Lecture 1: Lucas Model and Asset Pricing

Economics 714, Spring 2018

1 Asset Pricing

1.1 Lucas (1978) Asset Pricing Model

We assume that there are a large number of identical agents, modeled as a representative agents. This is an endowment economy with a single nonstorable consumption good (fruit), which is given off stochastically by productive units (trees) with net supply of 1.

The representative agent has standard time additively separable preferences, with a period utility function that satisfies the usual conditions: $u'(c) > 0, u''(c) < 0$, Inada: $\lim_{c \to 0} u'(c) = +\infty$.

We start with a market in the shares of trees. The owner of the tree receives a stochastic dividend s_t, whose realizations are governed by Markov transition function $Q(s, ds')$.

Thus we have the representative agent problem:

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + p_t a_{t+1} = (p_t + s_t) a_t$$

We could characterize optimal choices and equilibrium sequentially or recursively. Based on what you've learned previously, we will start with the recursive formulation.

But in order to solve agent optimization problem, we need structure on p_t.

Conjecture pricing function $p_t = p(s_t)$. Then can write Bellman equation for the representative agent:

$$v(a,s) = \max_{(c,a')} \left\{ u(c) + \beta \int v(a',s') Q(s,ds') \right\}$$

subject to:

$$c + p(s)a' \leq (p(s)+s)a, \quad c \geq 0, \quad 0 \leq a' \leq 1$$

1.2 Equilibrium

Definition A *recursive competitive equilibrium* is a continuous function $p(s)$ and a continuous, bounded function $v(a,s)$ such that:

1. $v(a,s)$ solves the Bellman equation
2. $\forall s, \ \ v(1,s)$ is attained by $c = s$, $a' = 1$.

Note that this definition builds in optimization and market clearing: representative agent holds all shares of the tree and consumes the fruit.

To characterize the solution, note that wealth on hand is what really matters $(p(s)+s)a$ rather than a and s separately. Then we can re-write Bellman equation:

$$v((p(s)+s)a) = \max_{(a')} \left\{ u((p(s)+s)a - p(s)a') + \beta \int v((p(s')+s')a') Q(s,ds') \right\}$$

Which gives the first order condition:

$$-u'(c(s))p(s) + \beta \int v'((p(s')+s')a')[p(s')+s']Q(s,ds') = 0$$

and the envelope condition:

$$v'((p(s)+s)a) = u'(c(s))$$

We can combine the two optimality conditions to get Euler equation:

$$u'(c(s)) = \beta \int u'(c(s'))\frac{p(s')+s'}{p(s)}Q(s,ds').$$

This expression may be more familiar if we write it in sequence notation. If $p_t = p(s_t)$, $R_{t+1} = \frac{p_{t+1}+s_{t+1}}{p_t}$:

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{t+1}]$$

This is the standard optimality condition for a consumption-savings problem. But what is different is our perspective here. Rather than taking returns as given, and determining consumption, we now use the Euler equation to determine equilibrium returns. Since this is an endowment economy, equilibrium consumption is given.

That is, equilibrium $a = a' = 1$, $c(s) = s$. So we can rewrite the (functional) Euler equation to determine the equilibrium pricing function:

$$p(s) = \beta \int \frac{u'(s')(p(s')+s')}{u'(s)}Q(s,ds')$$

1.3 Pricing General Claims

We have used the equilibrium of the model to determine a single price, the price of a tree. With trading in trees, markets are complete. However we can introduce additional (redundant) assets and use the same approach to determine their prices.

For this, suppose that the transition function Q has a density $f(s, s')$:

$$Q(s, s') = \int_{-\infty}^{s'} f(s, u)du$$

In general we can define a **pricing kernel** $q(s', s)$ such that the equilibrium price P^A of unit of the good in period $t+1$ when $s_{t+1} \in A$ conditional on $s_t = s$ is:

$$P^A(s) = \int_{s' \in A} q(s, s')ds'$$

In the Lucas model, the pricing kernel is:

$$q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s, s')$$

Roughly, this is the intertemporal marginal rate of substitution multiplied by the state probabilities.

Using the pricing kernel, we can find the equilibrium price of any contingent claim $g(s')$ one-period ahead:

$$\begin{aligned} p^g(s) &= \int q(s, s')g(s')ds' \\ &= \int \beta \frac{u'(s')}{u'(s)} g(s')f(s, s')ds' \\ &= E\left[\beta \frac{u'(s')}{u'(s)} g(s') \middle| s \right] \end{aligned}$$

The component $m = \beta \frac{u'(s')}{u'(s)}$ is called the **stochastic discount factor**, as it discounts random payoffs one period ahead to determine current period prices.

We can also introduce longer-lived assets, which are related on one-period assets via an **arbitrage** argument (that in equilibrium there can be no risk-free positive profits from a costless (self-financing) trading strategy). That is, we can replicate multi-period claims by rolling over one-period claims. Thus we can find the pricing kernel for multi-period claims by chaining together one-step claims:

$$q^j(s, s^j) = \int q(s, s')q^{j-1}(s', s^j)ds'$$

This multi-period perspective gives us another expression for the price of a tree. In fact, ownership of the tree is a claim to the entire dividend sequence $\{s_t\}$. Therefore we can define the price of the tree as:

$$p(s) \quad = \quad \beta \int \frac{u'(s')}{u'(s)} s' f(s, s')ds' + \beta^2 \int \frac{u'(s'')}{u'(s)} s'' f^2(s, s'')ds'' + \dots$$

Or in sequence notation:

$$p_t = E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j} \right]$$

Note that be working directly with an infinite horizon equilibrium, we rule out bubbles which may arise in finite-length markets.