation (Benhabib, Schmitt-Grohe, and Uribe (2001) and Bullard (2010)). We are interested in examining how the risky steady state of the deflationary equilibrium differs from its deterministic steady state.

We find that the rate of inflation is higher at the risky steady state than at the deterministic steady state in the deflationary equilibrium. This result arises because the distribution of future inflation is asymmetric if the variance of shocks driving fluctuations in the model is sufficiently large. When a negative shock hits the economy under the deflationary equilibrium, inflation declines, but the policy rate remains at the ELB. When a positive shock hits the economy under the deflationary equilibrium, inflation increases. If the size of the positive shock is sufficiently large, the policy rate also increases and partially offsets the increase in inflation. This asymmetry breaks the certainty equivalence and creates a discrepancy between the risk steady state and the deterministic steady state. Interestingly, this effect of uncertainty on the steady state of the deflationary equilibrium is the opposite of that on the steady state of the target equilibrium examined in Hills, Nakata, and Schmidt (2019).

We propose a novel concept, *the risk-adjusted Fisher relation*, to visualize the effect of uncertainty on the steady-state inflation rate and the policy rate. The standard Fisher relation is the relationship between the rate of inflation and the policy rate implied by the Euler equation in the absence of any exogenous shocks. The risk-adjusted Fisher relation is the relationship between the rate of inflation and the policy rate implied by the Euler equation when there are exogenous shocks in the economy but their realized values are zero. That is, even though realized shock values are zero, agents in the model are aware that shocks can hit the economy in the future and form expectations accordingly. The risky steady states are given by the intersection of the risk-adjusted Fisher relation and the truncated Taylor rule, whereas the deterministic steady states are given by the intersection of the standard Fisher relation and the truncated Taylor rule. We show that the risk-adjusted Fisher relation lies below the standard Fisher relation, which implies the result described in the previous paragraph.

When the target rate of inflation set by the central bank is positive, our result implies that the rate of inflation can be—somewhat paradoxically—positive at the risky steady state of the deflationary equilibrium if the degree of uncertainty is sufficiently large. This situation is consistent with the Japanese economy in the 2010s when the rate of inflation hovered around a slightly positive level while the policy rate was constrained at the ELB.

We also find that the policy rate can be positive at the risky steady state of the deflationary equilibrium, contrary to the conventional image associated with the steady state of the deflationary equilibrium featuring the binding ELB constraint. We also find that the ELB constraint can bind at the risky steady state of the target equilibrium, contrary to the conventional image associated with the steady state of the target equilibrium featuring a positive policy rate. All in all, our conventional images about the deflationary and target equilibria may not be valid once we take uncertainty into account.

We use a New Keynesian model with a three-state shock in the main body of the paper. We derive both analytical and numerical results. However, all the qualitative results also hold in the model with an AR(1) shock. The results from the model with an AR(1) shock are discussed in the Appendix.

The steady-state inflation analysis is important because we still do not know much about the determinants of the long-term behavior of inflation. For example, in the U.S., inflation was persistently below the target rate of 2 percent in the 2010s. Even when the economy was strong and the unemployment rate was hovering below 4 percent in the last few years of the 2010s, the rate of inflation struggled to move up to the target rate, making policymakers concerned about the possibility that the long-term inflation expectations were anchored at a level below the target rate of 2 percent. This concern eventually led to the adoption of the flexible average inflation targeting framework in the summer of 2020 after the Federal Reserve concluded its first Strategic Review.

As another example, in Japan, the rate of inflation hovered well below the target rate of 2 percent in the 2010s even with a highly accommodative monetary policy and a sustained period of low unemployment rates. Such economic development led to the concern that long-term inflation expectations had become firmly anchored at around zero percent in Japan, a concern that has been dispelled only by strong inflationary pressures in the aftermath of a once-in-a-century-type pandemic.

In the standard New Keynesian model without uncertainty, the rate of inflation eventually returns to either the target rate set by the central bank—if the economy is in the target equilibrium—or a negative rate consistent with the inverse of the subjective discount factor of the household—if the economy is in the deflationary equilibrium. Thus, the standard model is unsatisfactory in explaining the long-term trend in inflation observed in some advanced economies—even if the theory is often useful in explaining the deviation of inflation from the long-run trend. Our paper shows that uncertainty can help bridge the gap between the model and reality in terms of the long-term trend of inflation.

Our paper is related to the literature analyzing the interaction of uncertainty and the ELB. See, for example, Basu and Bundick (2017), Evans, Fisher, Gourio, and Krane (2015), Nakata (2016), Nakata (2017), Plante, Richter, and Throckmorton (2018), among others. Adam and Billi (2007) and Nakov (2008) also discussed the interaction of uncertainty and the ELB in their early prescient work on optimal policy—though that was not their main focus. Our paper is closest to Hills, Nakata, and Schmidt (2019) who analyze how uncertainty affects the steady state of the target equilibrium. They find that the rate of inflation at the risky steady state of the target equilibrium is lower than that at the deterministic steady state—which corresponds to the target rate of inflation set by the central bank—and showed that the model can quantitatively explain the below-target rate of inflation in the U.S. in the second half of 2010s. Our paper is different from these papers because we study the interaction of uncertainty and the ELB in the deflationary equilibrium. We also formalize a novel concept—the risk-adjusted Fisher relation—which was only casually alluded to in Hills, Nakata, and Schmidt (2019).

We contribute to the literature on the deflationary equilibrium in the New Keynesian model. Some researchers analyze what policies may (or may not) eliminate this LT (Alstadheim and Henderson (2006); Armenter (2017); Benhabib, Schmitt-Grohe, and Uribe (2002); Coyle and Nakata (2020); Nakata and Schmidt (2022); Schmidt (2016); Schmitt-Grohe and Uribe (2013); Sugo and Ueda (2008); and Tamanyu (2022)). Others focus on the dynamics in and out of this LT (Aruoba, Cuba-Borda, and Schorfheide (2018); Bilbiie (2022); Cuba-Borda and Singh (2019); Hirose (2007); Hirose (2020); Mertens and Ravn (2014); Mertens and Williams (2021); Schmitt-Grohe and Uribe (2017). Our paper differs from these papers because we focus on the interaction of uncertainty and the ELB in the deflationary equilibrium.

Our analysis shares the same spirit of—and complements—the analysis of Takahashi and Takayama (2025). They point out that the lack of (sizable) deflation in Japan while the ELB was binding is inconsistent with the deflationary steady state of the standard New Keynesian model and show that a perpetual-youth monetary model can help bridge the gap between the model and the data. We point out that a positive inflation rate in Japan during the 2010s is inconsistent with the standard deflationary steady state and show that the introduction of uncertainty can help bridge the gap between the model and the data.

We also contribute to the empirical literature measuring the long-term or underlying trend in inflation. Examples include Bryan and Cecchetti (1994), Chan, Clark, and Koop (2018), Clark and Doh (2014), Cogley, Primiceri, and Sargent (2010), Chan, Koop, and Potter (2013), Kozicki and Tinsley (2012), Mertens (2016), Nason and Smith (2021), Rudd (2020), and Stock and Watson (2016), among many others. These papers often find a persistent deviation of the long-run trend in inflation from the target rate set by the central bank or the inverse of the household's discount factor. We contribute to this literature by providing a theoretical model that can account for such a persistent deviation.

The rest of the paper is organized as follows. Section 2 presents the model and defines key concepts. Section 3 discusses the main results. Section 4 defines and analyzes the risk-adjusted Fisher Relation. Section 5 discusses a few interesting configurations of inflation and the policy rate at the risky steady state. Section 6 concludes.

2 Model and Key Concepts

2.1 Model

We use a standard New Keynesian model formulated in discrete time with an infinite horizon (Woodford (2003) and Galí (2015)). We work with the equilibrium conditions of the model in a loglinear form, except for the nonlinearity associated with the ELB constraint in the policy rate. The use of this semi-loglinear form allows us to derive analytical results and to emphasize that the effect of uncertainty analyzed is solely driven by the ELB constraint, not by other nonlinear features of the original New Keynesian model. The equilibrium conditions of the model are given by:

$$y_t = \mathbb{E}_t[y_{t+1}] - \sigma\left[i_t - \mathbb{E}_t[\pi_{t+1}] - (r^* + \delta_t)\right] \tag{1}$$

$$\pi_t = \kappa y_t + \beta \mathbb{E}_t[\pi_{t+1}] \tag{2}$$

$$i_t = \max\left[0, r^* + \phi_\pi \pi_t\right] \tag{3}$$

y_t, π_t, and i_t are the output gap, the inflation rate, and the nominal interest rate on the one-period risk-free government bond. We will refer to this interest rate as the policy rate. Equation (1) is the Euler equation, equation (2) is the Phillips Curve, and equation (3) is the truncated Taylor Rule.

$\beta \in (0, 1)$ denotes the subjective discount factor of the representative household. $\sigma > 0$ is the intertemporal elasticity of substitution in consumption; κ is the slope of the New Keynesian Phillips Curve; r^* is the long-run natural rate of interest; ϕ_π is the coefficient on inflation in the truncated Taylor Rule. δ_t is an exogenous shock. We assume that the distribution of δ_t is symmetric so as to analyze the effect of a mean-preserving spread of the shock on the economy. In the main analysis, we assume that δ_t is i.i.d. and takes three values. Specifically, it takes the values of c, 0, and $-c$ in the high, middle, and low states, respectively:

$$\delta_H = c, \;\; \delta_M = 0, \;\; \delta_L = -c \tag{4}$$

The assumption of i.i.d. means that the transition probabilities are given by the following:

$$\text{Prob}\,(\delta = c) = \frac{1 - p_M}{2} \tag{5}$$

$$\text{Prob}\,(\delta = 0) = p_M \tag{6}$$

$$\text{Prob}\,(\delta = -c) = \frac{1 - p_M}{2} \tag{7}$$

2.2 Target and Deflationary Equilibria

The recursive equilibrium of the model is defined in the standard way as a set of policy functions for the output gap, the inflation rate, and the policy rate satisfying the Euler equation, the Phillips curve, and the truncated Taylor Rule. These policy functions are functions of the exogenous variable. We use $\{i(\cdot),\, \pi(\cdot),\, y(\cdot)\}$ to denote them. In the model where the exogenous shock takes three values, the recursive equilibrium is given by a vector, $\{y_H,\, \pi_H,\, i_H,\, y_M,\, \pi_M,\, i_M,\, y_L,\, \pi_L,\, i_L\}$, which satisfies the Euler equation, the Phillips curve, and the truncated Taylor rule in H, M, and L states.

Given that there are three states and that the ELB either binds or does not bind in each of the three states, there are eight potential equilibria. As we will show later, the number of equilibria is either zero, one, or two, depending on the parameter values. In our paper, our focus is on situations where there are two equilibria. Comparing the inflation rate across the two equilibria, one equilibrium features a higher inflation rate than the other equilibrium. We will call the equilibrium with a higher inflation rate the target equilibrium and the equilibrium with a lower inflation rate the deflationary equilibrium. We will denote the target equilibrium by $\{i^{TE}(\cdot),\, \pi^{TE}(\cdot),\, y^{TE}(\cdot)\}$ and the deflationary equilibrium by $\{i^{DE}(\cdot),\, \pi^{DE}(\cdot),\, y^{DE}(\cdot)\}$. In the three-state model, we will also denote them by $\{y_H^{TE},\, \pi_H^{TE},\, i_H^{TE},\, y_M^{TE},\, \pi_M^{TE},\, i_M^{TE},\, y_L^{TE},\, \pi_L^{TE},\, i_L^{TE}\}$ and $\{y_H^{DE},\, \pi_H^{DE},\, i_H^{DE},\, y_M^{DE},\, \pi_M^{DE},\, i_M^{DE},\, y_L^{DE},\, \pi_L^{DE},\, i_L^{DE}\}$.

2.3 Deterministic and Risky Steady States

Deterministic Steady State: Generically speaking, a deterministic steady state of the model is where the economy is at in the absence of any exogenous shocks. In our model with the three-state shock, a deterministic steady state of the model is given by the output gap, the inflation rate, and the policy rate if $c = 0$. If $c = 0$, the values of these variables are identical in all three states. To highlight the difference with the risky steady state, it is useful to characterize a deterministic steady state as $\{y_M,\, \pi_M,\, i_M\}$ when $c = 0$.

When there are two equilibria, there are two deterministic steady states: one associated with the target equilibrium and one associated with the deflationary equilibrium. We will denote the deterministic steady state associated with the target equilibrium by $\{y_{DSS}^{TE},\, \pi_{DSS}^{TE},\, i_{DSS}^{TE}\}$. We will denote the deterministic steady state associated with the deflationary equilibrium by $\{y_{DSS}^{DE},\, \pi_{DSS}^{DE},\, i_{DSS}^{DE}\}$. Because the deterministic steady state is identical to the value of the model's variables in the middle state, we can also state that the deterministic steady states associated with the target and deflationary equilibria are given by $\{y_M^{TE}$, π_M^{TE}, $i_M^{TE}\}$ and $\{y_M^{DE}$, π_M^{DE}, $i_M^{DE}\}$ when $c = 0$.

Risky Steady State: Generically speaking, a risky steady state of the economy is a point at which the economy eventually converges when exogenous shocks exist and are at its steady state values (Coeurdacier, Rey, and Winant (2011)). The key difference with the deterministic steady state is that even though the exogenous shocks are at their steady-state values, agents in the model are aware of the possibility that they may take other values in the next period and optimize accordingly. In a model in which there is no endogenous state variable, a risky steady state of the model is given by policy functions evaluated at exogenous shocks taking their steady-state values. In our model, that means the output gap, the inflation rate, and the policy rate in the M state when $\delta = 0$ ($\{y_M$, π_M, $i_M\}$).

When there are two equilibria, there are two deterministic steady states: one associated with the target equilibrium and one associated with the deflationary equilibrium. We will denote the risky steady state associated with the target equilibrium by $\{y_{RSS}^{TE}$, π_{RSS}^{TE}, $i_{RSS}^{TE}\}$. We will denote the risky steady state associated with the deflationary equilibrium by $\{y_{RSS}^{DE}$, π_{RSS}^{DE}, $i_{DSS}^{DE}\}$. Because the risky steady state is identical to the value of the model's variables in the middle state, we can also write the deterministic steady states associated with the target and deflationary equilibria by $\{y_M^{TE}$, π_M^{TE}, $i_M^{TE}\}$ and $\{y_M^{DE}$, π_M^{DE}, $i_M^{DE}\}$.

3 Results

3.1 Allocations without Uncertainty

When there is no uncertainty, it is straightforward to compute allocations for the target and deflationary equilibria—which correspond to the deterministic steady states of those two equilibria. The allocations in the target equilibrium are given by

$$y_{DSS}^{TE} = 0, \quad \pi_{DSS}^{TE} = 0, \quad i_{DSS}^{TE} = r^* \tag{8}$$

and the allocations in the deflationary equilibrium are given by

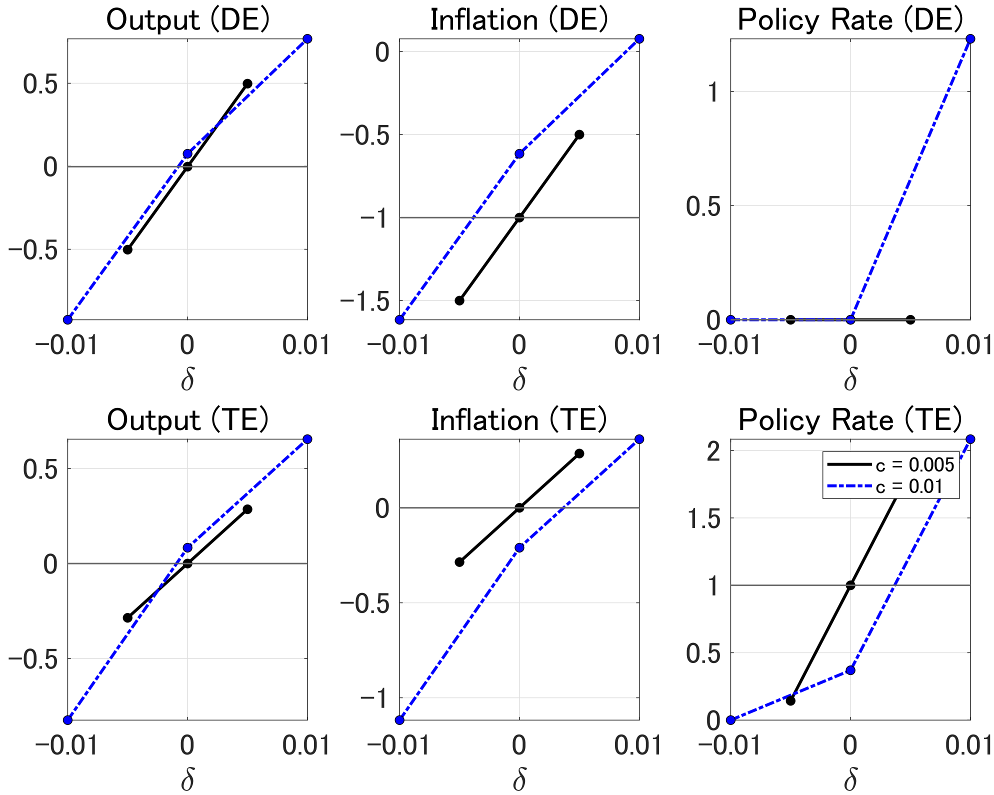
$$y_{DSS}^{DE} = -\frac{r^*(1-\beta)}{\kappa}, \quad \pi_{DSS}^{DE} = -r^*, \quad i_{DSS}^{DE} = 0 \tag{9}$$

3.2 Allocations with Uncertainty

We now examine the effect of uncertainty on steady-state allocations by comparing allocations with varying degrees of uncertainty.

The upper panels in Figure 1 show the policy functions for the output gap, inflation, and the policy rate in the deflationary equilibrium. Solid black and dashed blue lines represent the case of low and high uncertainty, respectively.

Figure 1: Policy functions: Moderate ϕ_π



When the degree of uncertainty is low, the policy functions are symmetric. The magnitudes of the increases in output and inflation when δ rises from 0 to c are the same as those when δ declines from 0 to $-c$. This symmetry arises because the policy rate remains at the ELB in all three states. The policy rate is still at the ELB even in the high state because the increase in inflation is small when c is small. As a result, the allocations in the middle state—the risky steady state allocations—are identical to those in the deterministic steady state, which are indicated by thin black horizontal lines. Note that the expected inflation is identical to the actual inflation when the economy is in the middle state in this symmetric case.

When the degree of uncertainty is high, the policy functions are asymmetric. When δ rises from 0 to c, output and inflation increase. If the increase in inflation is sufficiently large, the Taylor rule implies a positive policy rate. Thus, the policy rate also increases and partially offsets the increase in inflation. When δ declines from 0 to $-c$, output and inflation decline. Because the policy rate is bounded below by the ELB in the middle state, the declines in output and inflation are not met by a reduction in the policy rate no matter how large the shock size is.

Due to this asymmetry, certainty equivalence breaks down. In particular, all else equal, the introduction of a high degree of uncertainty pushes down the expected inflation in the middle state. In the deflationary equilibrium, a decline in the expected inflation—somewhat paradoxically—increases the actual inflation, as pointed out by Bilbiie (2022), Mertens and Ravn (2014), and Nakata and Schmidt (2022). In a rational-expectations equilibrium, an increase in the actual inflation pushes up inflation expectations, which further pushes up the actual inflation. All told, the rate of inflation in the middle state—which is equivalent to the rate of inflation at the risky steady state—is higher than that at the deterministic steady state in the deflationary equilibrium.

This effect of uncertainty in the steady-state inflation in the deflationary equilibrium is the opposite of that in the target equilibrium analyzed by Hills, Nakata, and Schmidt (2019). As shown in the lower panels of Figure 1—which shows the policy functions for the target equilibrium—the introduction of a high degree of uncertainty breaks certainty equivalence: When δ rises from 0 to c, output and inflation increase. However, their increases are partially offset by the corresponding increase in the policy rate. When δ declines from 0 to $-c$, output and inflation decline. If the decline in inflation is sufficiently large, the policy rate faces the ELB constraint. This asymmetry pushes down the expected inflation, which in turn lowers the actual inflation rate in the target equilibrium.

We can also theoretically prove these effects of uncertainty on the state-state inflation numerically described above.

Proposition 1. *When c is sufficiently small, the rate of inflation at the risky steady state coincides with that at the deterministic steady state in the deflationary equilibrium. When c is sufficiently large, the rate of inflation is higher at the risky steady state than at the deterministic steady state in the deflationary equilibrium.*

Proposition 2. *When c is sufficiently small, the rate of inflation at the risky steady state coincides with that at the deterministic steady state in the target equilibrium. When c is sufficiently large, the rate of inflation is lower at the risky steady state than at the deterministic steady state in the target equilibrium.*

The proofs for these propositions are in Appendix B.

4 Risk-adjusted Fisher Relation

4.1 Heuristics

It is common to visualize the deterministic steady states of the New Keynesian model using the Fisher relation and the truncated Taylor rule (Benhabib, Schmitt-Grohe, and Uribe (2001) and Bullard (2010)). The Fisher relation is a relationship between the rate of inflation and the policy rate in the absence of uncertainty. If there is no uncertainty in our three-state model, the Euler equation in the middle state is given by

$$y_M = \mathbb{E}[y] - \sigma\left[i_M - \mathbb{E}_M[\pi] - r^*\right] \tag{10}$$

In the absence of uncertainty, the expected output gap and the expected rate of inflation are identical to y_M and π_M, respectively. As a result, the Euler equation above can be written as

$$i_M = r^* + \pi_M \tag{11}$$

We call this relationship the Fisher relation. The intersections of the Fisher relation and the truncated Taylor rule are the model's deterministic steady states.

If we take uncertainty into account and if the degree of uncertainty is sufficiently large, the policy functions are asymmetric. As a result, the expected output gap and the expected rate of inflation are no longer identical to the output gap and the rate of inflation in the middle state. Let's denote the wedge between the actual rate of inflation and the expected rate of inflation as

$$h_y := \frac{y_H + y_L}{2} - y_M \tag{12}$$

$$h_\pi := \frac{\pi_H + \pi_L}{2} - \pi_M \tag{13}$$

Due to this wedge, a relationship between the rate of inflation and the policy rate is different from the Fisher relation described above. Specifically, the Euler equation becomes

$$\begin{aligned}
y_M &= \mathbb{E}[y] - \sigma\left[i_M - \mathbb{E}_M[\pi] - r^*\right] \\
&= p_M y_M + (1-p_M)\frac{y_H + y_L}{2} - \sigma\left[i_M - \left(p_M \pi_M + (1-p_M)\frac{\pi_H + \pi_L}{2}\right) - r^*\right] \\
&= p_M y_M + (1-p_M)(y_M + h_y) - \sigma\left[i_M - (p_M \pi_M + (1-p_M)(\pi_M + h_\pi)) - r^*\right] \\
&= y_M + (1-p_M)h_y - \sigma\left[i_M - (\pi_M + (1-p_M)h_\pi) - r^*\right] \\
i_M &= r^* + \pi_M + (1-p_M)(\frac{1}{\sigma}h_y + h_\pi) \qquad (14)
\end{aligned}$$

We call this relationship between π_M and i_M as the risky-adjusted Fisher relation. The risky steady states of the model are given by the intersections of the risk-adjusted Fisher relation and the truncated Taylor rule.

To formally define the risk-adjusted Fisher relation is a bit complicated. The Fisher relation—whether standard or risk-adjusted—gives us *possible* combinations of π_M and i_M that can constitute a steady state. It has to be defined for all values of π_M. Except for one or two values of π_M, all other values are not consistent with either the target or deflationary equilibrium. Yet, to compute the wedge between the actual and expected rates of inflation and the wedge between the actual output gap and the expected output gap, we need to compute policy functions for inflation and output that are consistent with values of π_M that do not materialize in equilibrium. Because the policy functions depend on the value of π_M, the wedge term in the risk-adjusted Fisher relation also depends on the value of π_M.

In what follows, we formally define the risk-adjusted Fisher relation, examine its properties, and illustrate it numerically.

4.2 Definition

The risk-adjusted Fisher relation is defined as follows:

$$i_M = r^* + \pi_M + \sigma^{-1}\left(\mathbb{E}[y^{h,\pi_M}(\delta')|\delta = 0] - y_M\right) + \left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] - \pi_M\right) \tag{15}$$

$y^{h,\pi_M}(\cdot)$ and $\pi^{h,\pi_M}(\cdot)$ are hypothetical policy functions for output and inflation that would prevail if $\pi^{h,M}(\delta = 0) = \pi_M$—that is, if the rate of inflation at the risky steady state is π_M—and that satisfies the *relative Euler equation*—relative to the Euler equation in the middle state—the Phillips curve. The relative Euler equation is defined for all $x \neq 0$ and is given by

$$\begin{aligned}
&y^{h,\pi_M}(\delta = x) - y^{h,\pi_M}(\delta = 0) \\
&\quad= \mathbb{E}[y^{h,\pi_M}(\delta')|\delta = x] - \mathbb{E}[y^{h,\pi_M}(\delta')|\delta = 0] - \sigma[i^{h,\pi_M}(\delta = x) - i^{h,\pi_M}(\delta = 0)] \\
&\quad+ \sigma\left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = x] - \mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]\right) - x
\end{aligned} \tag{16}$$

In other words, the relative Euler equation is what we obtain when we subtract the Euler equation at $\delta = 0$ from the Euler equation at $\delta \neq 0$.

4.3 Properties

We can analytically show that the risk-adjusted Fisher relation takes the form of a piecewise linear function:

Proposition 3. *The risk-adjusted Fisher relation is the following piecewise linear function:*

$$i_M = \begin{cases} r^* + \pi_M & \text{if } \pi_M < \pi_{LB} \\ r^* + A + B\pi_M & \text{if } \pi_{LB} \leq \pi_M \leq \pi_B \\ r^* + C + D\pi_M & \text{if } \pi_B \leq \pi_M \leq \pi_{UB} \\ r^* + \pi_M & \text{if } \pi_{UB} < \pi_M \end{cases} \tag{17}$$

where

$$\pi_{LB} := -\frac{r^* + \kappa\phi_\pi\sigma c}{\phi_\pi}, \quad \pi_B := -\frac{r^*}{\phi_\pi}, \quad \pi_{UB} := -\frac{\frac{r^*}{\phi_\pi} + \kappa\sigma(r^* - c)}{\kappa\sigma\phi_\pi + 1}$$

and

$$
\begin{aligned}
A &= \frac{1}{2}(p_M - 1)\left(r^* + \kappa\sigma c + \frac{\kappa\sigma(1-\phi_\pi)(r^*-c)}{1+\kappa\sigma\phi_\pi}\right) \\
B &= \frac{1}{2}\left(p_M + 1 - \frac{(\phi_\pi - 1)(1-p_M)}{1+\kappa\sigma\phi_\pi}\right) \\
C &= \frac{1}{2}(1-p_M)(1+\kappa\sigma)\left(r^* - \frac{\kappa\sigma\phi_\pi}{1+\kappa\sigma\phi_\pi}c\right) \\
D &= \frac{1}{2}\phi_\pi(1-p_M)(1+\kappa\sigma) + 1
\end{aligned}
$$

Proof. See Appendix C. □

Note that π_B is the rate of inflation at which the policy rate implied by the Taylor rule is exactly zero.

According to this proposition, when π_M is sufficiently away from this π_B, the risk-adjusted Fisher relation is identical to the standard Fisher relation. This result arises for the following reasons. Given an arbitrary shock size, c, if π_M is sufficiently below π_B, then a positive realization of the shock would not push the rate of inflation by a sufficiently large amount to cause the policy rate to become positive. Thus, the distribution of future inflation is symmetric. Thus, the expected inflation and output terms in the Euler equation are the same as the inflation and output in the middle state, making the risk-adjusted Fisher relation the same as the standard Fisher relation.

Similarly, if π_M is sufficiently above π_B, a negative shock would not push down inflation by a sufficiently large amount to cause the policy rate to face the ELB, implying the symmetric distribution in future inflation and output. Accordingly, the risk-adjusted Fisher relation is the same as the standard Fisher relation.

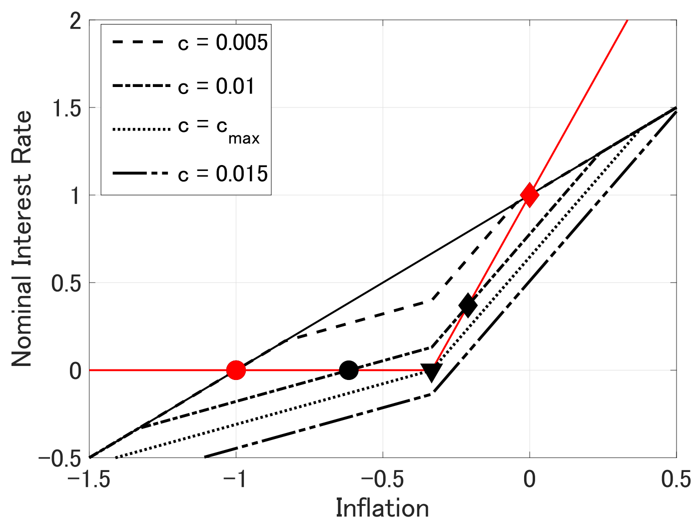
When π_M is sufficiently close to π_B, the risk-adjusted Fisher relation diverges from the standard Fisher relation. When π_M is below π_B but above π_{LB}, the policy rate is at the ELB in the middle and low states, but above the ELB in the high state, if the size of the shock—c—is sufficiently large. This asymmetry in the policy function for the policy rate means asymmetry in the policy functions for output and inflation. Thus, the expected output and inflation differ from output and inflation in the middle state. Accordingly, the risk-adjusted Fisher relation diverges from the standard Fisher relation.

Similarly, when π_M is above π_B but below π_{UB}, the policy rate is above the ELB in the middle and high states, but at the ELB in the low state if the size of the shock—c—is sufficiently large. This asymmetry breaks certainty equivalence, making the expected output and inflation differ from output and inflation in the middle state. Accordingly, the risk-adjusted Fisher relation diverges from the standard Fisher relation.

4.4 Numerical Illustration

Figure 2 shows the risk-adjusted Fisher relation with various degrees of uncertainty—various values for c—together with the truncated Taylor rule—shown by the solid red lines. When there is no uncertainty—when $c = 0$—the risk-adjusted Fisher relation is identical to the standard Fisher relation, as shown by the solid black line. The risky steady states are identical to the deterministic steady states in this case, as shown by the red and black diamonds in the figure. With $c = 0.005$, the risk-adjusted Fisher relation diverges from the standard Fisher relation, but only near the threshold value of π, as shown by the dashed black line. In this case, the risky steady states are also identical to the deterministic steady states.

Figure 2: Risk-adjusted Fisher Relation



With $c = 0.01$, the risk-adjusted Fisher relation shifts down further—shown by the dah-dotted line—from the standard Fisher relation. In this case, the risky steady states are no longer identical to the deterministic steady states. In particular, the risky steady state of the deflationary equilibrium—denoted by the black diamond—features a higher inflation rate than the deterministic steady state of the deflationary equilibrium. The risky steady state of the target equilibrium—denoted by the red diamond—features a lower inflation rate than the deterministic steady state of the target equilibrium. These results are consistent with the analytical results discussed in the previous section.

With $c = c_{max}$, the risk-adjusted Fisher relation touches the truncated Taylor rule at the threshold value of π_M. In this case, there is only one equilibrium in the model. If c increases further, the risk-adjusted Fisher relation does not intersect the truncated Taylor rule at all, which corresponds to the case with no equilibrium.

5 Interesting Cases

5.1 Positive Inflation in the Deflationary Equilibrium

The graphical illustration based on the risk-adjusted Fisher relation suggests that the risky steady state inflation rate of the deflationary equilibrium may take any values between $-r^*$ and π_B—the "bottom part" of the truncated Taylor rule. In models where the target rate of inflation is positive, the threshold value can be positive. As a result, if the degree of uncertainty is sufficiently large, the risky-steady-state inflation rate of the deflationary equilibrium becomes—somewhat paradoxically—positive.

Figure 3: Risk-adjusted Fisher Relation with Positive Inflation Target

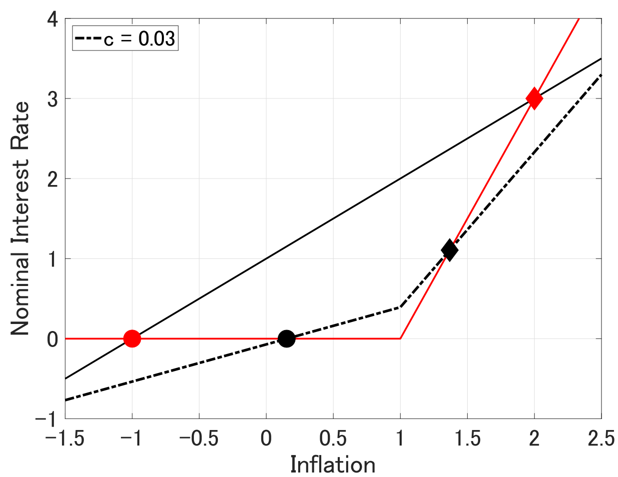
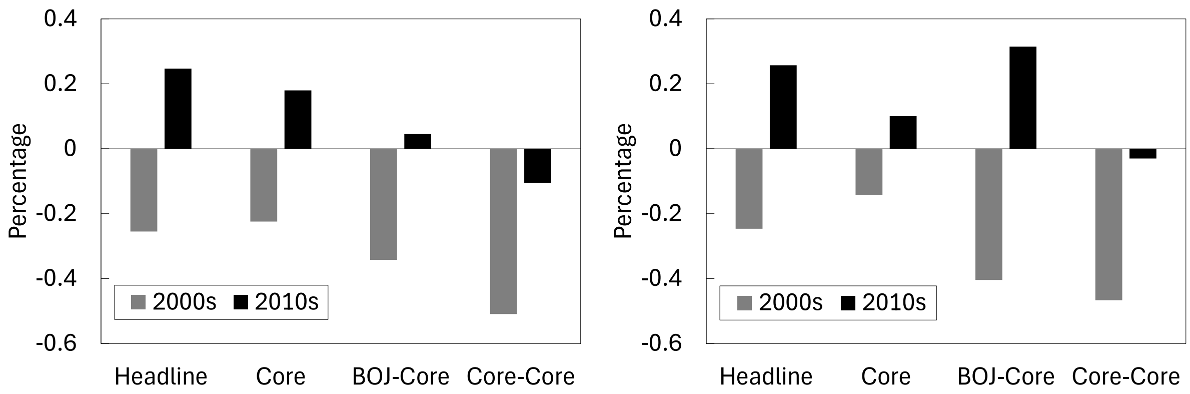


Figure 3 describes this case. In this example, the target rate of inflation is set to 2 percent, which is consistent with a positive threshold value of the inflation rate. Thus, with a sufficiently high degree of uncertainty, the inflation rate at the risky steady state is positive. The policy rate is still at the ELB, as in the standard example demonstrated in the previous section.

This configuration of the inflation rate and the policy rate is consistent with the Japanese economy in the 2010s. Since the late 1990s when the Bank of Japan first lowered the policy rate to the ELB, the Japanese economy has been described as being stuck in the deflationary equilibrium (Bullard (2010)). Such a description might have been adequate in the 2000s, but might not have been so in the 2010s. As shown by Figure 4, various measures of inflation rates in Japan hovered around a level slightly above zero percent in the 2010s, while it hovered around a level slightly below zero percent in the 2000s.

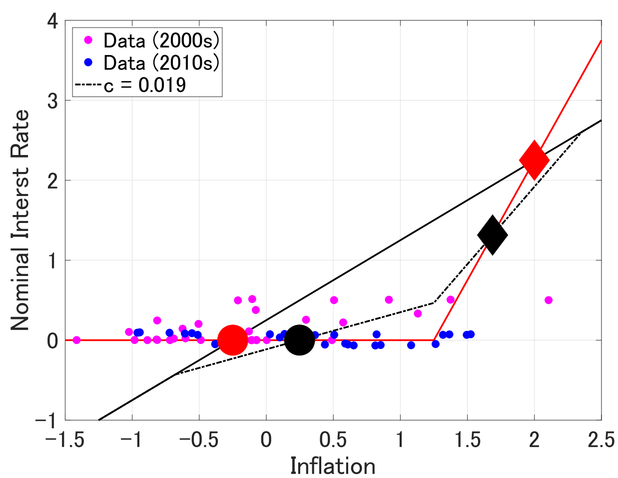
Figure 5 shows the scatter plot of the inflation rate and the policy rate together with the risk-adjusted Fisher relation and the truncated Taylor rule. Here, in computing these two theoretical constructs, we set r^* to 0.25 percent (annualized)—close to the upper estimate of the neutral rate in Nakano, Sugioka, and Yamamoto (2024). This figure reinforces the idea

Figure 4: Inflation in Japan: 2000s and 2010s



(a) Mean (b) Median

Figure 5: Risky Steady States and Inflation in Japan



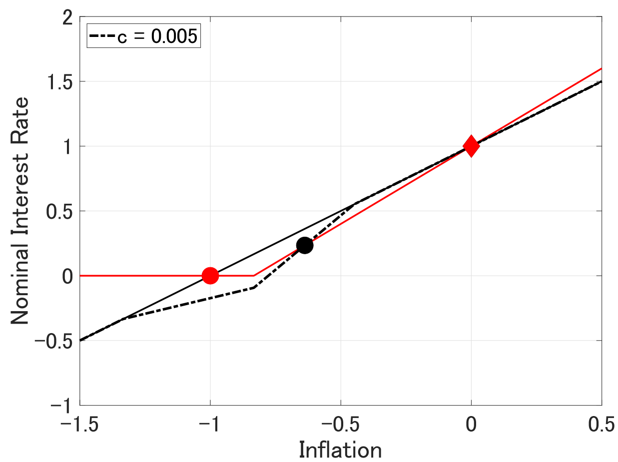
that the fluctuation around the risky steady state of the deflationary equilibrium—the black dot—may be a better description of the Japanese economy than the fluctuation around the deterministic steady state of the deflationary equilibrium—the red dot.

5.2 Positive Policy Rate in the Deflationary Equilibrium

Not only the target rate of the inflation examined above, the responsiveness of the policy rate to the inflation rate in the truncated Taylor rule—captured by the coefficient on inflation—affects the inflation rate of the risky steady state in interesting ways.

Figure 6 shows the case in which the coefficient on inflation is low. In this case, when the degree of uncertainty is sufficiently large, the risky steady state of the deflationary equilibrium can be at the part of the truncated Taylor rule featuring a positive policy rate and a below-target rate of inflation. We often associate the deflationary steady state with the binding ELB constraint. This example suggests that such an association is not warranted.

Figure 6: Risk-adjusted Fisher Relation:
Positive Policy Rate in the Risky Steady State of the Deflationary Equilibrium



It is interesting to note that the risky steady state of the target equilibrium typically features the positive policy rate and a below-target rate of inflation, as discussed in detail by Hills, Nakata, and Schmidt (2019) and reiterated in Section 3. Thus, the configuration of a positive policy rate and a below-target rate of inflation itself does not tell whether the economy is at the target equilibrium or the deflationary equilibrium.

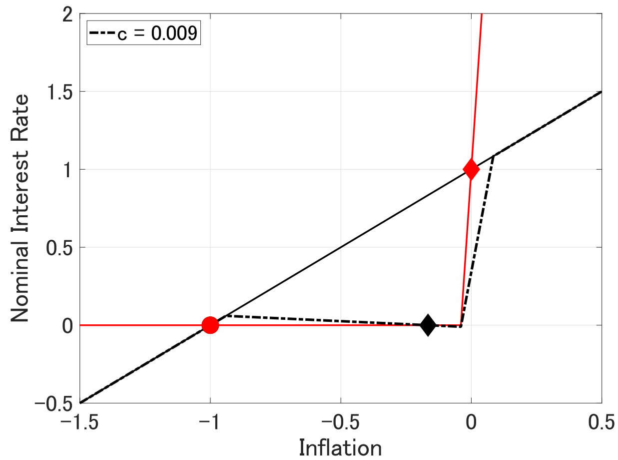
5.3 Binding ELB in the Target Equilibrium

The example above uncovered an interesting configuration of the policy rate and the inflation rate at the risky steady state of the deflationary equilibrium when the policy rate does not strongly respond to the rate of inflation. An interesting configuration can also emerge for the risky steady state of the target equilibrium when the policy rate strongly responds to the rate of inflation.

Figure 7 shows the case in which the coefficient on inflation is high. In this case, when the degree of uncertainty is sufficiently large, the risky steady state of the target equilibrium can be at the bottom part of the truncated Taylor rule featuring the binding ELB constraint and a below-target rate of inflation. We often associate the steady state of the target equilibrium with a positive policy rate. This example suggests that such an association is not warranted.

The risky steady state of the deflationary equilibrium typically features the binding ELB constraint and a below-target rate of inflation, as discussed in Section 3. Thus, the configuration of the binding ELB constraint policy rate and a below-target rate of inflation itself does not tell whether the economy is at the target equilibrium or the deflationary equilibrium.

Figure 7: Risk-adjusted Fisher Relation:
Binding ELB in the Risky Steady State of the Target Equilibrium



6 Conclusion

In this paper, we analyzed how uncertainty about the future course of the economy affects allocations under the deflationary equilibrium, paying particular attention to the steady-state allocation. We find that the rate of inflation is higher at the risky steady state than at the deterministic steady state in the deflationary equilibrium. When the target rate of inflation set by the central bank is positive, our result implies that the rate of inflation can be—somewhat paradoxically—positive at the risky steady state of the deflationary equilibrium if the degree of uncertainty is sufficiently large. This situation is consistent with the Japanese economy in the 2010s when the rate of inflation hovered around a slightly positive level while the policy rate was constrained at the ELB. Along the way, we also proposed a novel concept, the risk-adjusted Fisher relation, to visualize the effect of uncertainty on the steady-state inflation rate and the policy rate.

The long-run trend rate of inflation is either the target rate set by the central bank or the inverse of the subjective discount factor of the household in the New Keynesian model if we abstract away from uncertainty. Many researchers find that the long-run trend rate of inflation can deviate persistently from these two possibilities in reality. Our analysis shows that uncertainty can help bridge the gap between the model and reality in terms of the long-term trend of inflation.

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Technical Appendix for Online Publication

A Proofs Related to Equilibrium Existence

A.1 Equilibrium Conditions: State-by-State

Before we proceed to various proofs, it is useful to spell out the equilibrium conditions of our three-state model state-by-state.

$$y_H = \mathbb{E}[y_{t+1}] - \sigma\left[i_H - r^* - \mathbb{E}[\pi_{t+1}] - c\right] \tag{A1}$$
$$y_M = \mathbb{E}[y_{t+1}] - \sigma\left[i_M - r^* - \mathbb{E}[\pi_{t+1}]\right] \tag{A2}$$
$$y_L = \mathbb{E}[y_{t+1}] - \sigma\left[i_L - r^* - \mathbb{E}[\pi_{t+1}] + c\right] \tag{A3}$$

$$\pi_H = \kappa y_H + \beta\mathbb{E}[\pi_{t+1}] \tag{A4}$$
$$\pi_M = \kappa y_M + \beta\mathbb{E}[\pi_{t+1}] \tag{A5}$$
$$\pi_L = \kappa y_L + \beta\mathbb{E}[\pi_{t+1}] \tag{A6}$$

$$i_H = \max\left[0, r^* + \phi_\pi \pi_H\right] \tag{A7}$$
$$i_M = \max\left[0, r^* + \phi_\pi \pi_M\right] \tag{A8}$$
$$i_L = \max\left[0, r^* + \phi_\pi \pi_L\right] \tag{A9}$$

where $\mathbb{E}[x_{t+1}] := \frac{1-p_M}{2}x_H + p_M x_M + \frac{1-p_M}{2}x_L$ and $x \in \{y, \pi\}$.

A.2 Candidate Equilibria

There are 8 ($= 2^3$) possible equilibria in our model because the ELB constraint either binds or does not bind in each of the three states.

Definition A.1. *Let the equilibrium 1 be an equilibrium in which the ELB does not bind in any state.*

Definition A.2. *Let the equilibrium 2 be an equilibrium in which the ELB binds only in the low state.*

Definition A.3. *Let the equilibrium 3 be an equilibrium in which the ELB binds only in the middle state.*

Definition A.4. *Let the equilibrium 4 be an equilibrium in which the ELB binds only in the high state.*

Definition A.5. *Let the equilibrium 5 be an equilibrium in which the ELB binds in the low and middle state but does not bind in the high state.*

Definition A.6. *Let the equilibrium 6 be an equilibrium in which the ELB binds in the low and high state but does not bind in the middle state.*

Definition A.7. *Let the equilibrium 7 be an equilibrium in which the ELB binds in the middle and high state but does not bind in the low state.*

Definition A.8. *Let the equilibrium 8 be an equilibrium in which the ELB binds in all three states.*

A.3 Non-existence of Some Equilibria

Proposition A.1. *Candidate equilibria 3, 4, 6, and 7 do not exist for any* $c \geq 0$*.*

Proof. To begin, we can rewrite our system of nine equations and nine unknowns into a system of three equations and three unknowns. Our three unknowns will be $\{\pi_H, \pi_M, \pi_L\}$. To do so, rewrite the Phillips Curve as

$$y_j = \frac{\pi_j - \beta\mathbb{E}[\pi_{t+1}]}{\kappa}$$

for $j \in \{H, M, L\}$. Substitute the above equation and the Taylor rule into the Euler equation. We now have a system of three equations and three unknowns. To check if this is a valid equilibrium, we will construct the shadow policy rate as follows.

$$i_j^{shadow} = r^* + \phi_\pi \pi_j^*.$$

where π_j^* is the solution to the system of linear equations. If i_j^{shadow} is at odds with the equilibrium condition we assumed, then this cannot be a valid equilibrium.

In the remainder of the proof, we will consider these four candidate equilibria one by one.

Equilibrium 3

By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 3 is given by

$$\pi_L = \frac{-\sigma\left(p_M\sigma\left(r^*+c\right)\kappa + \left(r^*+c\right)p_M - c\right)\kappa\phi_\pi - \sigma\left(p_M r^* + c\right)\kappa - p_M r^*}{\left(1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi\right)\left(\kappa\phi_\pi\sigma + 1\right)}$$
$$\pi_M = -\frac{\left(\kappa\phi_\pi\sigma + \sigma\left(p_M - 1\right)\kappa + p_M\right)r^*}{1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi}$$
$$\pi_H = \frac{-\sigma\kappa\left(p_M\sigma\left(r^*-c\right)\kappa + \left(r^*-c\right)p_M + c\right)\phi_\pi - \sigma\left(p_M r^* - c\right)\kappa - p_M r^*}{\left(1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi\right)\left(\kappa\phi_\pi\sigma + 1\right)}$$

and the shadow policy rate evaluated at $c = 0$ is given by:

$$i_L^{shadow} = -\frac{r^*\left(\phi_\pi - 1\right)\left(\kappa\phi_\pi\sigma + 1\right)}{\left(1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi\right)\left(\kappa\phi_\pi\sigma + 1\right)}$$
$$i_M^{shadow} = -\frac{r^*\left(\phi_\pi - 1\right)\left(\kappa\phi_\pi\sigma + 1\right)}{1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi}$$
$$i_H^{shadow} = -\frac{r^*\left(\phi_\pi - 1\right)\left(\kappa\phi_\pi\sigma + 1\right)}{\left(1+\left(\kappa p_M\sigma + p_M - 1\right)\phi_\pi\right)\left(\kappa\phi_\pi\sigma + 1\right)}$$

Suppose that $(\kappa p_M \sigma + p_M - 1)\,\phi_\pi + 1 > 0$. Then, $i_L^{shadow}(c=0) < 0$. We can show that

$$\frac{\partial i_L^{shadow}}{\partial c} = -\frac{\kappa \phi_\pi \sigma}{\kappa \phi_\pi \sigma + 1} < 0$$

Thus, $i_L^{shadow} < 0$ for any c, which contradicts the equilibrium condition that ELB does not bind in L state.

Suppose that $(\kappa p_M \sigma + p_M - 1)\,\phi_\pi + 1 < 0$. Then, $i_M^{shadow}(c=0) > 0$. We can show that

$$\frac{\partial i_M^{shadow}}{\partial c} = 0$$

Thus, $i_M^{shadow} > 0$ for any c, which contradicts the equilibrium condition that ELB binds in the M state.

Equilibrium 4

By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 4 is given by

$$\pi_L = \frac{-\sigma^2 \phi_\pi (r^* + 2c)(p_M - 1)\,\kappa^2 - \sigma\left(\left(\left((r^* + 2c)\,p_M - r^*\right)\phi_\pi + p_M r^* - r^* - 2c\right)\kappa - r^*(p_M - 1)\right)}{(\kappa \phi_\pi \sigma + 1)\left(\sigma \phi_\pi (p_M - 1)\,\kappa - 2 + (p_M + 1)\,\phi_\pi\right)}$$

$$\pi_M = -\frac{(p_M - 1)\left(\kappa \sigma (r^* + c)\,\phi_\pi + r^*\right)(\sigma \kappa + 1)}{(\kappa \phi_\pi \sigma + 1)\left(-2 + \left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi\right)}$$

$$\pi_H = \frac{-\left((-2r^* - 2c)\,\phi_\pi + (p_M + 1)\,r^* + 2c\right)\sigma \kappa - r^*(p_M - 1)}{\sigma \phi_\pi (p_M - 1)\,\kappa - 2 + (p_M + 1)\,\phi_\pi}$$

and the shadow policy rate evaluated at $c = 0$ is given by:

$$i_L^{shadow} = \frac{2(\phi_\pi - 1)\,r^*}{\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2}$$

$$i_M^{shadow} = \frac{2(\phi_\pi - 1)\,r^*}{\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2}$$

$$i_H^{shadow} = \frac{2r^*(\phi_\pi - 1)(\kappa \phi_\pi \sigma + 1)}{\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2}$$

Suppose $\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2 > 0$. Then, $i_H^{shadow}(c=0) > 0$. We can show that

$$\frac{\partial i_H^{shadow}}{\partial c} = \frac{2(\phi_\pi - 1)\,\kappa \phi_\pi \sigma}{\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2} > 0$$

Thus, $i_H^{shadow} > 0$ for any c, which contradicts the equilibrium condition that ELB binds in H state.

Suppose that $\left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi - 2 < 0$. Then, $i_M^{shadow}(c=0) < 0$. We can show that

$$\frac{\partial i_M^{shadow}}{\partial c} = -\frac{(\sigma \kappa + 1)(p_M - 1)\,\kappa \sigma \phi_\pi^2}{\left(-2 + \left((p_M - 1)\,\sigma \kappa + p_M + 1\right)\phi_\pi\right)(\kappa \phi_\pi \sigma + 1)} < 0$$

Thus, $i_M^{shadow} < 0$ for any c, which contradicts the equilibrium condition that ELB does not bind in the M state.

Equilibrium 6

By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 6 is given by

$$\begin{aligned}
\pi_L &= \frac{-c\,\sigma^2\phi_\pi\left(p_M-1\right)\kappa^2-\left(\left(cp_M-r^*\right)\phi_\pi+p_Mr^*-c\right)\sigma\kappa-r^*\left(p_M-1\right)}{\sigma\phi_\pi\left(p_M-1\right)\kappa+p_M\phi_\pi-1}\\
\pi_M &= -\frac{r^*\left(\sigma\kappa+1\right)\left(p_M-1\right)}{-1+\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi}\\
\pi_H &= \frac{c\,\sigma^2\phi_\pi\left(p_M-1\right)\kappa^2-\left(\left(-cp_M-r^*\right)\phi_\pi+p_Mr^*+c\right)\sigma\kappa-r^*\left(p_M-1\right)}{\sigma\phi_\pi\left(p_M-1\right)\kappa+p_M\phi_\pi-1}
\end{aligned}$$

and the shadow policy rate evaluated at $c=0$ is given by:

$$\begin{aligned}
i_L^{shadow} &= \frac{r^*\left(\phi_\pi-1\right)\left(\kappa\phi_\pi\sigma+1\right)}{-1+\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi}\\
i_M^{shadow} &= \frac{r^*\left(\phi_\pi-1\right)}{-1+\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi}\\
i_H^{shadow} &= \frac{r^*\left(\phi_\pi-1\right)\left(\kappa\phi_\pi\sigma+1\right)}{-1+\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi}
\end{aligned}$$

Suppose that $\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi-1>0$. Then, $i_H^{shadow}(c=0)>0$. We can show that

$$\frac{\partial i_H^{shadow}}{\partial c}=\kappa\phi_\pi\sigma>0$$

Thus, $i_H^{shadow}>0$ for any c, which contradicts the equilibrium condition that ELB binds in H state.

Suppose that $\left(\sigma\left(p_M-1\right)\kappa+p_M\right)\phi_\pi-1<0$. Then, $i_M^{shadow}(c=0)<0$. We can show that

$$\frac{\partial i_M^{shadow}}{\partial c}=0$$

Thus, $i_M^{shadow}<0$ for any c, which contradicts the equilibrium condition that ELB does not bind in the M state.

Equilibrium 7

By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 7 is given by

$$\pi_L = \frac{-\sigma\left(p_M r^* + 2c + r^*\right)\kappa - r^*\left(p_M+1\right)}{\sigma\phi_\pi\left(p_M+1\right)\kappa + 2 + \left(p_M-1\right)\phi_\pi}$$
$$\pi_M = \frac{\sigma\left(c\sigma\left(p_M-1\right)\kappa + p_M c - 2r^* - c\right)\kappa\phi_\pi - r^*\left(\left(p_M-1\right)\sigma\kappa + p_M + 1\right)}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$
$$\pi_H = \frac{2c\,\kappa^2 p_M\phi_\pi\sigma^2 - \sigma\left(\left(\left(-2p_M c + 2c + 2r^*\right)\phi_\pi + p_M r^* - r^* - 2c\right)\kappa - r^*\left(p_M+1\right)\right)}{\sigma\phi_\pi\left(p_M+1\right)\kappa + 2 + \left(p_M-1\right)\phi_\pi}$$

and the shadow policy rate evaluated at $c = 0$ is given by:

$$i_L^{shadow} = -\frac{2r^*\left(\phi_\pi - 1\right)}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$
$$i_M^{shadow} = -\frac{2r^*\left(\phi_\pi - 1\right)\left(\kappa\phi_\pi\sigma + 1\right)}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$
$$i_H^{shadow} = -\frac{2r^*\left(\phi_\pi - 1\right)\left(\kappa\phi_\pi\sigma + 1\right)}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$

Suppose that $2+\left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi > 0$. Then, $i_L^{shadow}(c=0) < 0$. We can show

$$\frac{\partial i_L^{shadow}}{\partial c} = -\frac{2\kappa\phi_\pi\sigma}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi} < 0$$

Thus, $i_L^{shadow} < 0$ for any c, which contradicts the equilibrium condition that ELB does not bind in the L state.

Suppose that $2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi < 0$. Then, $i_M^{shadow}(c = 0) > 0$. It can be shown that

$$\frac{\partial i_M^{shadow}}{\partial c} = \frac{\sigma\kappa\phi_\pi^2\left(\sigma\kappa+1\right)\left(p_M-1\right)}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi} > 0$$

Thus, $i_M^{shadow} > 0$ for any c, which contradicts the equilibrium condition that ELB binds in the M state. □

A.4 Existence of Other Equilibria

Unless otherwise noted, we will assume that $\underline{\phi}_\pi < \phi_\pi < \overline{\phi}_\pi$ where

$$\underline{\phi}_\pi := \frac{2}{\sigma\kappa p_M - \sigma\kappa + p_M + 1}$$
$$\overline{\phi}_\pi := \frac{-2}{\sigma\kappa p_M + \sigma\kappa + p_M - 1}$$

We will also assume that $p_M > \left(\sigma\kappa - 1\right)/\left(\sigma\kappa + 1\right)$.

Proposition A.2. *Equilibrium 1 exists when $c \le \bar{c}$ where*

$$\bar{c} = \frac{r^*\left(\kappa\phi_\pi\sigma + 1\right)}{\kappa\phi_\pi\sigma}$$

Proposition A.3. *Equilibrium 2 exists when $\overline{c} \leq c \leq \tilde{c}$ where $\overline{c}$ is defined in the previous proposition and*

$$\tilde{c} = \frac{2r^* \left(\phi_\pi - 1\right) \left(\kappa \phi_\pi \sigma + 1\right)}{\kappa \left(\sigma \kappa + 1\right) \left(1 - p_M\right) \sigma \phi_\pi^2}$$

Proposition A.4. *Equilibrium 5 exists when $\underline{c} \leq c \leq \tilde{c}$ where $\tilde{c}$ is defined in the previous proposition and*

$$\underline{c} = \frac{r^* \left(\phi_\pi - 1\right)}{\kappa \phi_\pi \sigma}$$

Proposition A.5. *Equilibrium 8 exists when $c \leq \underline{c}$ where $\underline{c}$ is defined in the previous proposition.*

Proposition A.6. *There are two equilibria when $c < \tilde{c}$. There is one equilibrium when $c = \tilde{c}$. There is no equilibrium when $c > \tilde{c}$.*

Proof of Proposition A.2.: By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 1 is given by

$$\begin{aligned} \pi_L &= -\frac{\kappa \sigma}{\kappa \phi_\pi \sigma + 1} c \\ \pi_M &= 0 \\ \pi_H &= \frac{\kappa \sigma}{\kappa \phi_\pi \sigma + 1} c \end{aligned}$$

When $c = 0$, $\pi_H = \pi_M = \pi_L = 0$. According to the truncated Taylor rule, $i_H = i_M = i_L = r^*$. Thus, Equilibrium 1 exists when $c = 0$. when $c > 0$, $\pi_L < \pi_M < \pi_H$. Accordingly, to find the maximum value of c for which this equilibrium exists, we only need to check when i_L^{shadow} is at odd with our equilibrium condition.

i_L^{shadow} consistent with π_L above is given by

$$i_L^{shadow} = -\frac{\kappa \sigma \phi_\pi}{\kappa \phi_\pi \sigma + 1} c + r^*$$

If we find the value of c such that $i_L^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 1 exists when $c \leq \overline{c}$.

Proof of Proposition A.3.: By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 2 is given by

$$\begin{aligned} \pi_L &= \frac{\left(\kappa \sigma \left(r^* - c\right) \phi_\pi + r^*\right) \left(\sigma \kappa + 1\right) \left(1 - p_M\right)}{\left(\left(p_M - 1\right) \sigma \kappa + p_M + 1\right) \phi_\pi - 2} + \kappa \sigma (r^* - c) \\ \pi_M &= \frac{\left(\kappa \sigma \left(r^* - c\right) \phi_\pi + r^*\right) \left(\sigma \kappa + 1\right) \left(1 - p_M\right)}{\left(\kappa \phi_\pi \sigma + 1\right) \left(\left(\left(p_M - 1\right) \sigma \kappa + p_M + 1\right) \phi_\pi - 2\right)} \\ \pi_H &= \pi_M + \frac{\kappa \sigma c}{1 + \kappa \sigma \phi_\pi} \end{aligned}$$

When $c > 0$, $\pi_M < \pi_H$. Thus, to find the minimum and maximum on c which can be supported by this equilibrium, it suffices to find the value of c where $i_L^{shadow} = 0$ and $i_M^{shadow} =$

0, respectively. i_L^{shadow} and i_M^{shadow} consistent with π_L and π_M above are given by

$$i_L^{shadow} = \frac{2\left(\phi_\pi - 1\right)\left(\kappa\sigma\left(r^* - c\right)\phi_\pi + r^*\right)}{\left(\left(p_M - 1\right)\sigma\kappa + p_M + 1\right)\phi_\pi - 2}$$
$$i_M^{shadow} = \frac{\kappa\left(c\sigma\left(p_M - 1\right)\kappa + \left(p_M - 1\right)c + 2r^*\right)\sigma\phi_\pi^2 + \left(2r^* - 2\kappa r^*\sigma\right)\phi_\pi - 2r^*}{\left(\left(\left(p_M - 1\right)\sigma\kappa + p_M + 1\right)\phi_\pi - 2\right)\left(\kappa\phi_\pi\sigma + 1\right)}$$

If we find the value of c such that $i_L^{shadow} = 0$, that is the minimum degree of uncertainty consistent with this equilibrium. If we find the value of c such that $i_M^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 2 exists when $\bar{c} \leq c \leq \tilde{c}$.

Proof of Proposition A.4.: By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 5 is given by

$$\pi_L = \pi_M - \kappa\sigma c$$
$$\pi_M = \frac{-\left(c\sigma\left(p_M - 1\right)\kappa + p_M c + 2r^* - c\right)\sigma\kappa\phi_\pi - \left(\sigma\left(p_M - 1\right)\kappa + p_M + 1\right)r^*}{2 + \left(\sigma\left(p_M + 1\right)\kappa + p_M - 1\right)\phi_\pi}$$
$$\pi_H = \frac{\pi_M - \kappa\sigma(r^* - c)}{1 + \kappa\sigma\phi_\pi}$$

When $c > 0$, $\pi_L < \pi_M$. Thus, to find the minimum and maximum on c which can be supported by this equilibrium, it suffices to find the value of c where $i_M^{shadow} = 0$ and $i_H^{shadow} = 0$, respectively. i_M^{shadow} and i_H^{shadow} consistent with π_M and π_H above are given by

$$i_M^{shadow} = \frac{-\left(c\sigma\left(p_M - 1\right)\kappa + \left(p_M - 1\right)c + 2r^*\right)\kappa\sigma\phi_\pi^2 + \left(2\kappa r^*\sigma - 2r^*\right)\phi_\pi + 2r^*}{2 + \left(\sigma\left(p_M + 1\right)\kappa + p_M - 1\right)\phi_\pi}$$
$$i_H^{shadow} = \frac{\left(2c\sigma\kappa - 2r^*\right)\phi_\pi + 2r^*}{2 + \left(\sigma\left(p_M + 1\right)\kappa + p_M - 1\right)\phi_\pi}$$

If we find c such that $i_M^{shadow} = 0$, that is the maximum degre of uncertainty consistent with this equilibrium. If we find c such that $i_H^{shadow} = 0$, that is the minimum degree of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 5 exists when $\underline{c} \leq c \leq \tilde{c}$.

Proof of Proposition A.5.: By solving a system of linear equations defining the model's equilibrium conditions, inflation in Equilibrium 8 is given by

$$\pi_L = -r^* - \kappa\sigma c$$
$$\pi_M = -r^*$$
$$\pi_H = -r^* + \kappa\sigma c$$

When $c = 0$, $\pi_H = \pi_M = \pi_L = -r^*$. According to the shadow policy rate, $i_H^{shadow} = i_M^{shadow} = i_L^{shadow} = 0$. Thus, Equilibrium 8 exists when $c = 0$. When $c > 0$, $\pi_L < \pi_M < \pi_H$. Accordingly, to find the maximum value of c for which this equilibrium exists, we only need to compute when i_L^{shadow} is at odds with our equilibrium condition.

i_H^{shadow} consistent with π_H above is given by

$$i_H^{shadow} = \left(\kappa\sigma c - r^*\right)\phi_\pi + r^*$$

If we find c such that $i_H^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium.

Proof of Proposition A.6.: When $\underline{\phi}_\pi < \phi_\pi$, $\underline{c} < \overline{c}$ holds. By Proposition A.2 to Proposition A.5, $\underline{c} < \overline{c} < \tilde{c}$. According to Proposition A.2-A.5, Equilibrium 1 and Equilibrium 8 exist when $c \leq \underline{c}$, Equilibrium 1 and Equilibrium 5 exist when $\underline{c} \leq c \leq \overline{c}$, Equilibrium 2 and Equilibrium 5 exist when $\overline{c} \leq c \leq \tilde{c}$. When $c = \tilde{c}$, Equilibrium 2 and Equilibrium 5 coincide. Accordingly, there is one equilibrium. When $c > \tilde{c}$, there is no equilibrium.

B Proofs of the Main Propositions

Proposition 1. *When c is sufficiently small, the rate of inflation at the risky steady state coincides with that at the deterministic steady state in the deflationary equilibrium. When c is sufficiently large, the rate of inflation is higher at the risky steady state than at the deterministic steady state in the deflationary equilibrium.*

Proposition 2. *When c is sufficiently small, the rate of inflation at the risky steady state coincides with that at the deterministic steady state in the target equilibrium. When c is sufficiently large, the rate of inflation is lower at the risky steady state than at the deterministic steady state in the target equilibrium.*

In what follows, we will denote π_M in Equilibrium 2 and Equilibrium 5 as π_M^L and π_M^{LM}, respectively.

Proof of Proposition 1: By Proposition A.6, Equilibrium 1 and Equilibrium 8 exist when $c \leq \underline{c}$. Then, Equilibrium 8 is the deflationary equilibrium. Thus, rate of inflation at the risky steady state coincides with that at the deterministic steady state when $c \leq \underline{c}$.

By Proposition A.6, Equilibrium 1 and Equilibrium 5 exist when $\underline{c} \leq c \leq \overline{c}$, Equilibrium 2 and Equilibrium 5 exist when $\overline{c} \leq c \leq \tilde{c}$. When $c = \tilde{c}$, Equilibrium 2 and Equilibrium 5 coincide. Accordingly, Equilibrium 5 is deflationary equilibrium when $\underline{c} < c \leq \tilde{c}$.

π_M^{LM} evaluated at $c = \underline{c}$ is $-r^*$, which coincides with the rate of inflation at the deterministic steady state. It can be shown that

$$\frac{\partial \pi_M^{LM}}{\partial c} = \frac{(\sigma\kappa + 1)(1 - p_M)\kappa\sigma\phi_\pi}{2 + (\sigma(p_M + 1)\kappa + p_M - 1)\,\phi_\pi} > 0$$

Therefore, the rate of inflation is higher at the risky steady state than at the deterministic steady state in the deflationary equilibrium when $\underline{c} < c < \tilde{c}$.

Proof of Proposition 2: By Proposition A.6, Equilibrium 1 and Equilibrium 8 exist when $c \leq \underline{c}$, Equilibrium 1 and Equilibrium 5 exist when $\underline{c} \leq c \leq \overline{c}$. Then, Equilibrium 1 is the target equilibrium when $c \leq \overline{c}$. Thus, the rate of inflation at the risky steady state coincides with that at the deterministic steady state when $c \leq \overline{c}$.

By Proposition A.6, Equilibrium 2 and Equilibrium 5 exist when $\overline{c} \leq c \leq \tilde{c}$. When $c = \tilde{c}$, Equilibrium 2 and Equilibrium 5 coincide. Accordingly, Equilibrium 2 is the target equilibrium when $\overline{c} < c < \tilde{c}$.

π_M^L evaluated at $c = \overline{c}$ is 0 which coincides with the rate of inflation at the deterministic steady state. It can be shown that

$$\frac{\partial \pi_M^L}{\partial c} = \frac{\kappa(\sigma\kappa+1)(p_M-1)\sigma\phi_\pi}{(\kappa\phi_\pi\sigma+1)\left(((p_M-1)\sigma\kappa+p_M+1)\,\phi_\pi-2\right)} < 0$$

Therefore, the rate of inflation is lower at the risky steady state than at the deterministic steady state in the target equilibrium when $\overline{c} < c < \tilde{c}$.

C Proofs Related to the Risk-Adjusted Fisher Relation

Proposition 3. *The risk-adjusted Fisher relation is the following piecewise linear function:*

$$i_M = \begin{cases} r^* + \pi_M & \text{if } \pi_M < \pi_{LB} \\ r^* + A + B\pi_M & \text{if } \pi_{LB} \leq \pi_M \leq \pi_B \\ r^* + C + D\pi_M & \text{if } \pi_B \leq \pi_M \leq \pi_{UB} \\ r^* + \pi_M & \text{if } \pi_{UB} < \pi_M \end{cases} \tag{17}$$

where

$$\pi_{LB} := -\frac{r^* + \kappa\phi_\pi\sigma c}{\phi_\pi}, \quad \pi_B := -\frac{r^*}{\phi_\pi}, \quad \pi_{UB} := -\frac{\frac{r^*}{\phi_\pi} + \kappa\sigma(r^* - c)}{\kappa\sigma\phi_\pi + 1}$$

and

$$\begin{aligned} A &= \frac{1}{2}(p_M - 1)\left(r^* + \kappa\sigma c + \frac{\kappa\sigma(1-\phi_\pi)(r^*-c)}{1+\kappa\sigma\phi_\pi}\right) \\ B &= \frac{1}{2}\left(p_M + 1 - \frac{(\phi_\pi - 1)(1-p_M)}{1+\kappa\sigma\phi_\pi}\right) \\ C &= \frac{1}{2}(1-p_M)(1+\kappa\sigma)\left(r^* - \frac{\kappa\sigma\phi_\pi}{1+\kappa\sigma\phi_\pi}c\right) \\ D &= \frac{1}{2}\phi_\pi(1-p_M)(1+\kappa\sigma) + 1 \end{aligned}$$

Proof. As discussed in Section 4, given π_M, we need to compute the risk-adjustment term based on *hypothetical* policy functions that would prevail if $\pi^{h,\pi_M}(\delta = 0) = \pi_M$ and that satisfy the *relative* Euler equations, the Phillips curve, and the truncated Taylor rule.

For any given π_M, we will first solve for such hypothetical policy functions and then compute the risk-adjusted Fisher relation consistent with them. We will do so separately for each of the four ranges of π_M: (i) $\pi_M < \pi_{LB}$, (ii) $\pi_{LB} \leq \pi_M \leq \pi_B$, (iii) $\pi_B \leq \pi_M \leq \pi_{UB}$, and (iv) $\pi_{UB} < \pi_M$.

(i) $\pi_M < \pi_{LB}$

We construct the hypothetical policy functions consistent with π_M in the following four steps.

Step 1: Construct the hypothetical policy function of inflation.

Substituting the Phillips Curve and the truncated Taylor rule into the relative Euler equation, we obtain

$$\pi^{h,\pi_M}(\delta = c) = \pi_M + \kappa\sigma c$$
$$\pi^{h,\pi_M}(\delta = -c) = \pi_M - \kappa\sigma c$$

$\pi^{h,\pi_M}(\delta = 0)$ is given by

$$\pi^{h,\pi_M}(\delta = 0) = \pi_M$$

Step 2: Express the expected inflation as a function of π_M.

Using $\pi^{h,\pi_M}(\delta = c)$ and $\pi^{h,\pi_M}(\delta = -c)$, we obtain

$$\begin{aligned}\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] :=& \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M\pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c)\\ =&\pi_M\end{aligned}$$

Step 3: Construct the hypothetical policy function of output gap.

Substituting $\pi^{h,\pi_M}(\delta = c)$, $\pi^{h,\pi_M}(\delta = -c)$, and $\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]$ into rearranged Philips curves, we obtain

$$\begin{aligned}y^{h,\pi_M}(\delta = c) &= \frac{\pi^{h,\pi_M}(\delta = c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa}\\ &= \frac{1-\beta}{\kappa}\pi_M + \sigma c\\ y^{h,\pi_M}(\delta = 0) &= \frac{\pi^{h,\pi_M}(\delta = 0) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa}\\ &= \frac{1-\beta}{\kappa}\pi_M\\ y^{h,\pi_M}(\delta = -c) &= \frac{\pi^{h,\pi_M}(\delta = -c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa}\\ &= \frac{1-\beta}{\kappa}\pi_M - \sigma c\end{aligned}$$

Step 4: Construct the hypothetical policy function of policy rate.

According to the truncated Taylor rules, the policy function of the policy rate is given by

$$i^{h,\pi_M}(\delta = c) = 0$$
$$i^{h,\pi_M}(\delta = 0) = 0$$
$$i^{h,\pi_M}(\delta = -c) = 0$$

To summarize, the hypothetical policy functions are given by

$$
\begin{aligned}
\pi^{h,\pi_M}(\delta = c) &= \pi_M + \kappa\sigma c \\
\pi^{h,\pi_M}(\delta = 0) &= \pi_M \\
\pi^{h,\pi_M}(\delta = -c) &= \pi_M - \kappa\sigma c \\
y^{h,\pi_M}(\delta = c) &= \frac{1-\beta}{\kappa}\pi_M + \sigma c \\
y^{h,\pi_M}(\delta = 0) &= \frac{1-\beta}{\kappa}\pi_M \\
y^{h,\pi_M}(\delta = -c) &= \frac{1-\beta}{\kappa}\pi_M - \sigma c \\
i^{h,\pi_M}(\delta = c) &= 0 \\
i^{h,\pi_M}(\delta = 0) &= 0 \\
i^{h,\pi_M}(\delta = -c) &= 0
\end{aligned}
$$

By construction, they satisfy the relative Euler equation, the Phillips curve, and the truncated Taylor rule and $\pi^{h,\pi_M}(\delta = 0) = \pi_M$

Now that we have obtained the hypothetical policy functions consistent with π_M such that $\pi_{LB} \leq \pi_M \leq \pi_B$, we can compute the risk-adjusted Fisher relation.

$$
\begin{aligned}
i_M = r^* &+ \pi_M + \sigma^{-1}\left(\mathbb{E}[y^{h,\pi_M}(\delta')|\delta = 0] - y_M\right) \\
&+ \left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] - \pi_M\right) \\
= r^* &+ \pi_M + \sigma^{-1}\left(\frac{1-p_M}{2}y^{h,\pi_M}(\delta = c) + p_M y^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}y^{h,\pi_M}(\delta = -c) - y_M\right) \\
&+ \left(\frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M \pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c) - \pi_M\right) \\
= r^* &+ \pi_M + (y_M - y_M) + (\pi_M - \pi_M) \\
= r^* &+ \pi_M
\end{aligned}
$$

(ii) $\underline{\pi_{LB} \leq \pi_M \leq \pi_B}$

We first construct the hypothetical policy functions consistent with π_M in the following four steps.

Step 1: Construct the hypothetical policy function of inflation.

Substituting the Philips curve and the truncated Taylor rule into the relative Euler equation, we obtain

$$
\begin{aligned}
\pi^{h,\pi_M}(\delta = c) &= \frac{\pi_M - \kappa\sigma(r^* - c)}{1 + \kappa\sigma\phi_\pi} \\
\pi^{h,\pi_M}(\delta = -c) &= \pi_M - \kappa\sigma c
\end{aligned}
$$

$\pi^{h,\pi_M}(\delta = 0)$ is given by

$$\pi^{h,\pi_M}(\delta = 0) = \pi_M$$

Step 2: Express the expected inflation as a function of π_M.

Using $\pi^{h,\pi_M}(\delta = c)$ and $\pi^{h,\pi_M}(\delta = -c)$, we obtain

$$\begin{aligned}
\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] :=& \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M\pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c) \\
=& \frac{1}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\pi_M - \frac{1-p_M}{2}\left(\frac{\kappa\sigma(r^* - c)}{1+\kappa\sigma\phi_\pi} + \kappa\sigma c\right)
\end{aligned}$$

Step 3: Construct the hypothetical policy function of output gap.

Substituting $\pi^{h,\pi_M}(\delta = c)$, $\pi^{h,\pi_M}(\delta = -c)$, and $\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]$ into rearranged Philips curves, we obtain

$$\begin{aligned}
y^{h,\pi_M}(\delta = c) &= \frac{\pi^{h,\pi_M}(\delta = c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \left(\frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right) - \sigma\phi_\pi\right)\pi_M \\
&+ \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) - \sigma(r^* - c) \\
y^{h,\pi_M}(\delta = 0) &= \frac{\pi^{h,\pi_M}(\delta = 0) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) \\
y^{h,\pi_M}(\delta = -c) &= \frac{\pi^{h,\pi_M}(\delta = -c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) - \sigma c
\end{aligned}$$

Step 4: Construct the hypothetical policy function of policy rate.

Substituting $\pi^{h,\pi_M}(\delta = c)$ into the truncated Taylor rule, we obtain

$$\begin{aligned}
i^{h,\pi_M}(\delta = c) &= r^* + \phi_\pi\pi^{h,\pi_M}(\delta = c) \\
&= r^* + \phi_\pi\left(\frac{\pi_M - \kappa\sigma(r^* - c)}{1+\kappa\sigma\phi_\pi}\right)
\end{aligned}$$

According to the truncated Taylor, $i^{h,\pi_M}(\delta = 0)$ and $i^{h,\pi_M}(\delta = -c)$ are given by

$$i^{h,\pi_M}(\delta = 0) = 0$$
$$i^{h,\pi_M}(\delta = -c) = 0$$

To summarize, the hypothetical policy functions are given by

$$
\begin{aligned}
\pi^{h,\pi_M}(\delta = c) &= \frac{\pi_M - \kappa\sigma(r^* - c)}{1 + \kappa\sigma\phi_\pi} \\
\pi^{h,\pi_M}(\delta = 0) &= \pi_M \\
\pi^{h,\pi_M}(\delta = -c) &= \pi_M - \kappa\sigma c \\
y^{h,\pi_M}(\delta = c) &= \left(\frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right) - \sigma\phi_\pi\right)\pi_M \\
&\quad + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) - \sigma(r^* - c) \\
y^{h,\pi_M}(\delta = 0) &= \frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) \\
y^{h,\pi_M}(\delta = -c) &= \frac{1}{\kappa}\left(1 - \frac{\beta}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) - \sigma c \\
i^{h,\pi_M}(\delta = c) &= r^* + \frac{\phi_\pi\left(\pi_M - \kappa\sigma(r^* - c)\right)}{1+\kappa\sigma\phi_\pi} \\
i^{h,\pi_M}(\delta = 0) &= 0 \\
i^{h,\pi_M}(\delta = -c) &= 0
\end{aligned}
$$

By construction, they satisfy the relative Euler equation, the Phillips curve, and the truncated Taylor rule and $\pi^{h,\pi_M}(\delta = 0) = \pi_M$.

Now that we have obtained the hypothetical policy functions consistent with π_M such that $\pi_{LB} \leq \pi_M \leq \pi_B$, we can compute the risk-adjusted Fisher relation.

$$
\begin{aligned}
i_M &= r^* + \pi_M + \sigma^{-1}\left(\mathbb{E}[y^{h,\pi_M}(\delta')|\delta = 0] - y_M\right) \\
&\quad + \left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] - \pi_M\right) \\
&= r^* + \pi_M + \sigma^{-1}\left(\frac{1-p_M}{2}y^{h,\pi_M}(\delta = c) + p_M y^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}y^{h,\pi_M}(\delta = -c) - y_M\right) \\
&\quad + \left(\frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M\pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c) - \pi_M\right) \\
&= r^* + \pi_M + \sigma^{-1}\left(y_M - \frac{(1-p_M)\sigma}{2}\left(r^* + \frac{\phi_\pi\left(\pi_M - \kappa\sigma(r^* - c)\right)}{1+\kappa\sigma\phi_\pi}\right) - y_M\right) \\
&\quad + \frac{1}{2}\left(\frac{1-p_M}{1+\kappa\sigma\phi_\pi} + 1 + p_M\right)\pi_M - \frac{(1-p_M)\kappa\sigma}{2}\left(\frac{r^* - c}{1+\kappa\sigma\phi_\pi} + c\right) - \pi_M \\
&= r^* + \frac{1}{2}(p_M - 1)\left(r^* + \kappa\sigma c + \frac{\kappa\sigma(1-\phi_\pi)(r^* - c)}{1+\kappa\sigma\phi_\pi}\right) + \frac{1}{2}\left(p_M + 1 - \frac{(\phi_\pi - 1)(1-p_M)}{1+\kappa\sigma\phi_\pi}\right)\pi_M \\
&= r^* + A + B\pi_M
\end{aligned}
$$

(iii) $\pi_B \leq \pi_M \leq \pi_{UB}$

We construct the hypothetical policy functions consistent with π_M in the following four steps.

Step 1: Construct hypothetical policy functions of inflation.

Substituting the Philips cure and the truncated Taylor rule into the relative Euler equation, we obtain

$$\pi^{h,\pi_M}(\delta = c) = \pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}$$
$$\pi^{h,\pi_M}(\delta = -c) = (1+\kappa\sigma\phi_\pi)\pi_M + \kappa\sigma(r^* - c)$$

$\pi^{h,\pi_M}(\delta = 0)$ is given by

$$\pi^{h,\pi_M}(\delta = 0) = \pi_M$$

Step 2: Express the expected inflation as a function of π_M.

Using $\pi^{h,\pi_M}(\delta = c)$ and $\pi^{h,\pi_M}(\delta = -c)$, we obtain

$$\begin{aligned}
\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] :=& \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M\pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c) \\
=& \left(1 + \frac{1-p_M}{2}\kappa\sigma\phi_\pi\right)\pi_M + \frac{1-p_M}{2}\left(\kappa\sigma r^* - \frac{\kappa^2\sigma^2\phi_\pi c}{1+\kappa\sigma\phi_\pi}\right)
\end{aligned}$$

Step 3: Construct the hypothetical policy function of output gap.

Substituting $\pi^{h,\pi_M}(\delta = c)$, $\pi^{h,\pi_M}(\delta = -c)$, and $\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]$ into rearranged Philips curves, we obtain

$$\begin{aligned}
y^{h,\pi_M}(\delta = c) &= \frac{\pi^{h,\pi_M}(\delta = c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2}\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) + \frac{\sigma c}{1+\kappa\sigma\phi_\pi} \\
y^{h,\pi_M}(\delta = 0) &= \frac{\pi^{h,\pi_M}(\delta = 0) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2}\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) \\
y^{h,\pi_M}(\delta = -c) &= \frac{\pi^{h,\pi_M}(\delta = -c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\
&= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2} + \sigma\phi_\pi\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) + \sigma(r^* - c)
\end{aligned}$$

Step 4: Construct the hypothetical policy function of policy rate.

Substituting $\pi^{h,\pi_M}(\delta = c)$ into the truncated Taylor rule, we obtain

$$\begin{aligned}
i^{h,\pi_M}(\delta = c) &= r^* + \phi_\pi\pi^{h,\pi_M}(\delta = c) \\
&= r^* + \phi_\pi\left(\pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right)
\end{aligned}$$

According to the truncated Taylor rule, $i^{h,\pi_M}(\delta=0)$ and $i^{h,\pi_M}(\delta=-c)$ is given by

$$
\begin{aligned}
i^{h,\pi_M}(\delta=0) &= r^* + \phi_\pi \pi_M \\
i^{h,\pi_M}(\delta=-c) &= 0
\end{aligned}
$$

To summarize, the hypothetical policy functions are given by

$$
\begin{aligned}
\pi^{h,\pi_M}(\delta=c) &= \pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi} \\
\pi^{h,\pi_M}(\delta=0) &= \pi_M \\
\pi^{h,\pi_M}(\delta=-c) &= (1+\kappa\sigma\phi_\pi)\pi_M + \kappa\sigma(r^*-c) \\
y^{h,\pi_M}(\delta=c) &= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2}\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) + \frac{\sigma c}{1+\kappa\sigma\phi_\pi} \\
y^{h,\pi_M}(\delta=0) &= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2}\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) \\
y^{h,\pi_M}(\delta=-c) &= \left(\frac{1-\beta}{\kappa} - \frac{(1-p_M)\beta\sigma\phi_\pi}{2} + \sigma\phi_\pi\right)\pi_M + \frac{(1-p_M)\beta\sigma}{2}\left(\frac{\kappa\sigma\phi_\pi c}{1+\kappa\sigma\phi_\pi} - r^*\right) + \sigma(r^*-c) \\
i^{h,\pi_M}(\delta=c) &= r^* + \phi_\pi\left(\pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right) \\
i^{h,\pi_M}(\delta=0) &= r^* + \phi_\pi\pi_M \\
i^{h,\pi_M}(\delta=-c) &= 0
\end{aligned}
$$

By construction, they satisfy the relative Euler equation, the Phillips curve, and the truncated Taylor rule and $\pi^{h,\pi_M}(\delta=0)=\pi_M$.

Now that we have obtained the hypothetical policy functions consistent with π_M such that $\pi_B \le \pi_M \le \pi_{UB}$, we can compute the risk-adjusted Fisher relation.

$$
\begin{aligned}
i_M &= r^* + \pi_M + \sigma^{-1}\left(\mathbb{E}[y^{h,\pi_M}(\delta')|\delta=0] - y_M\right) \\
&\quad + \left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta=0] - \pi_M\right) \\
&= r^* + \pi_M + \sigma^{-1}\left(\frac{1-p_M}{2}y^{h,\pi_M}(\delta=c) + p_M y^{h,\pi_M}(\delta=0) + \frac{1-p_M}{2}y^{h,\pi_M}(\delta=-c) - y_M\right) \\
&\quad + \left(\frac{1-p_M}{2}\pi^{h,\pi_M}(\delta=c) + p_M\pi^{h,\pi_M}(\delta=0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta=-c) - \pi_M\right) \\
&= r^* + \pi_M + \sigma^{-1}\left(y_M + \frac{1-p_M}{2}\left(\sigma\phi_\pi\pi_M + \sigma(r^*-c) + \frac{\sigma c}{1+\kappa\sigma\phi_\pi}\right) - y_M\right) \\
&\quad + \left(1 + \frac{1-p_M}{2}\kappa\sigma\phi_\pi\right)\pi_M + \frac{1-p_M}{2}\left(\kappa\sigma r^* - \frac{\kappa^2\sigma^2\phi_\pi c}{1+\kappa\sigma\phi_\pi}\right) - \pi_M \\
&= r^* + \frac{1}{2}(1-p_M)(1+\kappa\sigma)\left(r^* - \frac{\kappa\sigma\phi_\pi}{1+\kappa\sigma\phi_\pi}c\right) + \left(\frac{1}{2}(1-p_M)(1+\kappa\sigma)\phi_\pi + 1\right)\pi_M \\
&= r^* + C + D\pi_M
\end{aligned}
$$

(iv) $\underline{\pi_{UB} < \pi_M}$

We construct the hypothetical policy functions consistent with π_M in the following four steps.

Step 1: Construct the hypothetical policy function of inflation.

Substituting the Philips curve and the truncated Taylor rule into the relative Euler equation, we obtain

$$\pi^{h,\pi_M}(\delta = c) = \pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}$$
$$\pi^{h,\pi_M}(\delta = -c) = \pi_M - \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}$$

$\pi^{h,\pi_M}(\delta = 0)$ is given by

$$\pi^{h,\pi_M}(\delta = 0) = \pi_M$$

Step 2: Express the expected inflation as a function of π_M.

Using $\pi^{h,\pi_M}(\delta = c)$ and $\pi^{h,\pi_M}(\delta = -c)$, we obtain

$$\begin{aligned}\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0] :=& \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = c) + p_M\pi^{h,\pi_M}(\delta = 0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta = -c) \\ =& \pi_M\end{aligned}$$

Step 3: Construct hypothetical policy functions of output gap.

Substituting $\pi^{h,\pi_M}(\delta = c)$, $\pi^{h,\pi_M}(\delta = -c)$, and $\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]$ into rearranged Philips curves, we obtain

$$\begin{aligned}y^{h,\pi_M}(\delta = c) &= \frac{\pi^{h,\pi_M}(\delta = c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\ &= \frac{1-\beta}{\kappa}\pi_M + \frac{\sigma c}{1+\kappa\sigma\phi_\pi} \\ y^{h,\pi_M}(\delta = 0) &= \frac{\pi^{h,\pi_M}(\delta = 0) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\ &= \frac{1-\beta}{\kappa}\pi_M \\ y^{h,\pi_M}(\delta = -c) &= \frac{\pi^{h,\pi_M}(\delta = -c) - \beta\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta = 0]}{\kappa} \\ &= \frac{1-\beta}{\kappa}\pi_M - \frac{\sigma c}{1+\kappa\sigma\phi_\pi}\end{aligned}$$

Step 4: Construct the hypothetical policy function of policy rate.

Substituting the hypothetical policy functions of inflation into the truncated Taylor rules, we obtain

$$
\begin{aligned}
i^{h,\pi_M}(\delta=c) &= r^* + \phi_\pi \pi^{h,\pi_M}(\delta=c) \\
&= r^* + \phi_\pi \left(\pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right) \\
i^{h,\pi_M}(\delta=0) &= r^* + \phi_\pi \pi_M \\
i^{h,\pi_M}(\delta=-c) &= r^* + \phi_\pi \pi^{h,\pi_M}(\delta=-c) \\
&= r^* + \phi_\pi \left(\pi_M - \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right)
\end{aligned}
$$

To summarize, the hypothetical policy functions are given by

$$
\begin{aligned}
\pi^{h,\pi_M}(\delta=c) &= \pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi} \\
\pi^{h,\pi_M}(\delta=0) &= \pi_M \\
\pi^{h,\pi_M}(\delta=-c) &= \pi_M - \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi} \\
y^{h,\pi_M}(\delta=c) &= \frac{1-\beta}{\kappa}\pi_M + \frac{\sigma c}{1+\kappa\sigma\phi_\pi} \\
y^{h,\pi_M}(\delta=0) &= \frac{1-\beta}{\kappa}\pi_M \\
y^{h,\pi_M}(\delta=-c) &= \frac{1-\beta}{\kappa}\pi_M - \frac{\sigma c}{1+\kappa\sigma\phi_\pi} \\
i^{h,\pi_M}(\delta=c) &= r^* + \phi_\pi \left(\pi_M + \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right) \\
i^{h,\pi_M}(\delta=0) &= r^* + \phi_\pi \pi_M \\
i^{h,\pi_M}(\delta=-c) &= r^* + \phi_\pi \left(\pi_M - \frac{\kappa\sigma c}{1+\kappa\sigma\phi_\pi}\right)
\end{aligned}
$$

By construction, they satisfy the relative Euler equation, the Phillips curve, and the truncated Taylor rule and $\pi^{h,\pi_M}(\delta=0)=\pi_M$

Now that we have obtained the hypothetical policy functions consistent with π_M such that $\pi_{UB} < \pi_M$, we can compute the risk-adjusted Fisher relation.

$$
\begin{aligned}
i_M &= r^* + \pi_M + \sigma^{-1}\left(\mathbb{E}[y^{h,\pi_M}(\delta')|\delta=0] - y_M\right) \\
&\quad + \left(\mathbb{E}[\pi^{h,\pi_M}(\delta')|\delta=0] - \pi_M\right) \\
&= r^* + \pi_M + \sigma^{-1}\left(\frac{1-p_M}{2}y^{h,\pi_M}(\delta=c) + p_M y^{h,\pi_M}(\delta=0) + \frac{1-p_M}{2}y^{h,\pi_M}(\delta=-c) - y_M\right) \\
&\quad + \left(\frac{1-p_M}{2}\pi^{h,\pi_M}(\delta=c) + p_M \pi^{h,\pi_M}(\delta=0) + \frac{1-p_M}{2}\pi^{h,\pi_M}(\delta=-c) - \pi_M\right) \\
&= r^* + \pi_M + (y_M - y_M) + (\pi_M - \pi_M) \\
&= r^* + \pi_M
\end{aligned}
$$

□

D Proofs Related to Positive Inflation Target

The model with a positive inflation target can be written in the following way.

$$y_t = \mathbb{E}_t[y_{t+1}] - \sigma\left[i_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - (r^* + \pi^*) - \delta_t\right] \tag{D10}$$

$$\hat{\pi}_t = \kappa y_t + \beta \mathbb{E}_t[\hat{\pi}_{t+1}] \tag{D11}$$

$$i_t = \max[0, r^* + \pi^* + \phi_\pi \hat{\pi}_t] \tag{D12}$$

where

$$\hat{\pi}_t = \pi_t - \pi^* \tag{D13}$$

Equation (D10) is the consumption Euler equation, equation (D11) is the standard New Keynesian Phillips Curve, and equation (D12) is the truncated Taylor rule.

Proposition D.1. *Let*

$$\tilde{c}_{targ} = -\frac{2(\pi^* + r^*)(\phi_\pi - 1)(\kappa\sigma\phi_\pi + 1)}{\kappa\phi_\pi^2\sigma(\kappa\sigma + 1)(p_M - 1)}$$

There are two equilibria when $c < \tilde{c}_{targ}$. There is one equilibrium when $c = \tilde{c}_{targ}$. There is no equilibrium when $c > \tilde{c}_{targ}$

Proof. By equation D10 to D13, the system of equations with a positive inflation target is identical to that with a zero inflation target, with π_t and r^* being replaced by $\hat{\pi}_t$ and $r^* + \pi^*$, respectively. Accordingly, this proposition is identical to A.6. □

Proposition 3: Suppose that $\pi^* > \frac{r^*}{\phi_\pi - 1}(> 0)$. If c is sufficiently large, inflation at the risky steady state in the deflationary equilibrium is positive.

Proof. Similar to the model with a zero inflation target, we define $\underline{c}_{targ}$ as the maximum value of c for the existence of equilibrium 1. By Proposition D.1, Equilibrium 8 is deflationary equilibrium when $c \leq \underline{c}_{targ}$. Thus $\pi_M = -r^* < 0$. By Proposition D.1, Equilibrium 5 is deflationary equilibrium when $\underline{c}_{targ} \leq c \leq \tilde{c}_{targ}$. π_M evaluated at $\underline{c}_{targ}$ is:

$$\pi_M = -r^* < 0$$

π_M evaluated at $\tilde{c}_{targ}$ is:

$$\pi_M = \pi^* - \frac{\pi^* + r^*}{\phi_\pi} > 0$$

It can be shown that

$$\frac{\partial \pi_M}{\partial c} = \frac{(\sigma\kappa + 1)(1 - p_M)\kappa\sigma\phi_\pi}{2 + (\sigma(p_M + 1)\kappa + p_M - 1)\phi_\pi} > 0$$

□

Accordingly, if c is sufficiently large, inflation at the risky steady state in deflationary equilibrium is positive.

E Proofs Related to Low ϕ_π

In this section, we will assume that $\phi_\pi < \underline{\phi}_\pi$. Key propositions from this subsection are as follows.

E.1 Existence of Equilibria: Low ϕ_π

Proposition E.1. *Equilibrium 1 exists when $c \leq \bar{c}$ where $\bar{c}$ is defined in Section A.*

Proposition E.2. *Equilibrium 2 exists when $\tilde{c} \leq c \leq \bar{c}$ where $\tilde{c}$ and $\bar{c}$ are defined in Section A.*

Proposition E.3. *Equilibrium 5 exists when $\underline{c} \leq c \leq \tilde{c}$ where $\tilde{c}$ and $\underline{c}$ are defined in Section A.*

Proposition E.4. *Equilibrium 8 exists when $c \leq \underline{c}$ where $\underline{c}$ is defined in Section A.*

Proposition E.5. *There are two equilibria when $c < \bar{c}$. There is one equilibrium when $c = \bar{c}$. There is no equilibrium when $c > \bar{c}$.*

Proof of Proposition E.1.: By Proposition A.2, the existence of Equilibrium 1 does not depend on the value of ϕ_π. Accordingly, Equilibrium 1 exists when $c \leq \bar{c}$.

Proof of Proposition E.2.: By Proposition A.3, it suffices to find the value of c where $i_L^{shadow} = 0$ and $i_M^{shadow} = 0$, respectively. i_L^{shadow} and i_M^{shadow} are given by

$$i_L^{shadow} = \frac{2(\phi_\pi - 1)(\kappa\sigma(r^* - c)\phi_\pi + r^*)}{((p_M - 1)\sigma\kappa + p_M + 1)\phi_\pi - 2}$$
$$i_M^{shadow} = \frac{\kappa(c\sigma(p_M - 1)\kappa + (p_M - 1)c + 2r^*)\sigma\phi_\pi^2 + (2r^* - 2\kappa r^*\sigma)\phi_\pi - 2r^*}{(((p_M - 1)\sigma\kappa + p_M + 1)\phi_\pi - 2)(\kappa\phi_\pi\sigma + 1)}$$

Note that the sign of the denominator is the opposite: when $\underline{\phi}_\pi < \phi_\pi < \overline{\phi}_\pi$ and when $\phi_\pi < \overline{\phi}_\pi$. If we find the value of c such that $i_L^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium. If we find the value of c such that $i_M^{shadow} = 0$, that is the minimum degree of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 2 exists when $\tilde{c} \leq c \leq \bar{c}$.

Proof of Proposition E.3.: By Proposition A.4, it suffices to find the value of c where $i_M^{shadow} = 0$ and $i_H^{shadow} = 0$, respectively. i_M^{shadow} and i_H^{shadow} are given by

$$i_M^{shadow} = \frac{-\left(c\sigma\left(p_M-1\right)\kappa + \left(p_M-1\right)c + 2r^*\right)\kappa\sigma\phi_\pi^2 + \left(2\kappa r^*\sigma - 2r^*\right)\phi_\pi + 2r^*}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$

$$i_H^{shadow} = \frac{\left(2c\sigma\kappa - 2r^*\right)\phi_\pi + 2r^*}{2 + \left(\sigma\left(p_M+1\right)\kappa + p_M - 1\right)\phi_\pi}$$

Note that the sign of the denominator is the same in both cases: when $\underline{\phi}_\pi < \phi_\pi < \overline{\phi}_\pi$ and when $\phi_\pi < \overline{\phi}_\pi$. If we find c such that $i_M^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium. If we find c such that $i_H^{shadow} = 0$, that is the minimum level of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 5 exists when $\underline{c} \leq c \leq \tilde{c}$.

Proof of Proposition E.4.: By Proposition A.5, the existence of Equilibrium 8 does not depend on the value of ϕ_π. Accordingly, Equilibrium 1 exists when $c \leq \underline{c}$.

Proof of Proposition E.5.: When $\phi_\pi < \underline{\phi}_\pi$, $\underline{c} < \tilde{c} < \bar{c}$ holds. According to Proposition E.1-Proposition E.4, Equilibrium 1 and Equilibrium 8 exist when $c \leq \underline{c}$, Equilibrium 1 and Equilibrium 5 exist when $\underline{c} \leq c \leq \tilde{c}$, Equilibrium 1 and Equilibrium 2 exist when $\tilde{c} \leq c \leq \bar{c}$. When $c = \bar{c}$, Equilibrium 1 and Equilibrium 2 coincide. Accordingly, there is one equilibrium. When $c > \bar{c}$, there is no equilibrium.

E.2 Proof of Main Proposition: Low ϕ_π

Proposition E.6. *Suppose that $\phi_\pi < \underline{\phi}_\pi$. If c is sufficiently large, the policy rate at the risky steady state in the deflationary equilibrium is positive.*

Proof. By Proposition E.5, Equilibrium 8 is the deflationary equilibrium when $c < \underline{c}$. Equilibrium 5 is the deflationary equilibrium when $\underline{c} \leq c \leq \tilde{c}$. Accordingly, $i_M = 0$ when $c \leq \tilde{c}$. When $\tilde{c} \leq c \leq \bar{c}$, Equilibrium 2 is the deflationary equilibrium. Accordingly, the policy rate at the risky steady state in the deflationary equilibrium is positive when $\tilde{c} \leq c \leq \bar{c}$. □

F Proofs Related to High ϕ_π

In this section, we will assume that $\overline{\phi}_\pi < \phi_\pi$. Key propositions from this subsection are as follows.

F.1 Existence of Equilibria: High ϕ_π

Proposition F.1. *Equilibrium 1 exists when $c \leq \bar{c}$ where $\bar{c}$ is defined in Section A.*

Proposition F.2. *Equilibrium 2 exists when $\bar{c} \leq c \leq \tilde{c}$ where $\tilde{c}$ and $\bar{c}$ are defined in Section A.*

Proposition F.3. *Equilibrium 5 exists when $\tilde{c} \leq c \leq \underline{c}$ where $\tilde{c}$ and $\underline{c}$ are defined in Section A.*

Proposition F.4. *Equilibrium 8 exists when $c \leq \underline{c}$ where $\underline{c}$ is defined in Section A.*

Proposition F.5. *There are two equilibria when $c < \underline{c}$. There is one equilibrium when $c = \underline{c}$. There is no equilibrium when $c > \underline{c}$.*

Proof of Proposition F.1.: By Proposition A.2, the existence of Equilibrium 1 does not depend on the value of ϕ_π. Accordingly, Equilibrium 1 exists when $c \leq \bar{c}$.

Proof of Proposition F.2.: By Proposition A.3, it suffices to find the value of c where $i_L^{shadow} = 0$ and $i_M^{shadow} = 0$, respectively. i_L^{shadow} and i_M^{shadow} are given by

$$i_L^{shadow} = \frac{2(\phi_\pi - 1)(\kappa\sigma(r^* - c)\phi_\pi + r^*)}{((p_M - 1)\sigma\kappa + p_M + 1)\phi_\pi - 2}$$

$$i_M^{shadow} = \frac{\kappa(c\sigma(p_M - 1)\kappa + (p_M - 1)c + 2r^*)\sigma\phi_\pi^2 + (2r^* - 2\kappa r^*\sigma)\phi_\pi - 2r^*}{(((p_M - 1)\sigma\kappa + p_M + 1)\phi_\pi - 2)(\kappa\phi_\pi\sigma + 1)}$$

Note that the sign of the denominator is the same in both cases: when $\underline{\phi}_\pi < \phi_\pi < \overline{\phi}_\pi$ and when $\phi_\pi < \overline{\phi}_\pi$. If we find c such that $i_L^{shadow} = 0$, that is the minimum degree of uncertainty consistent with this equilibrium. If we find c such that $i_M^{shadow} = 0$, that is the maximum degree of uncertainty supported by this equilibrium. Accordingly, Equilibrium 2 exists when $\bar{c} \leq c \leq \tilde{c}$.

Proof of Proposition F.3.: By Proposition A.4, it suffices to find the value of c where $i_M^{shadow} = 0$ and $i_H^{shadow} = 0$, respectively. i_M^{shadow} and i_H^{shadow} are given by

$$i_M^{shadow} = \frac{-(c\sigma(p_M - 1)\kappa + (p_M - 1)c + 2r^*)\kappa\sigma\phi_\pi^2 + (2\kappa r^*\sigma - 2r^*)\phi_\pi + 2r^*}{2 + (\sigma(p_M + 1)\kappa + p_M - 1)\phi_\pi}$$

$$i_H^{shadow} = \frac{(2c\sigma\kappa - 2r^*)\phi_\pi + 2r^*}{2 + (\sigma(p_M + 1)\kappa + p_M - 1)\phi_\pi}$$

Note that the sign of the denominator is opposite: when $\underline{\phi}_\pi < \phi_\pi < \overline{\phi}_\pi$ and when $\phi_\pi < \overline{\phi}_\pi$. If we find c such that $i_M^{shadow} = 0$, that is the minimum degree of uncertainty consistent with this equilibrium. If we find c such that $i_H^{shadow} = 0$, that is the maximum degree of uncertainty consistent with this equilibrium. Accordingly, Equilibrium 5 exists when $\tilde{c} \leq c \leq \underline{c}$.

Proof of Proposition F.4.: By Proposition A.5, the existence of Equilibrium 8 does not depend on the value of ϕ_π. Accordingly, Equilibrium 1 exists when $c \leq \underline{c}$.

Proof of Proposition F.5.: When $\overline{\phi}_\pi < \phi_\pi$, $\underline{c} < \bar{c}$ holds. By Proposition F.1 to Proposition F.4, $\underline{c} < \tilde{c} < \bar{c}$.

According to Proposition F.1-Proposition F.4, Equilibrium 1 and Equilibrium 8 exist when $c \leq \bar{c}$, Equilibrium 2 and Equilibrium 8 exist when $\bar{c} \leq c \leq \tilde{c}$, Equilibrium 5 and Equilibrium 8 exist when $\tilde{c} \leq c \leq \underline{c}$. When $c = \underline{c}$, Equilibrium 5 and Equilibrium 8 coincide. Accordingly, there is one equilibrium. When $c > \underline{c}$, there is no equilibrium.

F.2 Proof of Main Proposition: High ϕ_π

Proposition F.6. *Suppose that* $\overline{\phi}_\pi < \phi_\pi$. *If* c *is sufficiently large, the policy rate at the risky steady state in the target equilibrium is 0.*

Proof. By Proposition F.5, Equilibrium 1 is the target equilibrium when $c < \bar{c}$. Equilibrium 2 is the target equilibrium when $\bar{c} \leq c \leq \tilde{c}$. Accordingly, the policy rate at the risky steady state in the target equilibrium is positive when $c \leq \tilde{c}$. When $\tilde{c} \leq c \leq \underline{c}$, Equilibrium 5 is the target equilibrium. Accordingly, the policy rate at the risky steady state in the target equilibrium is zero when $\tilde{c} \leq c \leq \underline{c}$. □

G Model with an AR(1) Shock

In this section, we consider the model with an AR(1) shock process:

$$\delta_t = \rho\delta_{t-1} + \epsilon_t \tag{G1}$$

A recursive equilibrium for this stylized, semi-loglinear model is given by a set of policy functions $\{y(\cdot), \pi(\cdot), i(\cdot)\}$ that satisfies the Euler equation, the Phillips curve, and the truncated Taylor rule, as described in Section 2.

In solving the model, we approximate the AR(1) process of the exogenous shock using Markov chains via the Rouwenhorst approximation method. With this approximation, the model can be solved with linear algebra.

G.1 Solution Method for Policy Functions

Recall that the problem is to find a set of $\{y(\cdot), \pi(\cdot), i(\cdot)\}$ that satisfies the equilibrium conditions:

$$y_t = \mathbb{E}_t[y_{t+1}] - \sigma\left[i_t - \mathbb{E}_t[\pi_{t+1}] + \delta_t\right] \tag{G2}$$
$$\pi_t = \kappa y_t + \beta\mathbb{E}_t[\pi_{t+1}] \tag{G3}$$
$$i_t = \max\left[i_{ELB}, r^* + \phi_\pi(\pi_t)\right] \tag{G4}$$

Consider an n-state discretization of an AR(1) shock approximated via the Rouwenhorst method. The Rouwenhorst approximation method will yield an $n \times 1$ vector of grid points $[\delta_1, \ldots, \delta_n]$ and an $n \times n$ matrix, T, of transition probabilities:

$$T = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \tag{G5}$$

where t_i is the i^{th} row of T.

Given this n-state discretization, we are left with a series of n equations and n unknowns to solve for:

$$
\begin{aligned}
&y_1 = \mathbb{E}_1[y_{t+1}] - \sigma\left[i_1 - r^* - \mathbb{E}_1[\pi_{t+1}] + \delta_1\right] \\
&\vdots \\
&y_n = \mathbb{E}_n[y_{t+1}] - \sigma\left[i_n - r^* - \mathbb{E}_n[\pi_{t+1}] + \delta_n\right] \\
&\pi_1 = \kappa y_1 + \beta \mathbb{E}_1[\pi_{t+1}] \\
&\vdots \\
&\pi_n = \kappa y_n + \beta \mathbb{E}_n[\pi_{t+1}] \\
&i_1 = \max\left[i_{ELB}, r^* + \phi_\pi \pi_1\right] \\
&\vdots \\
&i_n = \max\left[i_{ELB}, r^* + \phi_\pi \pi_n\right]
\end{aligned}
$$

Here, $\mathbb{E}_i[\cdot]$ is the conditional expectation of our policy function, given state i. It is formally defined as the $t_i \cdot z$, where $z = [z_1, \cdots, z_n]^T$, for a given policy function.

Notice that, absent the ELB constraint, we are left with a linear-system of equations and can be solved for using basic matrix algebra. Let A be a matrix of coefficients, x be a vector of variables, b be a vector of coefficients, where

$$
A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \quad x = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ \pi_1 \\ \vdots \\ \pi_n \\ i_1 \\ \vdots \\ i_n \end{bmatrix} \quad b = \begin{bmatrix} \sigma(r^* - \delta_1) \\ \vdots \\ \sigma(r^* - \delta_n) \\ 0 \\ \vdots \\ 0 \\ r^* \\ \vdots \\ r^* \end{bmatrix}
$$

and

$$
\begin{array}{lll}
A_{1,1} = \mathbb{I}_n - T & A_{1,2} = -\sigma \cdot T & A_{1,3} = \sigma \cdot \mathbb{I}_n \\
A_{2,1} = -\kappa \cdot \mathbb{I}_n & A_{2,2} = \beta(\mathbb{I}_n - T) & A_{2,3} = 0 \cdot \mathbb{I}_n \\
A_{3,1} = 0 \cdot \mathbb{I}_n & A_{3,2} = -\phi_\pi \cdot \mathbb{I}_n & A_{3,3} = \mathbb{I}_n
\end{array}
$$

Here, $\mathbb{I}_n$ is the identity matrix of dimension n.

There are two algorithms to consider: one for the target equilibrium and the other for the deflationary equilibrium.

The algorithm to solve for the policy functions in the target equilibrium is as follows. Start by assuming that the ELB does not bind in any period and solve the linear system of equations. If $i_n < 0$ then assume that $i_n = 0$ and resolve the system of equations. If $i_{n-1} < 0$ then assume that $i_{n-1} = 0$ and resolve the system of equations. Continue this process until for all $j \in (1, \dots, n)$, $i_j \geq 0$.

The algorithm to solve for the policy functions in the deflationary equilibrium is as follows. Start by assuming that the ELB binds in all periods and solve the linear system of equations.

If the implied policy rate, $i_1^{imp} \equiv r^* + \phi_\pi \pi_1$ is higher than i_{ELB}, assume that $i_i \neq i_{ELB}$ and resolve the system of equations. If $i_2 > i_{ELB}$, then assume that $i_2^{imp} > i_{ELB}$ and resolve the system of equations. Continue this process until for all $j \in (1, \ldots, n)$, $i_j^{imp} > 0$.

G.2 Solution Method for the Risk-adjusted Fisher Relation

In this section, we present the details on how to solve for the risk-adjusted Fisher relation given a continuous AR(1) shock approximated using Markov chains via the Rouwenhorst method.

Again, consider an n-state discretization—where n is odd—of an AR(1) shock approximated via the Rouwenhorst method. There will be an $n \times 1$ vector of grid points $[\delta_1^n, \ldots, \delta_n^n]$ and an $n \times n$ matrix, T, of transition probabilities, where T is defined in the same way above.

To solve for the risk-adjusted Fisher relation, given the *candidate* π_{RSS}, we need to compute the risk-adjustment term based on *hypothetical* policy functions. These hypothetical policy functions must satisfy the following conditions: (i) $\pi^{h,\pi_M}(\delta = 0) = \pi_{RSS}$, (ii) the truncated Taylor rule, (iii) the Phillips curve, and (iv) the *relative* Euler equations. Unlike the three-state shock case, we will not present a full algebraic derivation, as the goal is to develop a general solution method for an n-state discretized shock. The goal is to re-frame the problem in terms of a system of equations, thus allowing us to take advantage of basic linear algebra techniques to solve for the hypothetical policy functions and the risk-adjusted Fisher relation.

Let x_M be the value a given policy function takes in the "middle state," where M is the $(n+1)/2^{th}$ position of our grid. Notice that by construction, the middle state is identical to the risky steady state, because the $(n+1)/2^{th}$ position of our vector of grid points is 0. Given this, observe that by rewriting the system in the following way we satisfy our conditions:

$$
\begin{aligned}
&y_1 - y_M = \mathbb{E}_1[y_{t+1}] - \mathbb{E}_M[y_{t+1}] - \sigma\left[i_1 - i_M - \mathbb{E}_1[\pi_{t+1}] - \mathbb{E}_M[\pi_{t+1}] + \delta_1\right] \\
&\vdots \\
&y_j - y_M = \mathbb{E}_j[y_{t+1}] - \mathbb{E}_M[y_{t+1}] - \sigma\left[i_j - i_M - \mathbb{E}_j[\pi_{t+1}] - \mathbb{E}_M[\pi_{t+1}] + \delta_j\right] \\
&\vdots \\
&y_n - y_M = \mathbb{E}_n[y_{t+1}] - \mathbb{E}_M[y_{t+1}] - \sigma\left[i_n - i_M - \mathbb{E}_n[\pi_{t+1}] - \mathbb{E}_M[\pi_{t+1}] + \delta_n\right] \\
&\pi_1 = \kappa y_1 + \beta \mathbb{E}_1[\pi_{t+1}] \\
&\vdots \\
&\pi_n = \kappa y_n + \beta \mathbb{E}_n[\pi_{t+1}] \\
&i_1 = \max\left[i_{ELB}, r^* + \phi_\pi \pi_1\right] \\
&\vdots \\
&i_n = \max\left[i_{ELB}, r^* + \phi_\pi \pi_n\right]
\end{aligned}
$$

for $j \neq M$. $\mathbb{E}_i[\cdot]$ is defined as above. Absent the ELB constraint, we are left with a linear system of equations that can be solved using matrix algebra.Let A be a matrix of coefficients, x be a vector of variables, b be a vector of coefficients, where

$$A = \begin{bmatrix} A_{1,1}^{(M)} - A_{y_M,1} & A_{1,2}^{(M)} - A_{y_M,2} & A_{1,3}^{(M)} - A_{y_M,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \quad x = \begin{bmatrix} y_1 - y_M \\ \vdots \\ y_j - y_M \\ \vdots \\ y_n \\ \pi_1 \\ \vdots \\ \pi_n \\ i_1 \\ \vdots \\ i_n \end{bmatrix} \quad b = \begin{bmatrix} -\sigma\delta_1 \\ \vdots \\ -\sigma\delta_j \\ \vdots \\ -\sigma\delta_n \\ 0 \\ \vdots \\ 0 \\ r^* \\ \vdots \\ r^* \end{bmatrix}$$

for $j \neq M$. $A_{1,1}^{(M)}$, $A_{1,2}^{(M)}$, and $A_{1,3}^{(M)}$ are matrices of dimension $n-1 \times n$. We use the notation $A_{i,j}^{(M)}$ to represent the matrix $A_{i,j}$ where M^{th} row has been removed. Similarly, $A_{y_M,1}$, $A_{y_M,2}$, and $A_{y_M,3}$ are matrices of dimension $n-1 \times n$ that take on the form:

$$A_{y_M,1} = [A_{1,1,M}, \overset{\times N-1}{\cdots\cdots}, A_{1,1,M}]'$$
$$A_{y_M,2} = [A_{1,2,M}, \overset{\times N-1}{\cdots\cdots}, A_{1,2,M}]'$$
$$A_{y_M,3} = [A_{1,3,M}, \overset{\times N-1}{\cdots\cdots}, A_{1,3,M}]'$$

We use the notation $A_{i,j,M}$ to denote the M^{th} row of $A_{i,j}$. Here, the M^{th} row of $A_{i,j}$ has been repeated $n-1$ times.

$A_{1,1}$, $A_{1,2}$, $A_{1,3}$, $A_{2,1}$, $A_{2,2}$, $A_{2,3}$, $A_{3,1}$, $A_{3,2}$, and $A_{3,3}$ are defined as before.

G.2.1 Algorithm for the risk-adjusted Fisher relation

The algorithm to solve for the risk-adjusted Fisher relation is as follows. For the *candidate* $\pi_{RSS} = \pi^{h,\pi_M}(\delta = 0)$, start by assuming that the ELB does not bind in any period and solve the linear system of equations. If $i_n < 0$ then assume that $i_n = 0$ and resolve the system of equations. If $i_{n-1} < 0$ then assume that $i_{n-1} = 0$ and resolve the system of equations. Continue this process until for all $j \in (1, \ldots, n)$, $i_j \geq 0$.

Upon the completion of this algorithm, given the candidate π_M, the hypothetical policy functions that have been solved for satisfy the following conditions needed to calculate the risk-adjustment term: (i) $\pi^{h,\pi_M}(\delta = 0) = \pi_M$, (ii) the truncated Taylor rule, (iii) the Phillips curve, and (iv) the *relative* Euler equations. From here, it is straightforward to compute the risk-adjustment term and the risk-adjusted Fisher relation.

G.3 Numerical Results

Table 1 lists the parameter values used for the numerical analysis.
Figure 8 presents the policy functions for output, inflation, and the policy rate. In the figure, the top and bottom rows are policy functions for the target and deflationary equilibria,

Table 1: Parameter Values for the Stylized Model

Parameter	Description	Parameter Value
β	Discount rate	$\frac{1}{1+0.0025}$
σ	Inverse intertemporal elasticity of substitution	1
κ	Slope of Phillips Curve	0.02
$400r^*$	Annualized Natural Rate of Interest	1%
ϕ_π	Coefficient on inflation in the Taylor rule	4
i_{ELB}	Effective lower bound	0
ρ	AR(1) coefficient for the demand shock	0.80
σ_ϵ	standard deviation of shocks to demand shock	$[0,\ \sigma_\epsilon^{max}]$

respectively. Consistent with the model with a three-state shock, uncertainty increases (decreases) the rate of inflation at the risky steady state in the deflationary (target) equilibrium. Figure 9 presents the risk-adjusted Fisher relation associated with this model.

Figure 8: Policy Functions for the Model with an AR(1) Shock

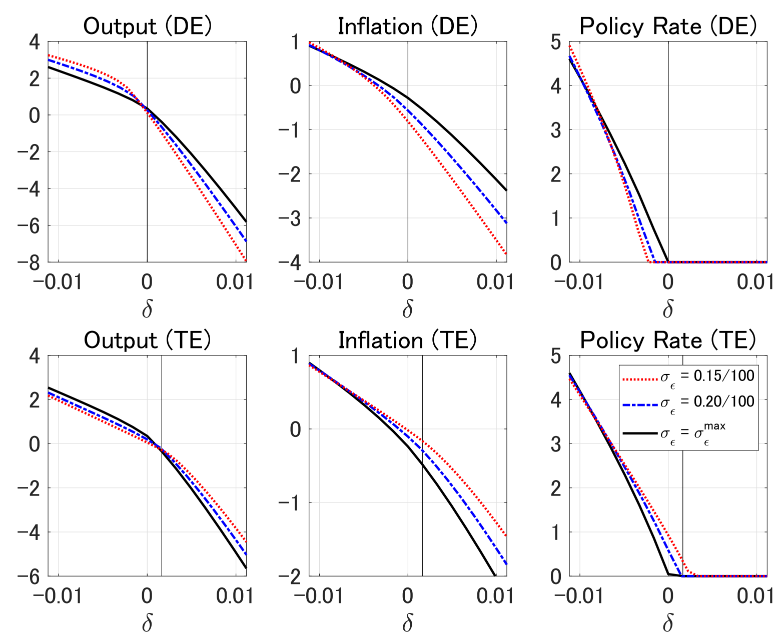

Figure 9: Risk-adjusted Fisher Relation for the Model with an AR(1) Shock