Lecture 7
Solow Model

Noah Williams

University of Wisconsin - Madison

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Solow Model Overview

- So far have analyzed the sources of growth in an economy, argued that only TFP growth provides sustainable growth.
- Now develop a model showing this and addressing:
  1. What is the relationship between the long-run standard of living and the saving rate, population growth rate, and rate of technical progress?
  2. How does economic growth change over time? (Speed up, slow down, stabilize?)
  3. Are there forces that allow poorer countries to catch up to richer ones?

Basic Assumptions of the Solow Growth Model

- Continuous time. (Book does discrete time.)
- Single good in the economy produced, constant technology.
- No government or international trade.
- All factors of production are fully employed.
- Labor force (population) grows at constant rate

$$n = \frac{\dot{N}}{N}$$

- Initial values for capital, K_0 and labor, N_0 given.

Production

- Cobb-Douglas aggregate production function:

$$Y(t) = z(t)F(K(t), N(t)) = z(t)K(t)^{\alpha}N(t)^{1-\alpha}$$

- For now suppose no TFP growth: $z(t) = 1$.
- Define per worker variables: $y = \frac{Y}{N}, k = \frac{K}{N}$. Then:

$$y = \frac{Y}{N} = \frac{K^{\alpha}N^{1-\alpha}}{N} = \left(\frac{K}{N}\right)^{\alpha}\left(\frac{N}{N}\right)^{1-\alpha} = k^{\alpha} = f(k)$$

- Per worker production function has decreasing returns to scale.
- Again constant returns implies $Y = rK + wN$.

Households

- Suppose households don't value leisure, inelastically supply 1 unit of labor. Aggregate labor supply then N.
- Assume result of household maximization (or social planner's) problem is to save a constant fraction s of income, consume $1 - s$.

$$C = (1-s)[rK + wN] = (1-s)Y$$

- Capital evolution:

$$K_{t+1} = (1-\delta)K_t + sY_t$$

- Rearrange and take limits as $\Delta t \to 0$:

$$\begin{aligned} K_{t+1} - K_t &= -\delta K_t + sY_t \\ \Rightarrow \dot{K} &= sY - \delta K \end{aligned}$$

Capital Accumulation

- Divide by N in the capital accumulation equation:

$$\frac{\dot{K}}{N} = sy - \delta k = sk^{\alpha} - \delta k$$

- Now remember that $k(t) = K(t)/N(t)$, so:

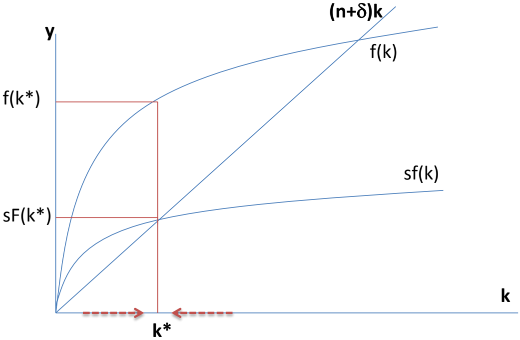
$$\begin{aligned} \dot{k} &= \frac{\dot{K}}{N} - \frac{\dot{N}K}{NN} \\ &= \frac{\dot{K}}{N} - nk \\ &= sk^{\alpha} - (\delta + n)k \end{aligned}$$

- The last line is the fundamental equation of the Solow Model

$$\boxed{\dot{k} = sk^{\alpha} - (\delta + n)\, k}$$

Graphical Analysis

- Change in k, $\dot{k}$ is given by difference of sk^{α} and $(\delta + n)k$
- If $sk^{\alpha} > (\delta + n)k$, then k increases.
- If $sk^{\alpha} < (\delta + n)k$, then k decreases.
- Unique positive steady state. (Trivial steady state at $k = 0$.)
- Positive steady state stable: if move away, will come back to it.



Steady state in the Solow model

Steady State Analysis

- Steady State: $\dot{k} = 0$
- Solve for steady state

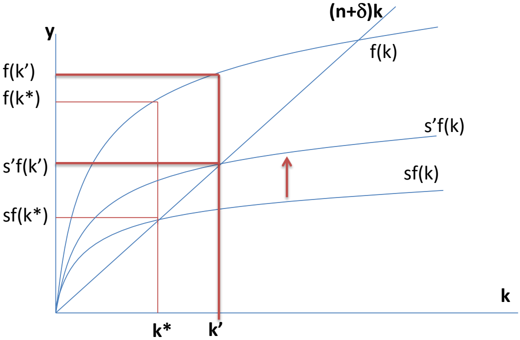
$$0 = s\,(k^*)^{\alpha} - (n+\delta)k^* \Rightarrow k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Steady state output per worker $y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$
- Steady state consumption per worker:

$$c^* = (1-s)(k^*)^{\alpha}$$

Comparative Statics

- Suppose that of all a sudden saving rate s increases to $s' > s$. Suppose that economy was initially at its old steady state with saving rate s.
- $(n + \delta)k$ curve does not change.
- $sf(k)$ shifts up to $s'f(k)$
- New steady state: higher capital and output per worker.
- Capital stock increases monotonically from old to new steady state.



Increase in savings rate in the Solow model

Evaluating the Basic Solow Model

- Why are some countries rich (have high per worker GDP) and others are poor (have low per worker GDP)?
- Solow model: if all countries are in their steady states, then:
    1. Rich countries have higher saving (investment) rates than poor countries.
    2. Rich countries have lower population growth rates than poor countries.
- Data seem to support this prediction of the model.
- But no growth in the steady state: only transitional dynamics.
- Illuminates why capital accumulation has an inherit limitation as a source of economic growth. (Recall Singapore example.)

Figure 7.2 Real Income Per Capita vs. Investment Rate

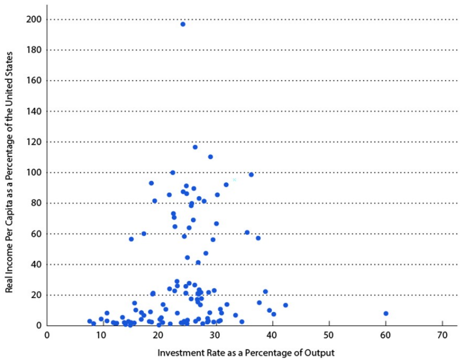
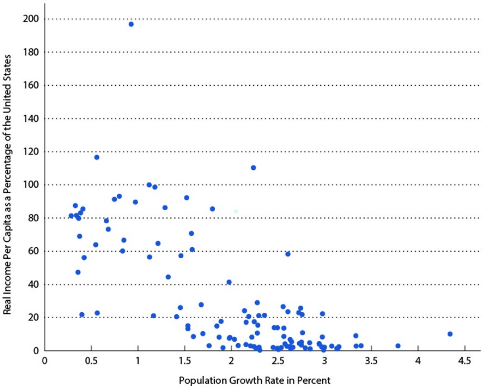


Figure 7.3 Real Income Per Capita vs. the Population Growth Rate



Introducing Technological Progress

- Re-introduce TFP. No limits to innovation.
- Slightly simpler to introduce as labor-augmenting technical change. Aggregate production function:

$$Y = K^{\alpha} (AN)^{1-\alpha}$$

- Equivalent to $zK^{\alpha}N^{1-\alpha}$ with $z = A^{1-\alpha}$.
- Key assumption: constant rate of technological progress:

$$\frac{\dot{A}}{A} = g > 0$$

- TFP growth is exogenous here.

Balanced Growth Path

- Situation in which output per worker, capital per worker, and consumption per worker grow at constant rates.
- For Solow model, y, k, c all grow same at constant rate g.
- Why? In balanced growth path g_k constant, but:

$$\dot{k} = sy - (n+d)k \Rightarrow g_k = sy/k - (n+d)$$

So y/k constant $\Rightarrow g_y = g_k$. But then:

$$y = \frac{Y}{N} = \frac{K^\alpha (AN)^{1-\alpha}}{N} = k^\alpha A^{1-\alpha}$$

- Take logs and differentiate:

$$g_y = \alpha g_k + (1-\alpha) g_A$$

But we showed $g_k = g_y$ and assumed $g_A = g$, so $g_k = g_y = g_A = g$.

Analysis of Extended Model

- In BGP all per-capita variables grow at rate g. Want to work with variables that are constant in long run. Define:

$$\tilde{y} = \frac{y}{A} = \frac{Y}{AN}$$
$$\tilde{k} = \frac{k}{A} = \frac{K}{AN}$$

- Repeat the analysis with new variables:

$$\tilde{y} = \tilde{k}^{\alpha}$$
$$\dot{\tilde{k}} = s\tilde{y} - (n+g+\delta)\tilde{k}$$
$$\dot{\tilde{k}} = s\tilde{k}^{\alpha} - (n+g+\delta)\tilde{k}$$

Balanced Growth Path Analysis I

- Solve for $\tilde{k}^*$ analytically

$$
\begin{aligned}
0 &= s\tilde{k}^{*^{\alpha}} - (n+g+\delta)\tilde{k}^* \\
\tilde{k}^* &= \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

- Therefore

$$
\tilde{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}
$$

Balanced Growth Path Analysis II

$$k(t) = A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y(t) = A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$K(t) = N(t)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

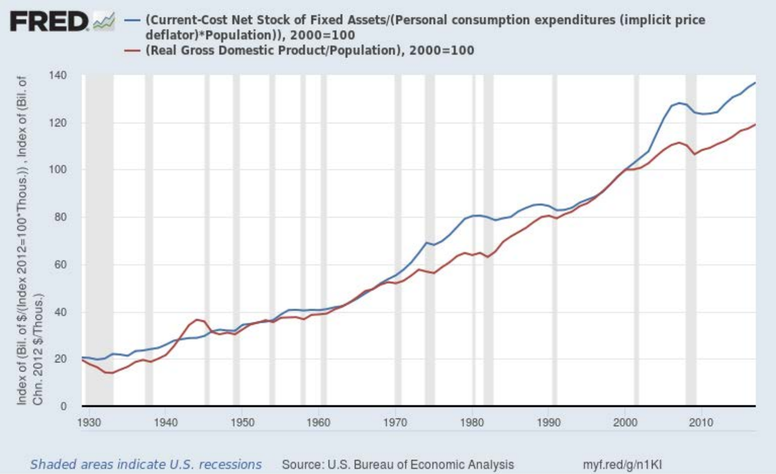
$$Y(t) = N(t)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Growth Facts

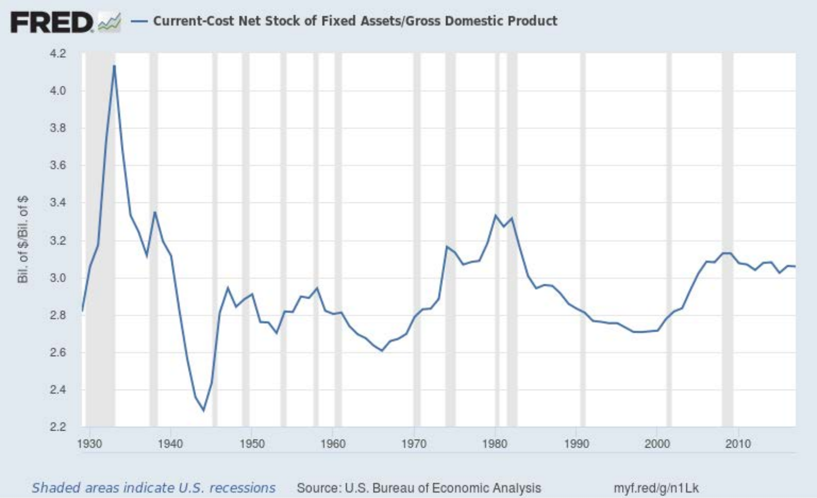
Stylized facts, originally due to Kaldor (1957). Empirical regularities of the growth process for the US and for most other industrialized countries

1. Output (real GDP) per worker $y = \frac{Y}{N}$ and capital per worker $k = \frac{K}{N}$ grow over time at relatively constant and positive rate.
2. They grow at similar rates, so that the ratio between capital and output, $\frac{K}{Y}$ is relatively constant over time
3. The real return to capital r (and the real interest rate $r - \delta$) is relatively constant over time.
4. The capital and labor shares are roughly constant over time.

Capital and Output Growth



Capital/Output Ratio



Decline in the Labor Share

- Recent work has documented a decline in the labor share over the past 25 years. Karabarbounis and Neiman (2014) show that this has happened around the globe, Elsby, Hobijn, and Sahin (2013) analyze the US
- Potential explanations:
  - A reduction in the price of investment goods, mostly due to IT, has induced firms to shift away from labor and toward capital.
  - The industry composition of total output has changed, with much of the decline in labor dominated by manufacturing and trade sectors.
  - In the U.S., there is evidence that companies have off-shored more of the labor-intensive part of their supply chains

Labor Share of Income in the U.S.

Chart 1: U.S. Nonfarm Business Sector Labor Share, 1947–2017

Notes: Gray bars denote National Bureau of Economic Research (NBER)-defined recessions.
Sources: Bureau of Labor Statistics (BLS) and NBER.

Evaluation of the Model: Growth Facts

- The Solow model matches the stylized facts.
- Output and capital per worker grow at the same constant, positive rate in BGP of model. In long run model reaches BGP.
- Capital-output ratio $\frac{K}{Y}$ constant along BGP
- Interest rate constant in balanced growth path
- Capital share equals α, labor share equals $1 - \alpha$ in the model (always, not only along BGP)
- Success of the model along these dimensions, but source of growth, technological progress, is left unexplained.