Lecture 2: Dynamics, Completeness, Asset Pricing

Economics 714, Spring 2018

1 Linear Expectational Difference Equations

1.1 Setup

In the last class with linear utility we derived the linear expectational difference equation:

$$p_t = \beta E_t(p_{t+1} + s_{t+1}).$$

This type of equation arises frequently in linearized equilibrium models, which is a common solution technique. The stability criteria for this type of equation determine whether equilibrium exists and is unique. Let's consider slightly more general form:

$$p_t = aE_t p_{t+1} + cs_t.$$

where a and c are arbitrary constants. Here s_t is a predetermined or state variable, p_t is an endogenous or equilibrium or "jump" variable.

If $|a| < 1$, as it will be in our case with $a = \beta \in (0, 1)$, we can solve this equation forward:

$$p_t = c\sum_{j=0}^{\infty} a^j E_t s_{t+j} + \lim_{T\to\infty} a^T E_t p_{t+T}$$

1.2 Bubbles and Fundamentals

We can decompose the general solution into the fundamental and the bubble component

$$p_t = p_t^f + b_t$$

Fundamentals solution:

$$p_t^f = c \sum_{j=0}^{\infty} a^j E_t s_{t+j}$$

Bubble component:

$$b_t = \lim_{T \to \infty} a^T E_t p_{t+T}$$

Note that bubbles explode in expectation:

$$E_t b_{t+j} = \left(\frac{1}{a}\right)^j b_t$$

We argued before that our infinite horizon model ruled out bubbles. We'll return to that later. In the absence of bubbles, knowledge of the process for s_t will determine the price. For example suppose:

$$s_{t+1} = \rho s_t + w_{t+1},$$

where $0 < \rho < 1$ and $E_t w_{t+1} = 0$. Then we have:

$$p_t = \frac{c}{1 - a\rho} s_t.$$

The hallmark of rational expectations models is that they satisfy the Lucas critique (changes in the driving processes change the equilibrium) and cross-equation restrictions (equilibrium reduced form parameters depend on underlying structural parameters), as here.

1.3 Indeterminacy

What if $|a| > 1$? That's not relevant in our case, but could hold in general. Then we couldn't solve forward, and in that case equilibrium would no longer be unique.

For example, say $s_t \equiv 1$ and $c = 0$.

$$E_t p_{t+1} = \frac{1}{a} p_t$$

so

$$p_{t+1} = \frac{1}{a} p_t + e_{t+1},$$

where e_{t+1} is an expectational error, defined by $E_t e_{t+1} = 0$. But it otherwise arbitrary: a **sunspot**. Then we can solve backward:

$$p_{t+1} = \left(\frac{1}{a}\right)^{t+1} p_0 + \sum_{j=0}^{t+1} \left(\frac{1}{a}\right)^j e_{t+1-j}$$

Thus we now have a continuum of equilibria defined by the (arbitrary) initial condition p_0 and the specification of the sunspot process $\{e_t\}$.

The same basic approach extends to multivariate models, where there we use an eigenvector/eigenvalue decomposition. For determinacy, need as many eigenvalues $|\lambda_i| > 1$ as there are predetermined variables.

2 Asset Pricing

2.1 Compete markets

Now we return to the Lucas model. We have used the equilibrium of the model to determine a single price, the price of a tree. With trading in trees, markets are **complete**. By this we mean that by trading in shares of the tree the representative agent could achieve any state-contingent payoff.

2.2 Pricing General Claims

However we can introduce additional (redundant) assets and use the same approach to determine their prices. For this, suppose that the transition function Q has a density $f(s, s')$:

$$Q(s, s') = \int_{-\infty}^{s'} f(s, u) du$$

In general we can define a **pricing kernel** $q(s', s)$ such that the equilibrium price P^A of unit of the good in period $t+1$ when $s_{t+1} \in A$ conditional on $s_t = s$ is:

$$P^A(s) = \int_{s' \in A} q(s, s') ds'$$

In the Lucas model, the pricing kernel is:

$$q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s, s')$$

Roughly, this is the intertemporal marginal rate of substitution multiplied by the state probabilities.

Using the pricing kernel, we can find the equilibrium price of any contingent claim $g(s')$ one-period ahead:

$$\begin{aligned} p^g(s) &= \int q(s, s') g(s') ds' \\ &= \int \beta \frac{u'(s')}{u'(s)} g(s') f(s, s') ds' \\ &= E\left[\beta \frac{u'(s')}{u'(s)} g(s') \middle| s \right] \end{aligned}$$

The component $m = \beta \frac{u'(s')}{u'(s)}$ is called the **stochastic discount factor**, as it discounts random payoffs one period ahead to determine current period prices.

We can also introduce longer-lived assets, which are related on one-period assets via an **arbitrage** argument (that in equilibrium there can be no risk-free positive profits from a costless (self-financing) trading strategy). That is, we can replicate multi-period claims by rolling over one-period claims. Thus we can find the pricing kernel for multi-period claims by chaining together one-step claims:

$$q^j(s, s^j) = \int q(s, s')q^{j-1}(s', s^j)ds'$$

This multi-period perspective gives us another expression for the price of a tree. In fact, ownership of the tree is a claim to the entire dividend sequence $\{s_t\}$. Therefore we can define the price of the tree as:

$$p(s) \quad = \quad \beta \int \frac{u'(s')}{u'(s)} s' f(s, s')ds' + \beta^2 \int \frac{u'(s'')}{u'(s)} s'' f^2(s, s'')ds'' + \dots$$

Or in sequence notation:

$$p_t = E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j} \right]$$

Note that by working directly with an infinite horizon equilibrium, we rule out bubbles.

2.3 Risk Corrections

Risk free rate:

$$1 = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R \right] \quad \Rightarrow R = \frac{1}{E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right]}$$

or $R = 1/E_t m_{t+1}$.

For general payoff x_{t+1},

$$\begin{aligned} p_t &= E_t(m_{t+1}x_{t+1}) \\ &= E_t m_{t+1} E_t x_{t+1} + cov_t(m_{t+1}, x_{t+1}) \\ &= \frac{E_t x_{t+1}}{R} + cov_t(m_{t+1}, x_{t+1}) \end{aligned}$$