Welfare Measurements with Heterogenous Agents

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Abstract

The canonical infinite horizon framework with heterogenous consumers, used in macro and financial literature, lacks a preference-based welfare index that produces consistent normative predictions for different policies. In particular, the classic preference-based indices, such as equivalent or compensating variations, do not aggregate and they are not additive on the set of policies. This paper offers a positive result. We show that for arbitrary heterogenous von Neumann Morgenstern preferences with common discount factor, an equivalent (compensating) variation is nearly additive and admits a representative agent representation, as long as consumers are *patient*. Therefore, this index generates consistent quantitative comparisons of welfare effects in a wide variety of problems studied in the macro and finance literature. These problems include, among others, predictoins regarding welfare impacts of fiscal or monetary policies, costs of real business cycles, or welfare costs of policies implemented in financial markets.

Key words: Preference-based welfare, additivity, aggregation

JEL classification numbers: D43, D53, G11, G12, L13

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1 Introduction

Policymakers often chose among multiple variants of an economic policy. Consider, for example, a design of a stimulus package in response to some adverse economic shock. Available alternatives can differ in size, duration, targeted consumers, etc. How should a decision-maker quantify welfare effects of available scenarios? In macroeconomic literature, the canonical framework to evaluate economic policies is a stochastic infinite-horizon model with heterogeneous consumers. This framework ranks policies according to average utility in a population, Aiyagari (1994), Heathcote et al. (2009). However, the utilitarian (cardinal) approach requires interpersonal comparability of unobservable utilities and, hence, is not in line with the ordinal paradigm of the modern economic theory.

Unfortunately, the existing economic theory does not offer a functional *preference-based* criterion that gives consistent normative predictions for multiple policies. In particular, in economies with many heterogeneous consumers, policies rarely benefit all the agents, and Pareto rankings are highly incomplete and thus uninformative. A *consumption equivalent*, an index defined within the representative agent model, Lucas (1987), does not have interpretation in terms of the consumers' underlying (heterogenous) preferences. Importantly, the classic index, *equivalent variation*, Hicks (1939) is ill-behaved: welfare effects and associated normative recommendations depend on the assumed *status quo* or the order in which the alternatives are implemented. Moreover, for realistic preferences, the index does not aggregate. This paper offers a positive result. We show that equivalent variation acquires the above-mentioned desirable properties for transient policies when agents are sufficiently patient. We next explain our result in detail.

We revisit the properties of equivalent variation in the context of a small open economy. The welfare index is defined as follows. Fix some consumption bundle x that gives a welfare unit. Consider some factual and counterfactual policies. For an individual consumer, equivalent variation is given by a transfer of numeraire x, which makes the factual policy equally attractive as the counterfactual one, assuming factual prices. The aggregate effect is a sum of the effects for all consumers. Note that the aggregate index is not affected by monotonic transformations of utility functions, and hence it does not require comparability of consumers' utilities. Since welfare is expressed in real terms (consumption x), its value is invariant to price normalizations. Finally, the index is Paretian: whenever a counterfactual policy improves all consumers' preferences, the welfare effect is positive.

We are interested in two important properties of an index: additivity and aggregation. An index is deemed *additive*, if for any three policies p, p' and p'' the welfare impact of policy p'' relative to p coincides with the sum of the effects of policy p' relative to p and policy p'' relative to p'. Note that for an index with this property, welfare effects are invariant to the assumed *status quo* or the sequence in which policies are implemented. As a result welfare impacts of different variants of a policy can be unambiguously determined. Similarly, for a policy that affects several aspects of an economy, the total welfare effect can be decomposed into individual components. Finally, equivalent variation coincides with the alternative preference-based index, *compensating variation* $CV_{p,p'} \equiv -EV_{p',p}$.[1] The second property is defined as follows. An index *aggregates*, whenever welfare effects are fully determined by aggregate income and prices associated with policies. With this property, welfare predictions are not affected by a redistribution of wealth among consumers. As a result, the economy admits the fiction of a representative agent.

It is well known that, in the infinite-horizon economy, equivalent variation is not additive for any standard preferences, and it aggregates only for Gorman form preferences, which are not observed empirically.[2] We show in a general framework that *equivalent variation is nearly additive, and it approximately aggregates on a set of transient policies when consumers are patient*. More precisely, each policy is associated with social surplus measurable with respect to aggregate income, such that the impact of a policy relative to its alternative is approximated well by a difference between the policies' surplus values. The approximation accuracy improves with a discount factor and is exact in the limit with a discount factor equal to one.

We demonstrate our result assuming a common discount factor among consumers.[3] Otherwise, the result is quite robust. We allow for arbitrary heterogenous von Neumann-Morgenstern preferences and heterogeneous (Markov) endowment flows. Since the policies are modeled as transient shocks to prices and endowments, the result facilitates comparisons of a wide variety of policies. The latter include technological or income shocks, sales and service taxes, lump-sum transfers, subsidies, public spending, social safety net programs, and many others.

Our theory has useful implications for policy-making. A discount factor is defined for a specified unit of time, e.g., a year. By redefining a period as a quarter, a month, or a day, one can make the empirical discount factor arbitrary close to one. Consequently, in empirical

[1]For an additive index a round trip from policy p to p' and back yields zero welfare change, and hence, $EV_{p,p'} = -EV_{p',p} \equiv CV_{p,p'}$.

[2]Preferences are in Gorman polar form if they give rise to parallel Engel curves (lines) as, e.g., in the case of homogenous CRRA preferences, Gorman (1964). With Gorman preferences, redistribution of wealth does not affect aggregate equivalent variation. Note that Gorman polar form preferences are non-generic. They are also very restrictive in terms of implied consumers' behavior: for fixed prices, consumers spend additional dollars on the same consumption bundles regardless of their level of wealth, age, gender, or other characteristics. Not surprisingly, the empirical literature consistently rejected the Gorman hypothesis; see, e.g., Carroll (2000); Blundell et al. (2007) and Lewbel and Pendakur (2009).

[3]This assumption is technical and standard in the macro literature. It allows us to have a well-defined limit experiment with respect to one parameter, β, ignoring potential effects of the relative speed of convergence of heterogeneous discount factors.

studies, one can achieve the desired accuracy by considering policies with sufficiently short duration. In Section 4, we apply this logic to the Polish economy. We use a stylized model of an economy with the preference and income distributions extracted from the available micro-econometric data. We give normative predictions for different variants of a stimulus package and contrast them with the value derived using surplus approximation. Our simulations suggest that for the annual discount factor, $\beta = 0.97$, often assumed in the literature, and for the economically relevant policies that affect the economy within the first four (twelve) quarter(s), the approximation error is no greater than 1.5% (5%) of the actual value of the welfare effect.

The paper also contributes to aggregation literature. The recent empirical studies strongly reject the hypothesis of Gorman form preferences and thus refute the existence of a representative agent, Carroll (2000); Blundell et al. (2007) and Lewbel and Pendakur (2009). We partly quantify these findings by showing that, at least for transient policies, a representative agent approach may be justified, even if the underlying consumers' preferences themselves violate the Gorman hypothesis.

The approximation result relies on the following mechanism.[4] In the considered framework, savings induced by transient policies are spent on consumption in many subsequent periods. With patient consumers, the differences in savings can have at the most negligible impact on the marginal utility of money. With a discount factor near one, the infinite horizon framework is virtually indistinguishable from a quasilinear economy in terms of observables: the frameworks give identical predictions regarding choices and preference-based welfare. In the quasilinear model, in turn, equivalent variation is well-behaved. Thus, as a byproduct, we offer a micro-foundation for the assumption of quasilinear preferences, sometimes made in the macro and financial literature.[5]

The rest of this paper is organized as follows. Section 2 explains the main idea within a simple example of an open economy. Section 3 states the approximation result, and Section 4

[4]Similar mechanism has been explored in a general equilibrium setting in Bewley (1976) to establish permanent income hypothesis, as well as in Levine and Zame (2002) to show that incomplete market converges to complete markets. In a context of individual consumer, Vives (1987) uses this observation to characterize marginal utility of money, and income effects in each market as a number of goods increases to infinity. These classic results, however, are not sufficient to establish convergence of equivalent variation, due to so-called fallacy of composition. For the discussion see Mas-Colell et al. (1995), p. 89.

[5]These, among others, include applications in monetary economics, Lagos and Wright (2005), optimal taxation literature, Barro (1979); Aiyagari et al. (2002); Bhandari et al. (2016), financial economics, Carvajal et al. (2012), and the labor literature based on the search model Mortensen and Pissarides (1994) that all critically rely on quasilinear preferences. Some of these papers assume preferences over the consumption and labor, captured by quasilinear instantaneous utilities wherein consumption enters the utility function linearly, period by period. In contrast, the disutility of labor is strictly concave. With a riskless asset, this formulation is equivalent to a specification with a single linear consumption good x_0 that summarizes expenditures in all the periods.

tests the approximation in the context of the Polish economy. Section 5 concludes.

2 Motivating Example

In this section we explain the key ideas in a simple example of an economy with two consumers. Each consumer $i = 1, 2$ maximizes CRRA preferences over infinite consumption streams, represented by utility

$$\max \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\theta^i} - 1}{1 - \theta^i},$$

facing the intertemporal budget constraint

$$\text{s.t. } c_t^i + q_{t+1} w_{t+1}^i = w_t^i + A_t^i l_t^i,$$

for each $t \geq 0$. In the small open economy prices of bonds are formed in international markets. In each period a consumer inelastically supplies one unit of labor $l_t^i = 1$. A_t^i is labor productivity and w_t^i is wealth with which a consumer enters period t. In the absence of a policy (or, alternatively, for a neutral policy), the economy is stationary: labor productivity is $A_t^i = 2$ for $i = 1, 2$ and the price of a one-period bond is $q_{t+1} = \beta$ for all $t \geq 0$. The aggregate output in the economy in period t is $Y_t = \sum_{i=1,2} A_t^i l_t^i$.

We next verify the additivity and aggregation of equivalent variation in the example. Let the factual policy p be neutral, i.e., the fundamentals are not affected by shocks. We consider two alternative counterfactual scenarios. For policy p' period-zero productivity of both consumers becomes $A_0^{i\prime} = 3$. For p'' productivity of consumer one, is $A_0^{1\prime\prime} = 4$, while for consumer two it is $A_0^{2\prime\prime} = 2$. In both scenarios, the price of a bond increases to $q_1' = q_1'' = 2$. Note that with inelastic labor supply, both counterfactual scenarios result in the same aggregate output, $Y_0 = 6$. The policies, however, differ in how they allocate income among consumers.

We summarize numerical simulations in Tables 1-2. The top four rows report the values of welfare indices for different discount factors and welfare numeraire given by unit of consumption in period two. The next two rows quantify the degree to which the predictions violate the aggregation and additivity property (these values are zero whenever the index satisfies the assumptions). Finally, the last row measures how far the equivalent variation is from its limit.

For parameter values $\theta^i = 2$ for $i = 1, 2$, preferences are in Gorman polar form and equivalent variation is independent of income distributions. The index aggregates and alternative counterfactual policies generate identical welfare predictions, $EV_{p,p'} = EV_{p,p''}$. Indeed the aggregation gap, that quantifies the departure from the aggregation benchmark is zero,

Table 1: Welfare effects for homogenous preferences

β	0.5	0.7	0.9	0.95	0.98	0.99	Limit
$EV_{p,p'}$	3.5556	2.1652	1.5327	1.4305	1.3765	1.3596	1.3431
$CV_{p,p'}$	2.0000	1.6671	1.4334	1.3864	1.3600	1.3515	1.3431
$EV_{p,p''}$	3.5556	2.1652	1.5327	1.4305	1.3765	1.3595	1.3431
$CV_{p,p''}$	2.0000	1.6671	1.4334	1.3864	1.3600	1.3515	1.3431
$Ag\%$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$Ad\%$	0.4375	0.2300	0.0648	0.0308	0.0120	0.0059	0.0000
$L\%$	1.6472	0.6120	0.1411	0.0650	0.0248	0.0122	0.0000

Note: The table reports welfare effects for risk aversion coefficients $\theta^1 = \theta^2 = 2$ and x given by a unit of consumption in period $t = 2$. The first two rows report equivalent and compensating variation for policy p'. The two subsequent rows give for policy p''. The last three rows report three gaps: the aggregation gap, $Ag\% \equiv (EV_{p,p''} - EV_{p,p'})/EV_{p,p'}$ the additivity the gap $Ad\% \equiv (EV_{p,p'} - CV_{p,p'})/EV_{p,p'}$ and the limit gap $L\% \equiv (EV_{p,p'} - L)/L$, where L is the limit value of equivalent variation as $\beta \to 1$

$Ag\% = 0$, for all $\beta < 1$. Note, however, that the equivalent and compensating variations are not equal to each other, $EV_{p,p'} \neq CV_{p,p'}$. This shows that the index is not additive. Consequently, the addititivity gap, that ofmeasures non-additivity is non-zero, $|Ad\%| > 0$.

For heterogeneous preferences, equivalent variation does not exhibit any of the considered properties. Table 2 reports the values of welfare effects and gaps for risk aversion $\theta^1 = 0.5$ and $\theta^2 = 5$. Indeed, equivalent variations diverge in the two scenarios, and they differ from the corresponding compensating variations. As a result the aggregation and additivity gaps are nonzero. In summary, the welfare indices are not additive and aggregate only in the particular instance of homogenous preferences.

We now show that equivalent variation becomes additive and it aggregates with sufficiently patient consumers. For this, we consider the limit values of the welfare effects as the discount factor approaches one. Clearly, utility attained under the considered policies becomes unbounded, and the framework does not permit welfare comparisons in terms of limit cardinal utilities. The equivalent variation, however, is preference-based, and, as such, it has well-behaved limits. We report limit values in the last columns of Tables 1-2. For both types of preferences satisfy our properties, and the additivity and aggregation gaps converge to zero.

It is clear from the example that in a small open economy with patient consumers, the classic preference index is well-behaved. In the next section, we formalize this idea in the

Table 2: Welfare effects for heterogenous preferences

β	0.5	0.7	0.9	0.95	0.98	0.99	Limit
$EV_{p,p'}$	6.9098	3.9849	2.5231	2.2769	2.1456	2.1043	2.0641
$CV_{p,p'}$	2.9964	2.6775	2.2793	2.1721	2.1073	2.0857	2.0641
$EV_{p,p''}$	7.3271	4.1319	2.5530	2.2902	2.1505	2.1067	2.0641
$CV_{p,p''}$	2.9964	2.6775	2.2793	2.1721	2.1073	2.0857	2.0641
$Ag\%$	0.0604	0.0369	0.0119	0.0058	0.0023	0.0011	0.0000
$Ad\%$	0.5664	0.3281	0.0966	0.0461	0.0178	0.0088	0.0000
$L\%$	2.3476	0.9305	0.2223	0.1031	0.0395	0.0194	0.0000

Note: the table reports welfare effects for risk aversion coefficients $\theta^1 = \frac{1}{2}, \theta^2 = 5$ and x given by unit of consumption in period $t = 2$. The first two rows report equivalent and compensating variation for policy p'. The two subsequent rows give for policy p''. The last three rows report three gaps: the aggregation gap, $Ag\% \equiv (EV_{p,p''} - EV_{p,p'})/EV_{p,p'}$ the additivity the gap $Ad\% \equiv (EV_{p,p'} - CV_{p,p'})/EV_{p,p'}$ and the limit gap $L\% \equiv (EV_{p,p'} - L)/L$, where L is the limit value of equivalent variation as $\beta \to 1$.

framework with general von Neumann-Morgenstern preferences and productivity shocks. We also derive a simple formula for the limit index.

3 The main result

3.1 Small Open Economy

Consider an infinite-horizon economy with $i = 1, 2, ..., I$ consumers. Each consumer i has preferences over random, strictly positive consumption flows $c^i = \{c_t^i\}_{t=0}^{\infty}$, represented by the expected utility function

$$U^i\left(c^i\right) = E\sum_{t=0}^{\infty}\beta^t u^i(c_t^i). \tag{1}$$

Instantaneous utility function satisfies the standard assumptions: $u^i : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave, and satisfies Inada conditions. Consumers have common discount factors β. Otherwise, preferences can be heterogeneous.

Each consumer is endowed with one unit of time per period, that can be used to produce output. Individual output in period t is given by $y_t^i = A_t^i f^i(l_t^i)$ where labor choice is $l_t^i \in [0, 1]$. Production function $f^i(\cdot)$ is non-negative and strictly increasing and process of labor productivity $A^i = \{A_t^i\}_t$ is determined by some fundamental stochastic process $s = \{s_t\}_t$.

Consumers hedge idiosyncratic productivity shocks by trading assets in international markets that are dynamically complete. In state s_t a consumer faces the usual intertemporal budget constraint

$$c_t^i + E(q_{t+1} w_{t+1}^i | s_t) \leq w_t^i + A_t^i f^i(l^i).$$

Random variable w_{t+1}^i is wealth in different states of the next period chosen by a consumer and q_{t+1} are prices of the corresponding state-contingent claims. Both random processes are adapted with respect to natural filtration of process s.

Consumers enter period zero, state s_0 with no wealth.[6] The fundamentals evolve according to an underlying Markov chain $s = \{s_t\}_t$ with finite state space $\mathcal{S} = \{0, 1, 2, ..., S\}$. For such process an event in period t is identified by history $h_t = \{s_0, s_1, ..., s_t\}$. A transition matrix has real eigenvalues, for which process has a unique stationary distribution with full support. We denote stationary a state variable $\bar{s}$. For the neutral policy, (in absence of shocks), the fundamentals satisfy the following assumption

Assumption 1. *(Stationarity) There exist functions $q : \mathcal{S} \to \mathbb{R}_{++}$ and $A^i : \mathcal{S} \to \mathbb{R}_{++}$ for each i such that the price of the contingent consumption in s_{t+1}, in state s_t is given by $q_{s_{t+1}} | s_t = \beta q(s_{t+1})/q(s_t)$ and the realization of productivity of consumer i in state s_t is $A^i(s_t)$.*

Consumers' productivities can be arbitrarily correlated with each other as well as with international prices. The assumption on prices is standard in the literature — it assures that optimal consumption flows are martingales.

Economic policies are broadly defined as arbitrary perturbations of fundamentals, namely, domestic productivity and international prices. Formally, a policy is represented by a tuple of random processes $p = (\Delta^q, \{\Delta^{A^i}\}_i)$, adapted to the natural filtration of underlying Markov chain s. Note that the admissible shocks can be history dependent: under policy p, in the event following history $h_{t+1} = \{s_0, s_1, ..., s_t, s_{t+1}\}$ the price of contingent consumption adjusts to

$$q_{s_{t+1}} | s_t = \beta \frac{q(s_{t+1}) + \Delta^q_{h_{t+1}}}{q(s_t) + \Delta^q_{h_t}} > 0,$$

where h_t is a truncation of h_{t+1}. Similarly, productivity of consumer i after history h_t is perturbed to $A_{h_t} = A^i(s_t) + \Delta^{A^i}_{h_t} > 0$. Under neutral policy aggregate income $Y = \{Y_t\}_t$, after history h_t is given by $Y_{h_t} = \sum_i A^i(s_t) f^i(1)$. For policy p aggregate income is perturbed by $\Delta^Y = \{\Delta^Y_t\}_t$ where $\Delta^Y_{h_t} \equiv \sum_i \Delta^{A^i}_{h_t} f^i(1)$.

We restrict attention to policies, for which effects on fundamentals vanish over time. In particular, we make the following assumption:

Assumption 2. *(Vanishing shocks) Consider policy p. There exist constants $C > 0$ and $\Delta \in (0, 1)$ such that $|\Delta^q_{h_t}| \leq C \times (\Delta)^t$ and $|\Delta^{A^i}_{h_t}| \leq C \times (\Delta)^t$ for all period t, histories h_t, and i.*

[6] The initial states of the consumers can potentially differ among consumers.

Collection of all policies that satisfy Assumption 2 is denoted by $\mathcal{P}$. The assumed exogenous price process arises naturally in a stochastic representative agent economy with a discount factor β.[7] Importantly, in this micro-foundation, the discount factor of international markets coincides with the one of the considered small open economy.

Similarly to the example, we measure welfare effects as a standard *aggregate equivalent variation*, i.e., a sum of equivalent variations for all consumers, $EV_{p,p'} \equiv \sum_i EV^i_{p,p'}$, where components $EV^i_{p,p'}$ are defined as sufficient transfers of welfare numeraire flow x. We provide the formal definition of the index in Appendix A.2.

We conclude this section by showing that the welfare index is uniquely defined for an arbitrary pair of policies.

Proposition 1. *For any pair $p, p' \in \mathcal{P}$ and discount factor $\beta < 1$ equivalent variation $EV_{p,p'} \in \mathbb{R}$ exists and it is unique.*

Proof of Proposition 1: The proof is in appendix A.2.

Consequently, the comparative statics of equivalent variation with respect to a discount factor is a well-posed problem.

3.2 Surplus Approximation

In this section we state our main result. For this we first derive a formula for approximate surplus. Let $\bar{q} \equiv q(\bar{s})$ and $\bar{A}^i \equiv A^i(\bar{s})$ be the stationary prices and productivity. Consider equality

$$E\left[\bar{q}u^{i\prime-1}(\bar{q}\lambda^i)\right] = E\left[\bar{q}\bar{A}^i f^i(1)\right], \tag{2}$$

where $u^{i\prime-1}$ is an inverse of the marginal utility. Equation (2) has a unique solution, $\bar{\lambda}^i$, that is strictly positive.[8]

For history $h_t = \{s_0, s_1, ..., s_t\}$ let $\bar{c}^i(\Delta^q_{h_t}) = u^{i\prime-1}([q(s_t)+\Delta^q_{h_t}]\bar{\lambda}^i)$ be an optimal consumption in event h_t assuming price perturbation $\Delta^q_{h_t}$ and marginal utility of money $\bar{\lambda}^i$. We next define two components of the surplus function for each event. The first component captures aggregate welfare gains/losses from trade, resulting from shocks in international markets. For h_t, the first component is

$$s_{trade}(\Delta^q_{h_t}) \equiv \sum_i \left(\frac{u^i(\bar{c}^i(\Delta^q_{h_t})}{\bar{\lambda}^i} - (q(s_t) + \Delta^q_{h_t})\bar{c}^i(\Delta^q_{h_t}) \right). \tag{3}$$

[7]Such prices will be observed in a representative agent economy with preferences represented by a function of the form (1), and the endowment following a Markov chain, perturbed by shocks satisfying the assumption analogous to Assumption 2.

[8]The left-hand side of equation (2) is a continuous function strictly decreasing in λ^i with range $\mathbb{R}_+$. The right-hand side, a real number, that is strictly positive. It follows that a solution exists and is unique. Scalar $\bar{\lambda}^i$ gives marginal utility of money—a Lagrangian multiplier — in the optimization problem, in which for which optimal consumption satisfies budget constraint in steady state, period-by-period.

For an individual consumer the surplus is geometrically represented by the area under the ($\bar{\lambda}$-normalized) marginal utility and the price of consumption. The aggregate surplus is then the sum of such areas for all consumers. The second component is an aggregate nominal income,

$$s_{income}(\Delta^q_{h_t}, \Delta^Y_{h_t}) \equiv (q(s_t) + \Delta^q_{h_t}) \times (Y_{h_t} + \Delta^Y_{h_t}). \tag{4}$$

For a given policy p the approximate social surplus is given by a sum of the two components for all date-events, normalized by the corresponding value for the neutral policy:[9]

$$S(\Delta^q, \Delta^Y) \equiv \sum_{t=0}^{\infty} E\left[s_{trade}(\Delta^q_t) - s_{trade}(0) + s_{income}(\Delta^q_t, \Delta^Y_t) - s_{income}(0,0)\right]. \tag{5}$$

Since for the admissible policies shocks are vanishing over time, the surplus is finite for any $p \in \mathcal{P}$. The formula has a simple form that is additive across consumers and histories. It is also measurable with respect to market level data: aggregate output and prices of contingent claims.

We are ready to state our theorem. Consider policies $p, p' \in \mathcal{P}$ and a welfare numeraire $x = \{x_t\}_t$ that takes zero value in all events, in which perturbations of prices for both policies are non-zero, i.e., $x_{h_t}\Delta^q_{h_t} = x_{h_t}\Delta^{q'}_{h_t} = 0$. We also require that the limit present value (as $\beta \to 1$) is positive and finite, i.e., $\bar{v}^x \equiv \sum_{t=0}^{\infty} E[q_t x_t] \in \mathbb{R}_{++}$. For example, if considered policies perturb prices after period zero, then the natural welfare numeraire is a unit of consumption in period zero. Below we state the main result of the paper.

Theorem 1. *Aggregate equivalent variation has an additive limit, measurable with respect to aggregate income.*

$$\lim_{\beta \to 1} EV_{p,p'} = \frac{S(\Delta^{q'}, \Delta^{Y'}) - S(\Delta q, \Delta Y)}{\bar{v}^x} \in \mathbb{R}.$$

Proof of Theorem 1: The proof is in the appendix.

For all policies in $\mathcal{P}$, the limit equivalent variation admits a surplus representation and hence hence is additive. Moreover, the surplus magnitude is measurable with respect to aggregate income. As it is clear from (5), the value of equivalent variations can be easily computed using the surplus formula.

The choice of welfare numeraire x affects predictions only up to a normalization constant. The choice of x is restricted in two ways. First, to eliminate the differential effects of shocks on the value of the numeraire, x is zero value in periods for which policies perturb prices. Also, the present value of the numeraire in the limit has to be bounded. Otherwise, the

[9]The surplus equation involves infinite sums, and without the neutral policy normalization surplus 5 would not be finite.

Table 3: Empirical distribution of parameters

Decile	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
$A^i_{s_t=1}$	0.70	0.76	0.84	0.93	0.98	1.03	1.08	1.14	1.20	1.26
θ^i	0.3420	0.7760	1.2010	1.6450	2.0790	2.5130	2.9470	3.3810	3.8150	4.2490

Note:

numeraire unit would be infinitely more preferred relative to the effects of transient policies, and the welfare effects would vanish. The restriction is satisfied for x that takes non-zero values in a finite number of periods. The assumption, however, rules out, for example, stationary consumption flows.

4 Accuracy of the approximation

As we explained in the introduction, one way of thinking about this result, instead of more patient, shorter policies.

In this section, we demonstrate how welfare approximation can be used to compare the impact of policies in practice. We also determine the scope of policies for which the approximation gives accurate predictions. For this, we consider a stylized model of the Polish economy. Why (1) reasonably small (2) integration with EU markets, (3) external EU transfers. We test the approximation accuracy within a context of a once-and-for-all intervention, a stimulus package of the EU in coronavirus response.

The basic framework is defined as follows. Consumers are heterogeneous in two dimensions, namely, productivity and risk aversion (CRRA preferences). We extract the distributions of the respective parameters A^i, θ^i from the available micro-econometric data for Poland. The marginal distributions of the parameters are in Table 3.[10] In the baseline scenario, we assume that risk aversion and productivity are independently distributed,[11] and the joint distribution can be derived as a product of the two marginals. In the appendix, we perform robustness checks in which we allow for non-zero correlations. These variations do not significantly change the quantitative predictions (confirm).

[10] We borrow the the productivity distribution from Bielecki, Makarski, Tyrowicz 2017. For risk aversion, we use micro-data on risk-taking preferences for Poland from Falk et al. (2018)) The survey measures risk attitude as ... we covert these values intro relative risk aversion θ using the following algorithm ... . In appendix X we explain in detailed the construction of Table 3.

[11] To the best of our knowledge there is no available micro-data on the joint distribution of risk aversion and income. GEEP reports that the correlation between the risk preference and (self-reported) math skills is relatively weak. If the latter is a good proxy of income, this suggests a low correlation of income and risk aversion.

We introduce uncertainty to our model via global and idiosyncratic shocks. International markets can be either in a state of expansion (normal state) or a recession. An individual consumer is employed or unemployed. As a result, in each period, a consumer can be in one of the four states: an employed consumer during an expansion ($s = 1$), an unemployed consumer during an expansion ($s = 2$), an employed consumer during a recession ($s = 3$), and an unemployed consumer during a recession ($s = 4$). In the states with normal economy, the price of consumption is $q_{s_t=1,2} = 1$, while during a recessions, it increases to $q_{s_t=3,4} = 1.03$.

We construct an empirical transition matrix for the four states as follows. For the states of the economy ($S^i_{econ} = N, R$) we use the Markov matrix from Krueger, Mitman, Perri (2016)

$$\mathcal{P}_{econ}(S'_{econ}|S_{econ}) = \begin{bmatrix} \pi_{N|N} & \pi_{R|N} \\ \pi_{R|N} & \pi_{R|R} \end{bmatrix} = \begin{bmatrix} 0.9910 & 0.0090 \\ 0.0455 & 0.9545 \end{bmatrix} \tag{6}$$

Apart from increasing the prices of contingent consumption, an economic downturn slowdown elevates the probability with which a consumer becomes unemployed. We derive the transition matrices for the employment status ($S^i_{ind} = E, U\}$) from Polish panel data on the activity in labour market.[12] When the global economy is in a normal state, the transition probabilities are

$$\mathcal{P}^i([S'_{ind}|S_{ind}]|S_{econ} = N)) = \begin{bmatrix} \pi^i_{E|E|N} & \pi^i_{U|E|N} \\ \pi^i_{U|E|N} & \pi^i_{U|U|N} \end{bmatrix} = \begin{bmatrix} 0.9757 & 0.0243 \\ 0.1720 & 0.8280 \end{bmatrix}. \tag{7}$$

During a recession the probabilities become

$$\mathcal{P}^i([S'_{ind}|S_{ind}]|S_{econ} = R)) = \begin{bmatrix} \pi^i_{E|E|R} & \pi^i_{U|E|R} \\ \pi^i_{U|E|R} & \pi^i_{U|U|R} \end{bmatrix} = \begin{bmatrix} 0.9785 & 0.0215 \\ 0.2460 & 0.7540 \end{bmatrix}. \tag{8}$$

A Markov transition matrix for the general process is obtained by element-wise multiplication of an augmented transition matrix for the states of the global economy, and the block matrix consists of the conditional probabilities of retaining and losing a job at the individual level. The combined matrix is given by

$$\mathcal{P}^i(s'|s) = \begin{bmatrix} 0.9669 & 0.0241 & 0.0088 & 0.0002 \\ 0.1705 & 0.8205 & 0.0015 & 0.0075 \\ 0.0445 & 0.0010 & 0.9340 & 0.0205 \\ 0.0112 & 0.0343 & 0.2348 & 0.7197 \end{bmatrix}. \tag{9}$$

The matrix yields unconditional employment rate of 0.8762, which matches the average BAEL rates in Poland. **Can we find condition distribution in recession.**

[12]BAEL - Population Economic Activity Research in Poland

Table 4: Aggregated EV convergence

	number of periods in the finite part of the model				
	1	2	4	8	12
EV($\beta = 0.9924$)	-0.005999	-0.016272	-0.045803	-0.125000	-0.216156
L	-0.006000	-0.016351	-0.046461	-0.129109	-0.227167
[EV($\beta = 0.9924$)/EV(QLA) -1] $\times$100	-0.00005	-0.48131	-1.41562	-3.18210	- 4.84732

In the experiment, we compare various recovery paths from the slowdown triggered by the COVID-19 pandemic. Within each homogenous group of consumers, the initial employment rate is equal to 0.85(?), the empirical value in January 2020. For the employed consumers, initial beliefs are $\pi_0^i = [0, 0, 1, 0]$ while for the unemployed ones they are $\pi_0^i = [0, 0, 0, 1]$. Finally, we assume a quarterly discount factor $\beta = 0.9924$, which corresponds to the annual value of 0.97, typically assumed in the context of the Polish economy.

The factual policy is an economic recovery path without any intervention. In this scenario, in normal times, the productivity of an employed worker, $A^i_{s_t=1}$, is reported in the first row of Table 3. When a recession hits the economy, the productivity is uniformly reduced according to $A^i_{s_t=2} = 0.9614 \times A^i_{s_t=1}$.[13] In the two cases left - the unemployed agent receives a constant unemployment benefit, regardless of the state of the global economy - his productivity is $A_{s_t=3,4} = 0.15$. A counterfactual stimulus package proposed by the EU aims at preserving jobs (albeit at a lower productivity level). We model this by setting $A^{i\prime}_{s_t=3,4} = 0.25 \times A^i_{s_t=1}$. Since the stimulus is financed from the European budget, its costs are omitted in analyses. We consider the policy variants that last $T = 1, 2, 4, 8, 12$ quarters.

Predictions: Table 5 reports aggregated (weighted) EV convergence depending on β and on the number of periods in the finite part taken in the model, where we could clearly see that increasing β or decreasing the number of periods cause even better convergence.

For the limit value of the welfare effects we can unambiguously

5 Discussion

5.1 Identification of surplus

Policy makers *a priori* rarely have access to information regarding consumers preferences. In the companion paper Weretka (2018a), we show that, within a dynamic quasilinear economy,

[13]This drop reflects the reduction in productivity and consequently in wages during economic downturns. The magnitude of the reduction for the Polish economy has been estimated in KMP).

Table 5: Aggregated EV convergence

β	number of periods in the finite part of the model 1	3	6	12
	EV(β)			
0.9500	0.00014859	0.00042460	0.00079221	0.00140196
0.9700	0.00014857	0.00043273	0.00082813	0.00152332
0.9924	0.00014856	0.00044230	0.00087330	0.00169939
QLA	0.00014856	0.00044566	0.00088986	0.00177010
	[EV(β)/EV(QLA) -1] $\times$100			
0.95	0.0221	-4.7262	-10.9744	-20.7965
0.97	0.0078	-2.9015	- 6.9370	-13.9408
0.9924	0.0005	-0.7536	-1.8613	- 3.9935

the social surplus can be easily and accurately non-parametrically recovered from the prices of a relatively small portfolio of securities, realized under a factual policy (technically from a single observation of the equilibrium correspondence). Because the infinite horizon framework considered in this paper converges in terms of equilibrium allocation, prices, and social welfare to some quasilinear limit, our identification result straightforwardly carries over to the canonical consumption-based infinite horizon framework considered in this paper.

References

Aiyagari, S. R. (1994): "Uninsured idiosyncratic risk and aggregate saving," *The Quarterly Journal of Economics*, 109, 659–684.

Aiyagari, S. R., A. Marcet, T. J. Sargent, and J. Seppälä (2002): "Optimal Taxation without State-contingent Debt," *Journal of Political Economy*, 110, 1220–1254.

Barro, R. J. (1979): "On the Determination of the Public Debt," *Journal of Political Economy*, 87, 940–971.

Bewley, T. (1976): "The permanent income hypothesis: A theoretical formulation." Tech. rep., HARVARD UNIV CAMBRIDGE MASS.

Bhandari, A., D. Evans, M. Golosov, and T. J. Sargent (2016): "Fiscal Policy and Debt Management with Incomplete Markets," *The Quarterly Journal of Economics*, 132, 617–663.

Blundell, R., X. Chen, and D. Kristensen (2007): "Semi-nonparametric IV estimation of shape-invariant Engel curves," *Econometrica*, 75, 1613–1669.

Carroll, C. D. (2000): "Requiem for the representative consumer? Aggregate implications of microeconomic consumption behavior," *American Economic Review*, 90, 110–115.

Carvajal, A., M. Rostek, and M. Weretka (2012): "Competition in Financial Innovation," *Econometrica*, 80, 1895–1936.

Falk, A., A. Becker, T. Dohmen, B. Enke, D. Huffman, and U. Sunde (2018): "Global evidence on economic preferences," *The Quarterly Journal of Economics*, 133, 1645–1692.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): "Quantitative macroeconomics with heterogeneous households," *Annu. Rev. Econ.*, 1, 319–354.

Hicks, J. R. (1939): *Value and Capital.*

Lagos, R. and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113, 463–484.

Levine, D. K. and W. R. Zame (2002): "Does market incompleteness matter?" *Econometrica*, 70, 1805–1839.

Lewbel, A. and K. Pendakur (2009): "Tricks with Hicks: The EASI demand system," *American Economic Review*, 99, 827–63.

Lucas, R. E. (1987): *Models of Business Cycles*, vol. 26, Basil Blackwell Oxford.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*, New York: Oxford University Press.

Mortensen, D. T. and C. A. Pissarides (1994): "Job creation and job destruction in the theory of unemployment," *The review of economic studies*, 61, 397–415.

Vives, X. (1987): "Small Income Effects: A Marshallian Theory of Consumer Surplus and Downward Sloping Demand," *The Review of Economic Studies*, 54, 87–103.

Weretka, M. (2018a): "Normative Analyses in Efficient Markets," *Economic Theory (forthcoming).*

——— (2018b): "Ordial Minimum Theorem," *working paper.*

A Appendices

Proof of Theorem 1: The proof of the theorem is structured as follows. In section ...

A.1 A static problem

In this section we use the standard arguments to recast the recursive problem of a consumer from Section 3.1 as a static choice of a consumption flow $c_i = \{c_t^i\}_t$, among consumptions flows adapted with respect to a natural filtration of s

$$X^i \equiv \{c^i | c_t^i > 0 \text{ for all } t \text{ and } U^i(c^i) \in \mathbb{R}\},$$

given endowments and prices of state contingent claims.

First observe that with no disutility of labor and non-satiated preferences, a consumer is going to supply the maximal amount of labor, $l_t^i = 1$. Consequently, for each policy income is effectively determined by endowment flow $e^i = \{e_t^i\}_t$, for each history given by $e_{h_t}^i \equiv (A^i(s_t) + \Delta_{h_t}^{A^i}) f^i(1) > 0$.

In the recursive problem, consider an event followed by history $h_t = \{s_0, s_1, ..., s_t\}$. Using a rollover trading strategy of contingent claims, a consumer can transfer one unit of consumption to this event, paying in terms of consumption in s_0

$$\begin{aligned} Price(h_t) &= \pi_{s_1} q_{s_1}|_{s_0} \times \pi_{s_2} q_{s_2}|_{s_1} \times ... \times \pi_{s_t} q_{s_t}|_{s_{t-1}} \\ &= \pi_{s_1}|_{s_0} \beta \frac{q(s_1) + \Delta_{h_1}^q}{q(s_0) + \Delta_{h_0}^q} \times \pi_{s_2}|_{s_1} \beta \frac{q(s_2) + \Delta_{h_2}^q}{q(s_1) + \Delta_{h_1}^q} \times ... \times \pi_{s_t}|_{s_{t-1}} \beta \frac{q(s_t) + \Delta_{h_t}^q}{q(s_{t-1}) + \Delta_{h_{t-1}}^q} \\ &= \pi_{h_t} \beta^t \frac{q(s_t) + \Delta_{h_t}^q}{q(s_0) + \Delta_{h_0}^q}. \end{aligned}$$

where π_{h_t} is the unconditional probability of event followed by h_t. For all histories one can normalize prices $Price(h_t)$ by factor $1/(q(s_0) + \Delta_{h_0}^q)$. The recursive problem with dynamic trading strategy is equivalent to a static choice of a lifetime consumption plan in period zero. The process of state contingent prices $\zeta = \{\zeta_t\}_t$ for history h_t is given by $\zeta_{h_t} \equiv q(s_t) + \Delta_{h_t}^q > 0$. More specifically, for policy p, and with additional transfer of α units of welfare numeraire x, budget constraint is given by

$$b_p^i(c^i, \beta) \equiv E \sum_{t=0}^{\infty} \beta^t \zeta_t (c_t^i - e_t^i - \alpha x_t) \leq 0.$$

Budget set is a collection of measurable, strictly positive processes that satisfy budget constraint, i.e., $B_p^i \equiv \{c^i | b_p^i(c^i, \beta) \leq 0\}$. The recursive consumer's problem is then equivalent to a static problem

$$\max_{c^i \in B_p^i \cap X^i} E \sum_{t=0}^{\infty} \beta^t u^i(c_t^i). \tag{10}$$

Throughout the appendix we use the static formulation of the problem. We also adopt the following notation: for policy, p, and discount factor β, present value of flow c^i is defined as

$$PV^{\beta,p}(c^i) \equiv E\sum_{t=0}^{\infty}\beta^t\zeta_t c_t^i.$$

For the welfare numeraire and the individual endowment we uses the following compact notation: $v^x \equiv PV^{\beta,p}(x)$ and $v^{e^i} \equiv PV^{\beta,p}(e^i)$, respectively. Observe that under Assumption 2 present value of endowment is an increasing sequence of sums, bounded from above and consequently $v^{e^i} \in \mathbb{R}_{++}$. Similarly, it is straightforward to show that for any policy $p \in \mathcal{P}$ by Assumptions 1-2 there exist scalars $0 < \underline{\zeta} < \overline{\zeta}$ and $0 < \underline{e} < \overline{e}$ such that for all periods t, all histories h_t and all consumers i, one has $\underline{\zeta} < \zeta_{h_t} < \overline{\zeta}$ and $\underline{e} < e^i_{h_t} < \overline{e}$.

A.2 Equivalent variation

We first state a definition of equivalent variation in terms of preferences. Then we reformulate the definition using a utility representation. Consider an abstract problem of a consumer with set of alternatives $X^i \subset \mathbb{R}^N$, where N can be finite of infinite. Let B^i_p be a budget set associated with factual policy p and let $\Psi^i_{p'}$ be the upper countur set of an optimal alternative that is attained under the counterfactual policy p'. Equivalent variation is a minimal transfer of welfare numeraire $x \in X^i$, shifting B^i_p that allows to attain a bundle in $\Psi^i_{p'}$. Formally, the equivalent variation is a solution to the following problem:

$$EV^i_{p,p'} \equiv \min_{z\in X^i,\alpha\in\mathbb{R}} \alpha, \tag{11}$$

subject to $z \in \Psi^i_{p'}$ and $z \in B^i_p + \alpha x$. We say that equivalent variation is attained at $\bar{z} \in X^i$ if tuple $(\bar{z}, EV^i_{p,p'})$ is a solution to program (11). Note that equivalent variation is defined in real terms (upper countur set and budget set) and hence it is not affect by normalization of utility or prices.

The consumer preferences considered in this paper admit a strictly monotone, continuous utility representation and the budget sets are determined by linear inequality constraints. In this instance equivalent variation can be simplified as follows. Define value function

$$V^i(p,\alpha) = \max_{c^i\in X^i} E\sum_{t=0}^{\infty}\beta^t u^i(c^i_t) \text{ s.t. } E\sum_{t=0}^{\infty}\beta^t\zeta_t c^i_t \leq E\sum_{t=0}^{\infty}\beta^t\zeta_t(e^i_t + \alpha x_t) = v^{e^i} + \alpha v^x. \tag{12}$$

For policies p, p' equivalent variation is given by as a solution to the following equation

$$V^i(p, EV^i_{p,p'}) = V^i(p', 0). \tag{13}$$

Next, we prove Proposition 1 by showing that equivalent variation is well defined for any pair of policies p, p' that satisfy our assumptions.

Proof of Proposition 1:

Step 1. In this step we characterize properties of function $V^i(p,\cdot)$. Derivative $u^{i\prime}:\mathbb{R}_{++}\to\mathbb{R}_{++}$ is a continuous and strictly decreasing bijection, therefore its inverse $u^{i\prime-1}$ is well-defined, is continuous and strictly decreasing. The solution to program (12), if it exists, satisfies the first order conditions in the standard Lagrangian problem. For an event identified by history h_t the first order condition with respect to consumption, $\beta^t\pi_{h_t}u^{i\prime}\left(c^i_{h_t}\right)=\pi_{h_t}\beta^t\zeta_{h_t}\lambda^i$, can be equivalently reformulated as $c^i_{h_t}=u^{i\prime-1}(\lambda^i\zeta_{h_t})$. Replacing the latter conditions in the budget constraint implicitly defines scalar λ^i,

$$E\sum_{t=0}^{\infty}\beta^t\zeta_t u^{i\prime-1}(\lambda^i\zeta_t)=v^{e^i}+\alpha v^x. \tag{14}$$

The limit sum on the left hand side is well defined for any $\lambda^i>0$ and $\beta\in(0,1)$, since the sequence is increasing in t and it is bounded from above by $E\sum_{t=0}^{\infty}\beta^t\overline{\zeta}u^{i\prime-1}(\lambda^i\underline{\zeta})=\frac{\overline{\zeta}u^{i\prime-1}(\lambda^i\underline{\zeta})}{1-\beta}<\infty$. Moreover the limit sum is a strictly decreasing continuous bijection in λ^i, mapping $\mathbb{R}_{++}\to\mathbb{R}_{++}$. Consequently equation (14) has a unique solution if and only if the constant on the right hand side is strictly positive, or, in terms of parameter, $\alpha>-v^{e^i}/v^x$. Given strictly convex separable preferences, solution $\lambda^i>0$, along with stochastic consumption flow $c^i=\{c^i_t\}_{t=0}^{\infty}$ defined as $c^i_t=u^{i\prime-1}(\lambda^i\zeta_t)$ satisfy necessary and sufficient conditions for optimality. For policy p, solution is uniformly bounded from above by $\bar{u}^{i\prime-1}(\lambda^i\underline{\zeta})<\infty$ and from below by $u^{i\prime-1}(\lambda^i\overline{\zeta})>0$. As a result, for any $\alpha>-v^{e^i}/v^x$ limit $V^i(p,\alpha)=E\sum_{t=0}^{\infty}\beta^t u^i(u^{i\prime-1}(\lambda^i\zeta_t))$ exists. Moreover, since $V^i(p,\cdot)$ is a sum of continuous bijections, and itself it is a continuous bijection mapping $(-v^{e^i}/v^x,\infty)\to(\inf_{c^i}u^i(c^i)/(1-\beta),\infty)$. Importantly, the target set is independent of a particular policy.

Step 2. By the previous step $V^i(p',0)\in(\inf_{c^i}u^i(c^i)/(1-\beta),\infty)$. Since $V^i(p,\cdot)$ is a bijection, its inverse exists and is a bijection as well. It follows that equation (13) has the unique solution, given by $EV^i_{p,p'}=V^{i,-1}(p,V^i(p',0))\in(-v^{e^i}/v^x,\infty)$. □

A.3 Temporary policies.

Fix $\tau<\infty$. In this and the next section we characterize equivalent variation for a set of temporary policies $\mathcal{P}^\tau\subset\mathcal{P}$, restricted to ones whose effects vanish in finite time τ, i.e., for which $\Delta^q_{h_t}=0$ and $\Delta^{A^i}_{h_t}=0$ for all $t>\tau$, h_t and i. We extend our characterization to all polices in $\mathcal{P}$ in Section A.5.

We first introduce a reduced from of the static problem from Section A.1. For any $w^i\in\mathbb{R}$, consider the following problem

$$v^i(w^i)\equiv\max_{\{c^i_t\}_{t=\tau+1}^{\infty}}E\sum_{t=\tau+1}^{\infty}\beta^t u^i(c^i_t),\ \text{s.t. } E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t c^i_t\le E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t e^i_t+w^i \tag{15}$$

Since by assumption prices and endowments after τ are the same for all considered policies, function $v^i(\cdot)$ is independent of a particular policy. The set of consumption flows that satisfy budget constraint is empty, whenever borrowing constraint fails, i.e., $w^i \leq \underline{w}^i \equiv -E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t e_t^i$. The next lemma shows the converse: the domain is non-empty, and the solution is uniquely defined whenever the borrowing constraint is satisfied.

Lemma 1. *Program (15) has a unique solution if and only if* $w^i > \underline{w}^i$*.*

Proof of Lemma 1:

We essentially follow the steps of the proof of Proposition 1. By Inada assumption constraints $c_t^i > 0$ are not binding for $t > \tau$ and the solution to the program is given by the first order conditions in the Lagrangian problem. For each t and history h_t optimal consumptions satisfies $c_{h_t}^i = u^{i\prime -1}(\lambda^i\zeta_{h_t})$ where shadow price λ^i can be derived from the budget constraint (multiplied by constant $1-\beta$):

$$\eta\left(\beta,\lambda^i\right) \equiv (1-\beta)\,E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t u^{i\prime -1}(\lambda^i\zeta_t) = (1-\beta)\,w^i + (1-\beta)\,E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t e_t^i. \tag{16}$$

For any fixed $\lambda^i > 0$, the left hand side is a limit of an increasing sequence bounded from above, and, hence it is well defined and finite. Function $\eta(\beta,\cdot)$, is a strictly decreasing bijection, mapping $\mathbb{R}_{++} \to \mathbb{R}_{++}$. The right-hand side of the equality gives a real number. Equation (16) has a solution if and only if the constant is strictly positive, or $w^i > \underline{w}^i$. Given strictly convex separable preferences solution $\lambda^i > 0$, along with random consumption flow $c^i = \{c_t^i\}_{t=\tau+1}^{\infty}$ such that $c_t^i = u^{i\prime -1}(\lambda^i\zeta_t)$ satisfy necessary and sufficient conditions of optimality. $\square$

For the set of policies truncated to the first τ periods the reduced-form problem consists of three elements: consumption space, preferences and budget set correspondence defined as follows. Consider consumption flows $c^i = (w^i, \{c_t^i\}_{t=0}^{\tau})$ where $\{c_t^i\}_{t=0}^{\tau}$ is a stochastic process that satisfies the respective measurability conditions with respect to $\{s\}_{t=0}^{\tau}$. In the reduced-form model consumption space is $\tilde{X}^i \equiv \{c^i | w^i > \underline{w}^i \text{ and } c_t^i > 0 \text{ for } t = 0,1,...,\tau\}$. Reduced-form preferences, over consumption flows in the reduced from are represented by utility function

$$\tilde{U}^i(c^i) \equiv v^i(w^i) + E\sum_{t=0}^{\tau}\beta^t u^i(c_t^i). \tag{17}$$

Finally, for policy p and a monetary transfer αv^x, in the reduced form problem budget set $\tilde{B}_p^i$ is derived from the constraint

$$\tilde{b}_p^i\left(c^i,\beta\right) \equiv w^i + \alpha v^x + E\sum_{t=0}^{\tau}\beta^t\zeta_t(c_t^i - e_t^i) \leq 0. \tag{18}$$

By $\tilde{EV}^i_{p,p'}$ we denote an equivalent variation in the reduced-form, expressed in monetary units w^i, or $\tilde{x} = (1, 0, ..., 0)$. We next demonstrate the sufficiency of the reduced-form problem for equivalent variation in the infinite horizon model.

Lemma 2. *Fix $\beta \in (0, 1)$ and welfare numeraire x for which $v^x < \infty$. Equivalent variation in the infinite horizon problem is well defined if and only if equivalent variation is well defined in the reduced form. Moreover, the indices are related accordingly:*

$$EV^i_{p,p'} = \frac{\tilde{EV}^i_{p,p'}}{v^x}.$$

Proof of Lemma 2:

We demonstrate the lemma in three steps. We show the equivalence of the two representations of the problem in terms of budget sets (Step 1), optimal choices (Step 2), and welfare index (Step 3).

For a stochastic process $c^i = \{c^i_t\}^\infty_{t=0} \in X^i$ in the infinite horizon problem (henceforth referred to as IH), define a corresponding reduction $c^{i-} \equiv (w^i_{c^{i-}}, \{c^{i-}_t\}^\tau_{t=0}) \in \tilde{X}^i$ as follows: $w^i_{c^{i-}} \equiv E\sum^\infty_{t=\tau+1} \beta^t\zeta_t(c^i_t - e^i_t)$ is the value of consumption in periods after τ and $c^{i-}_t = c^i_t$ for $t = 0, 1, ..., \tau$. For a process $c^i = (w^i, \{c^i_t\}^\tau_{t=0}) \in \tilde{X}^i$ in the reduced form (RF) define its extension $c^{i+} \equiv \{c^{i+}_t\}^\infty_{t=0} \in X^i$ as $c^{i+}_t \equiv c^i_t$ for $t = 0, 1, \ldots, \tau$ while $\{c^{i+}_t\}^\infty_{t=\tau+1}$ is a solution to Program (15) given w^i.

Step 1. We first demonstrate the equivalence of the two representations in terms of budget sets, shifted by vector αx.

Claim 1. *Suppose consumption flow in IH satisfies $c^i \in X^i \cap (B^i_p + \alpha x)$. Reduction c^{i-} is well-defined in RF and satisfies $c^{i-} \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x\tilde{x})$. Conversely, for $c^i \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x\tilde{x})$ in RF, its extension, c^{i+}, is well-defined and satisfies $c^{i+} \in X^i \cap (B^i_p + \alpha x)$.*

Proof of Claim 1:

Fix $c^i \in X^i \cap (B^i_p + \alpha x)$ in IH. Since $c^i \in X^i$, for all $t = 0, \ldots, \tau$ one has $c^i_t > 0$ and $w^i_{c^{i-}} \equiv E\sum^\infty_{t=\tau+1} \beta^t\zeta_t(c^i_t - e^i_t) > -E\sum^\infty_{t=\tau+1} \beta^t\zeta_t e^i_t = \underline{w}^i$. Moreover, $c^i \in B^i_p + \alpha x$ and hence $E\sum^\infty_{t=0} \beta^t\zeta_t(c^i_t - e^i_t - \alpha x_t) \leq 0$. This implies

$$w^i_{c^{i-}} \equiv E\sum^\infty_{t=\tau+1} \beta^t\zeta_t(c^i_t - e^i_t) < E\sum^\tau_{t=0} \beta^t\zeta_t e^i_t + \alpha v^x < \infty.$$

Consequently $\underline{w}^i < w^i_{c^{i-}} < \infty$, and reduction $c^{i-} \in \tilde{X}^i$ is well-defined. Moreover,

$$w^i_{c^{i-}} - \alpha v^x + E\sum^\tau_{t=0} \beta^t\zeta_t(c^{i-}_t - e^i_t) = E\sum^\infty_{t=0} \beta^t\zeta_t(c^i_t - \alpha x_t - e^i_t) \leq 0$$

where the last inequality holds since $c^i \in B^i_p + \alpha x$. It follows that $c^{i-} \in \tilde{B}^i_p + \alpha v^x\tilde{x}$.

Next fix $c^i \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x \tilde{x})$ in RF. Since $c^i \in \tilde{X}^i$, one has $c^i_t > 0$ for $t = 0, \ldots, \tau$. Moreover, $w^i > \underline{w}^i$, and by Lemma 1 solution to Program (15) exists $\{c^{i+}_t\}^{\infty}_{t=\tau+1}$ that are strictly positive. It follows that extension $c^{i+} \in X^i$ is well-defined. Moreover,

$$\begin{aligned} E\sum_{t=0}^{\infty}\beta^t\zeta_t(c^{i+}_t - e^i_t - \alpha x_t) &= E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t(c^{i+}_t - e^i_t) - \alpha E\sum_{t=0}^{\infty}\beta^t\zeta_t x_t + E\sum_{t=0}^{\tau}\beta^t\zeta_t(c^i_t - e^i_t) \\ &\leq w^i - \alpha v^x + E\sum_{t=0}^{\tau}\beta^t\zeta_t(c^i_t - e^i_t) \leq 0, \end{aligned}$$

where the last inequality holds by the fact that $c^i \in \tilde{B}^i_p + \alpha v^x \tilde{x}$. Therefore, the extension of the consumption profile satisfies $c^{i+} \in B^i_p + \alpha x$. □

Step 2. We next demonstrate the equivalence of the two formulations in terms of optimal choices.

Claim 2. *Suppose c^i is optimal in IH on set $B^i_p \cap X^i$. Then reduction c^{i-} is well-defined and optimal on $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. Conversely, if in RF c^i is optimal on $\tilde{B}^i_p \cap \tilde{X}^i$ then c^{i+} is well-defined and optimal on $B^i_p \cap X^i$ in IH.*

Proof of Claim 2:

Let c^i be optimal on set $B^i_p \cap X^i$ in IH. Since $c^i \in B^i_p \cap X^i$, by Step 1 reduction is well-defined and $c^{i-} \in \tilde{B}^i_p \cap \tilde{X}^i$. Suppose c^{i-} is not optimal on this set. This implies that there exists $y^i \in \tilde{B}^i_p \cap \tilde{X}^i$ strictly preferred to c^{i-}. By Step 1 extension y^{i+} is well-defined and satisfies $y^{i+} \in B^i_p \cap X^i$. Finally,

$$\begin{aligned} U^i(y^{i+}) &= \sum_{t=0}^{\infty}\beta^t u^i(y^{i+}_t) = v^i(w^i_{y^i}) + E\sum_{t=0}^{\tau}\beta^t u^i(y^i_t) \\ &> v^i(w^i_{c^{i-}}) + E\sum_{t=0}^{\tau}\beta^t u^i(c^{i-}_t) \geq E\sum_{t=0}^{\infty}\beta^t u^i(c^i_t) = U^i(c^i), \end{aligned}$$

where $w^i_{y^i}$ and $w^i_{c^{i-}}$ are the first components of vectors y^i and c^{i-}, respectively, and the strict inequality holds by the fact that y^i is strictly preferred to c^{i-} in RF. This contradicts optimality of c^i on $B^i_p \cap X^i$.

Let c^i be optimal on set $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. Since $c^i \in \tilde{B}^i_p \cap \tilde{X}^i$, by Step 1 extension is well-defined and $c^{i+} \in B^i_p \cap X^i$. Suppose c^{i+} is not optimal on this set. It follows that there exists $y^i \in B^i_p \cap X^i$ strictly preferred to c^{i+}. By Step 1 reduction y^{i-} is well-defined and satisfies $y^{i-} \in \tilde{B}^i_p \cap \tilde{X}^i$. Finally,

$$\begin{aligned} \tilde{U}^i(y^{i-}) &= v^i(w^i_{y^{i-}}) + E\sum_{t=0}^{\tau}\beta^t u^i(y^i_t) \geq \sum_{t=0}^{\infty}\beta^t u^i(y^i_t) \\ &> E\sum_{t=0}^{\infty}\beta^t u^i(c^{i+}_t) = v^i(w^i_{c^i}) + E\sum_{t=0}^{\tau}\beta^t u^i(c^i_{\tau+1}c^i_t) = \tilde{U}^i\left(c^i\right) \end{aligned}$$

where the strict inequality holds by the fact that y^i is strictly preferred to c^{i+} in IH. This contradicts the optimality of c^i on $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. $\square$

Step 3. Finally, we demonstrate the equivalence of the frameworks in terms of equivalent variation.

Claim 3. *Suppose equivalent variation $EV^i_{p,p'}$ in IH is attained on z^i. Then in RF equivalent variation is attained on z^{i-} and satisfies $\tilde{EV}^i_{p,p'} = v^x \times EV^i_{p,p'}$. Conversely, if in RF equivalent variation $\tilde{EV}^i_{p,p'}$ is attained on z^i, then in IH equivalent variation is attained on z^{i+} and satisfies $EV^i_{p,p'}(\beta) = \tilde{EV}^i_{p,p'}/v^x$.*

Proof of Claim 3:

Suppose in IH equivalent variation $\alpha \equiv EV^i_{p,p'}$ is attained on z^i. By definition of equivalent variation $z^i \in B^i_p + \alpha x$ and $z^i \in \Psi^i_{p'} \subset X^i$. By Step 1 reduction $z^{i-} \in \tilde{X}^i$ is well-defined and it satisfies $z^{i-} \in \tilde{B}^i_p + \alpha v^x \tilde{x}$. Let $o^i \in \Psi_{p'}$ be an optimal choice in IH under policy p'. By definition of upper countur set, $U^i(z^i) \geq U^i(o^i)$. By Step 2 reduction o^{i-} is well-defined and optimal under p' in RF. Then

$$\begin{aligned} \tilde{U}^i(z^{i-}) &= v^i(w^i_{z^{i-}}) + E\sum_{t=0}^{\tau} \beta^t u^i(z^{i-}_t) \geq E\sum_{t=0}^{\infty} \beta^t u^i(z^i_t) \\ &\geq E\sum_{t=0}^{\infty} \beta^t u^i(o^i_t) = v^i(w^i_{o^{i-}}) + E\sum_{t=0}^{\tau} \beta^t u^i(o^{i-}_t) = \tilde{U}^i(o^{i-}). \end{aligned}$$

Consequently $z^{i-} \in \tilde{\Psi}^i_{p'}$ in RF. It follows that $(z^{i-}, \alpha v^x)$ satisfy constraints of Program (11) within RF. Suppose that $(z^{i-}, \alpha v^x)$ does not solve this program. It follows that there exists $z^{i\prime} \in \tilde{\Psi}^i_{p'} \subset \tilde{X}^i$ in RF satisfying $z^{i\prime} \in \tilde{B}^i_p + \alpha' v^x \tilde{x}$ for some $\alpha' < \alpha$. By Step 1, extension to IH of $z^{i\prime+} \in X^i$ is well-defined and satisfies $z^{i\prime+} \in B^i_p + \alpha' x$. Next, let $o^i \in \tilde{\Psi}_{p'}$ be an optimal choice in RF under policy p'. Then

$$\begin{aligned} U^i(z^{i\prime+}) &= \sum_{t=0}^{\infty} \beta^t u^i(z^{i\prime+}_t) = v^i(w^i_{z^{i\prime}}) + E\sum_{t=0}^{\tau} \beta^t u^i(z^{i\prime}_t) \\ &\geq v^i(w^i_{o^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(o^i_t) = \sum_{t=0}^{\infty} \beta^t u^i(o^{i+}_t) = U^i(o^{i+}), \end{aligned}$$

and hence $z^{i\prime+} \in \Psi^i_{p'}$ in the IH problem. Thus $(z^{i\prime+}, \alpha')$ satisfies constraints of Program (11) in IH and gives a smaller value, contradicting that (z^i, α) is a solution to a minimization problem. It follows that $\tilde{EV}^i_{p,p'} = v^x \times EV^i_{p,p'}$.

Suppose equivalent variation $\tilde{EV}^i_{p,p'}$ is attained on z^i in RF and let $\alpha \equiv \tilde{EV}^i_{p,p'}/v^x$. By the definition of equivalent variation, $z^i \in \tilde{B}^i_p + \alpha v^x \tilde{x}$ and $z^i \in \tilde{\Psi}^i_{p'} \subset \tilde{X}^i$. By Lemma 1 extension $z^{i+} \in X^i$ is well-defined and satisfies $z^{i+} \in B^i_{p'} + \alpha x$. Let o^i be an optimal choice in RF under policy p'. By Lemma 2, extension o^{i+} is well-defined and optimal in IH under p' as well. Then

$$\begin{aligned} U^i(z^{i+}) &= E\sum_{t=0}^{\infty}\beta^t u^i(z_t^{i+}) = v^i(w_{z^i}^i) + E\sum_{t=0}^{\tau}\beta^t u^i(z_t^i) \\ &\geq v^i(w_{o^i}^i) + E\sum_{t=0}^{\tau}\beta^t u^i(o_t^i) \geq E\sum_{t=0}^{\infty}\beta^t u^i(o_t^{i+}) = U^i(o^{i+}) \end{aligned}$$

which implies that $z^{i+} \in \Psi_{p'}^i$ in IH. It follows that (z^{i+}, α) satisfy constraints of Program (11) in IH. Suppose that (z^{i+}, α) is not a solution to the problem in IH. It follows that there exists $z^{i\prime} \in \Psi_{p'}^i \subset X^i$ satisfying $z^{i\prime} \in B_{p'}^i + \alpha' x$ for some $\alpha' < \alpha$. By Lemma 1, reduction of $z^{i\prime}$ to RF, $z^{i\prime-} \in \tilde{X}^i$ is well-defined and satisfies $z^{i\prime-} \in \tilde{B}_{p'}^i + \alpha' v^x \tilde{x}$. Let o^i be an optimal choice in IH under policy p'. By Lemma 2, reduction o^{i-} is well-defined and optimal in RF under p' as well.

Moreover,

$$\begin{aligned} \tilde{U}^i\left(z^{i\prime-}\right) &= v^i(w_{z^{i\prime-}}^i) + E\sum_{t=0}^{\tau}\beta^t u^i(z_t^{i\prime}) = E\sum_{t=0}^{\infty}\beta^t u^i(z_t^{i\prime}) \\ &\geq E\sum_{t=0}^{\infty}\beta^t u^i(o_t^{i+}) = v^i(w_{o^i}^i) + E\sum_{t=0}^{\tau}\beta^t u^i(o_t^i) = \tilde{U}^i(o^{i-}), \end{aligned}$$

hence $z^{i\prime-} \in \tilde{\Psi}_{p'}^i$ in RF. Thus $(z^{i\prime-}, \alpha' v^x)$ satisfied constraints of Program (11) in RF and attains a smaller value than $\tilde{EV}_{p,p'}^i = \alpha v^x$, contradicting that $z^i, \tilde{EV}_{p,p'}^i$ is a solution in the reduced form problem. It follows that $EV_{p,p'}^i = \tilde{EV}_{p,p'}^i / v^x$. □

The three claims jointly imply the result in Lemma 2. □

Corollary 1. *Fix $\beta \in (0,1)$. For any policies $p, p' \in \mathcal{P}^\tau$ equivalent variation in the reduced form, $\tilde{EV}_{p,p'}^i$ problem is well-defined.*

Proof of Corollary 1: In the infinite horizon problem pick x that pays one unit in $\tau+1$ and zero otherwise. Note that $v^x = E\beta^{\tau+1}\zeta_{\tau+1} \in \mathbb{R}_{++}$. By Proposition 1 equivalent variation is well defined in IH. Then by Lemma 2 equivalent variation in the reduced form exists and is equal to $\tilde{EV}_{p,p'}^i = EV_{p,p'}^i \times v^x$.

A.4 Ordinal convergence

In this section we argue that the reduced form preferences (locally, on a compact box) continuously transform into the quasilinear limits as consumers become patient. As a result the indifference curves become aligned with the quasilinear ones. In the reduced form model consider arbitrary compact box that gives a collection of measurable flows $c^i = (w^i, \{c^i_t\}^{\tau}_{t=0})$ defied as

$$\tilde{X}^{i,b} = \{c^i_t | \underline{w}_b \leq w^i < \overline{w}_b \text{ and } \underline{c}_b \leq c^i_t \leq \underline{c}_b \text{ for all } t = 0, ..., \tau\},$$

where finite bounds satisfy $\underline{w}_b < \overline{w}_b$ and $0 < \underline{c}_b < \underline{c}_b$.

For $\underline{\beta} \in (0,1)$ define weakly-better-than-c^i correspondence, mapping $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \rightrightarrows X^{i,b}$ as follows:

$$\Psi^i(c^i, \beta) \equiv \{c^{i\prime} \in X^{i,b} | c^{i\prime} \succsim^i_\beta c^i\},$$

where preferences $\succsim^i_\beta$, for all $\beta < 1$ are represented by utility function (17). For $\beta = 1$ preferences $\succsim^i_1$ are given by the quasilinear utility

$$\tilde{U}^{i,Q}(c^i) = \bar{\lambda}^i w^i + E \sum_{t=0}^{\tau} u^i(c^i_t), \tag{19}$$

where $\bar{\lambda}^i > 0$ solves equality (2).

Observe that for some values of a discount factors, one might have, $\underline{w}_b < \underline{w}^i$ and the reduced-form preferences, and hence, the weakly-better-than-c^i correspondence might not be well-defined on the entire domain. The next lemma shows that for sufficiently patient consumers, the correspondence is well-defined. Moreover, the result demonstrates that the preferences continuously transforms into the quasilinear ones.

Lemma 3. *There exists threshold $\underline{\beta} \in (0,1)$, such that weakly-better-than-c^i correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \rightarrow X^{i,b}$ is well-defined and continuous on its domain (including at $\beta = 1$).*

Proof of Lemma 3:

For considered policies fundamentals are not altered after period τ and so prices and endowments follow the underlying Markov process. In particular, for any $h_t = \{s_0, s_1, ..., s_t\}$ contingent prices and endowments are determined by a realization of a state in period t, i.e., $\zeta_{h_t} \equiv q(s_t)$ and $e^i_{h_t} = A^i(s_t) f^i(1)$. As a result, for all histories with the same last state are equivalent in terms of contingent prices and endowments. Let $\pi(t, s)$ be the unconditional probability of all histories h_t for which the realization of the Markov process in period t is $s_t = s$ (alternatively unconditional probability of state $s_t = s$ in period t). For the considered Markov chain, a transition matrix is diagonalizable with S independent eigenvectors and real eigenvalues. The largest eigenvalue is equal to one, while other, possibly repeated, eigenvalues $m = 2, 3, ..., S$ satisfy $|r_m| < 1$. It follows that the unconditional probability of s at t can be written as $\pi(t, s) = \bar{\pi}_s + \sum_{m=2}^{S} \gamma_m (r_m)^t v_{m,s}$ where $\bar{\pi}_s$ denotes the stationary probability of state s, derived from the eigenvector with the largest eigenvalue, $v_{m,s}$ is the s element of an eigenvector corresponding to r_m and γ_m is a constant that expresses the initial distribution in terms of eigenvector basis.

Step 1. We first show that the borrowing constrain is not binding on the box, for sufficiently high discount factor. Consider the bound in the the borrowing constraint

$$\begin{aligned}
\underline{w}^i &\equiv -E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t e_t^i = -\sum_{t\geq\tau+1}\beta^t\sum_{s=1}^{S}\pi(t,s)q(s)A^i(s)f^i(1) \\
&= = -\sum_{t\geq\tau+1}\beta^t\sum_{s=1}^{S}[\bar{\pi}_s+\sum_{m=2}^{S}\gamma_m\left(r_m\right)^t v_{m,s}]q(s)A^i(s)f^i(1) \\
&= -\sum_{t\geq\tau+1}\beta^t E(\bar{q}\bar{A}^i f^i(1)) - \sum_{m=2}^{S}\sum_{s=1}^{S}\gamma_m v_{m,s}q(s)A^i(s)f^i(1)\sum_{t\geq\tau+1}\left(r_m\beta\right)^t \\
&= -\frac{\beta^{\tau+1}}{(1-\beta)}E(\bar{q}\bar{A}^i f^i(1)) - \sum_{m=2}^{S} r_m^{\tau+1}\gamma_m\beta^{\tau+1}\frac{1}{1-r_m\beta}\sum_{s=1}^{S}v_{m,s}q(s)A^i(s)f^i(1).
\end{aligned}$$

By assumption $A^i(s)f^i(1)>0$ and $q(s)>0$, for any s therefore $E(\bar{q}\bar{A}^i f^i(1))>0$ and the first term in the equation converges to $-\infty$ as $\beta\to 1$. Since other eigenvalues are strictly smaller than one, one has $1/\left(1-r_m\beta\right)\to 1/\left(1-r_m\right)<\infty$ and therefore the second term converges to a finite limit. It follows that $\lim_{\beta\to 1}\underline{w}^i=-\infty$ and there exists $\beta^w<1$ such that for all $\beta\in[\beta^w,1)$ the borrowing constraint is satisfied for all $c^i\in X^{i,b}$ and correspondence $\Psi^i: X^{i,b}\times[\beta^w,1]\rightrightarrows X^{i,b}$ is well defined. In the next two steps we demonstrate that the correspondence is continuous.

Step 2. In this step we give an auxiliary result in which we characterize the slope of the value function $v^i(\cdot)$. In terms of eigenvalues of the transition matrix, function $\eta\left(\beta,\lambda^i\right)$ from (16) can be written as

$$\begin{aligned}
\eta\left(\beta,\lambda^i\right) &\equiv (1-\beta)E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t u^{i\prime-1}(\lambda^i\zeta_t) = (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^t\sum_{s=1}^{S}\pi(t,s)q(s)u^{i\prime-1}(\lambda^i q(s)) \\
&= (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^t\sum_{s=1}^{S}[\bar{\pi}_s+\sum_{m=2}^{S}\gamma_m\left(r_m\right)^t v_{m,s}]q(s)u^{i\prime-1}(\lambda^i q(s)) \\
&= (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^t E[\bar{q}u^{i\prime-1}(\bar{q}\lambda^i)] + (1-\beta)\sum_{m=2}^{S}\sum_{s=1}^{S}\gamma_m v_{m,s}q(s)u^{i\prime-1}(\lambda^i q(s))\sum_{t=\tau+1}\left(r_m\beta\right)^t \\
&= \beta^{\tau+1}E[\bar{q}u^{i\prime-1}(\bar{q}\lambda^i)] + \sum_{m=2}^{S}\omega_m\sum_{s=1}^{S}v_{m,s}q(s)u^{i\prime-1}(\lambda^i q(s)),
\end{aligned}$$

where corresponding weights ω_m are given by

$$\omega_m \equiv \gamma_m(r_m\beta)^{\tau+1}\frac{1-\beta}{1-r_m\beta}.$$

Since $|r_m| < 1$, for $m = 2, ..., S$ the weights are finite in a neighborhood of $\beta = 1$. Therefore, the weights, as well as function $\eta(\beta, \lambda)$ itself, are well-defined and differentiable with respect to β in the neighborhood of $\beta = 1$. Similarly, for arbitrary value $w^i \in \mathbb{R}$ the constant on the right hand side of equality (16) can be written as

$$(1-\beta) w^i + \beta^{\tau+1} E(\bar{q}\bar{A}^i f^i(1)) + \sum_{m=2}^{S} \omega_m \sum_{s=1}^{S} v_{m,s} q(s) A^i(s) f^i(1).$$

For $\beta = 1$ condition (16) reduces to $E(\bar{q} u^{i\prime -1}(\bar{q}\lambda^i)) = E(\bar{q}\bar{A}^i f^i(1))$ and is independent of initial wealth. By the arguments analogous to the ones in Lemma 1, this equation has unique solution denoted by $\bar{\lambda}^i$. Moreover, since $u^{i\prime -1}(\cdot)$ is strictly decreasing, the derivative $\partial\eta(1, \bar{\lambda}^i)/\partial\lambda^i = E(\bar{q}^2 u^{i\prime -1}(\bar{q}\bar{\lambda}^i)) < 0$ is non-zero. By the implicit function theorem there exists threshold $\beta^{w^i} < 1$, a neighborhood of $\bar{\lambda}^i$, denoted by V and a continuous bijection $\lambda^{w^i} : [\beta^{w^i}, 1] \to V$ such that $\lambda^{w^i}(\beta)$ is a unique solution to equation (16) for each $\beta \in [\beta^{w^i}, 1]$. Note that by continuity of this function $\lim_{\beta\to 1} \lambda^{w^i}(\beta) = \lambda^{w^i}(1) = \bar{\lambda}^i$ for arbitrary value w^i, i.e., the family of bijections $\lambda^{w^i}(\cdot)$ for various w^i has the same limit.

Step 3. Let $\beta^0, \beta^{\underline{w}_b}, \beta^{\overline{w}_b}$ be the thresholds from Step 2, derived for particular values of wealth equal to $0, \underline{w}_b$ and $\overline{w}_b$ respectively (recall that $\underline{w}_b$ and $\overline{w}_b$ are the bounds on wealth that define box $X^{i,b}$). Let functions $\lambda^0(\cdot)$, $\lambda^{\underline{w}_b}(\cdot)$ and $\lambda^{\overline{w}_b}(\cdot)$ be the corresponding bijections. Finally define $\underline{\beta} \equiv \max\{\beta^0, \beta^{\underline{w}_b}, \beta^{\overline{w}_b}, \beta^w\} \in (0, 1)$ where the last element is defined in Step 1.

We next define a monotonic transformation of the reduced-form utility function, that maps $\tilde{\tilde{U}}^i : X^{i,b} \times [\underline{\beta}, 1] \to \mathbb{R}$ as follows. For $\beta \in [\underline{\beta}, 1)$ let function $\tilde{\tilde{U}}^i(c^i, \beta) \equiv \tilde{U}^i(c^i) - v^i(0)$ where the latter utility function is defined in (17). For $\beta = 1$ function is given by $\tilde{\tilde{U}}^i(c^i, 1) \equiv \tilde{U}^{i,Q}$. Note that preferences represented by function $\tilde{\tilde{U}}^i$ coincide with the ones represented by $\tilde{U}^i$ and hence the function defines correspondence Ψ^i.

We now show that the representation $\tilde{\tilde{U}}^i(\cdot, \cdot)$ is jointly continuous. Clearly, $\tilde{\tilde{U}}^i(c^i, \beta)$ is jointly continuous for all $c^i, \beta \in X^{i,b} \times [\underline{\beta}, 1)$ by the standard maximum theorem. Therefore it suffices to verify joint continuity for the elements in the box for which $\beta = 1$. Consider an arbitrary sequence $\{c^{i,h}, \beta^h\}_{h=1}^{\infty} \subset X^{i,b} \times [\underline{\beta}, 1]$ such that $c^{i,h}, \beta^h \to \bar{c}, 1 \in X^{i,b} \times [\underline{\beta}, 1]$. By the envelope theorem, the derivative of the value function is given by the Lagrangian multiplier $\partial v^i(0)/\partial w^i = \lambda^0(\beta)$. Difference $v^i(w^i) - v^i(0)$ is strictly concave and it attains zero at $w^i = 0$. Hence, for any element of the sequence $h = 1, 2, ...$ utility function is bounded from above by

$$\tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \leq \max[\lambda^0(\beta^h) w^{i,h}; \bar{\lambda}^i w^{i,h}] + E \sum_{t=0}^{\tau} (\beta^h)^t u^i(c_t^{i,h}). \tag{20}$$

For all $\beta \in [\underline{\beta}, 1]$ function $\lambda^0(\beta)$, is well-defined and continuous and hence $\lim_{h\to\infty} \lambda^0(\beta^h) = \lambda^0(\lim_{h\to\infty} \beta^h) = \lambda^0(1) = \bar{\lambda}^i$. It follows that both elements of the max function have the same limit and $\lim_{h\to\infty} \tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \leq \bar{\lambda}^i \bar{w} + E \sum_{t=0}^{\tau} u^i(\bar{c}_t^i) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$.

By strict concavity of $v^i(\cdot)$ for all values of wealth $w^i \in [\underline{w}_b, 0]$ value function satisfies $v^i(w^i) - v^i(0) \geq \lambda^{\underline{w}_b}(\beta^h)w^i$, while for all $w^i \in [0, w_b]$ one has $v^i(w^i) - v^i(0) \geq \lambda^{\overline{w}_b}(\beta^h)w^i$ and hence

$$\tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \geq \min[\lambda^{\underline{w}_b}(\beta^h)w^{i,h}; \lambda^{\overline{w}_b}(\beta^h)w^{i,h}; \bar{\lambda}^i w^{i,h}] + E\sum_{t=0}^{\tau} \left(\beta^h\right)^t u^i(c_t^{i,h}) \tag{21}$$

Each of the three elements of the min function has the same limit. Taking the limit gives $\lim_{h\to\infty} \tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \geq \bar{\lambda}^i \bar{w}^i + E\sum_{t=0}^{\tau} u^i(\bar{c}_t^i) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$. Limits of inequalities (20) and (21) imply that $\lim_{h\to\infty} \tilde{\tilde{U}}^i\left(c^{i,h}, \beta^h\right) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$ and utility representation $\tilde{\tilde{U}}^i$ is jointly continuous on $X^{i,b} \times [\underline{\beta}, 1]$. Since for all $\beta \in [\underline{\beta}, 1]$ preferences $\succsim_\beta^i$ are strictly monotone and they admit jointly continuous utility representation, by Lemma 1 in Weretka (2018b) weakly-better-than-c^i correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \to X^{i,b}$ is continuous. $\square$

Define surplus function for consumer i as:

$$S^{i,\tau}(p) \equiv \sum_{t=0}^{\tau} E\left[u^i(u^{i\prime -1}(\zeta_t\bar{\lambda}^i))/\bar{\lambda}^i - \zeta_t u^{i\prime -1}(\zeta_t\bar{\lambda}^i) + \zeta_t e_t^i)\right] \tag{22}$$

Lemma 4. *Fix $\tau < \infty$. Consider arbitrary $p, p' \in \mathcal{P}^\tau$. In the reduced problem with quasi-linear preferences* (19) *equivalent variation is well-defined and given by*

$$\tilde{EV}_{p,p'}^{i,Q} = S^{i,\tau}(p') - S^{i,\tau}(p) \in \mathbb{R}.$$

Proof of Lemma 4:

In Step 1 we show that for the quasilinear preferences represented by $\tilde{U}^{i,Q}(c^i)$ optimal choice and equivalent variation on the unrestricted domain $\tilde{X}^{i,Q} \equiv \{c^i | w^i \in \mathbb{R} \text{ and } c_t^i > 0 \text{ for all } t \leq \tau\}$ are well-defined. For policy p', optimal choice $c^{i\prime}$ is uniquely defined by the necessary and sufficient conditions: consumption in the event after history h_t is given by $c_{h_t}^{i\prime} = u^{i\prime -1}(\bar{\lambda}^i\zeta'_{h_t})$ and consumption of wealth is determined from budget constraint $w_{c^{i\prime}}^i = -E\sum_{t=0}^{\tau}\zeta'_t(c_t^{i\prime} - e_t^{i\prime})$.

Next consider policies p and p'. Program (11) specializes to $\min_{z^i,\alpha} \alpha$ subject to two constraints $\tilde{U}^{i,Q}(z^i) \geq \tilde{U}^{i,Q}(c^{i\prime})$ and $w_{z^i}^i - \alpha + E\sum_{t=0}^{\tau}\zeta_t(z_t^i - e_t^i) \leq 0$. With strictly monotone preferences, both constraints must hold with equality. Solving the second equation for α and plugging it into the objective function reduces the problem to

$$\tilde{EV}_{p,p'}^{i,Q} = \min_{z^i} w_{z^i}^i + \sum_{t=0}^{\tau} E\zeta_t\left(z_t^i - e_t^i\right) \text{ s.t. } \tilde{U}^{i,Q}(z^i) = \tilde{U}^{i,Q}(c^{i\prime}). \tag{23}$$

Solution to program (23), denoted by z^{i*} is given by first order conditions: $z_{h_t}^{i*} = u^{i\prime -1}(\bar{\lambda}^i\zeta_{h_t})$ and $w_{z^{i*}}^i = w_{c^{i\prime}}^i + \frac{1}{\bar{\lambda}^i}E\sum_{t=0}^{\tau}(u^i(c_t^{i\prime}) - u^i(z_t^{i*}))$. Under Inada assumptions these conditions define unique $z^{i*} \in X^{i,Q}$. Plugging $z^{i*}, c^{i\prime}$ in objective function (23) gives

$$\begin{aligned}\tilde{EV}^{i,Q}_{p,p'} &= w^i_{c^{i\prime}} + E\sum_{t=0}^{\tau} \frac{u^i(c^{i\prime}_t) - u^i(z^{i*}_t))}{\bar{\lambda}^i} + \sum_{t=0}^{\tau} E\zeta_t \left(z^{i*}_t - e^i_t\right) \\ &= E\sum_{t=0}^{\tau} \frac{u^i(c^{i\prime}_t) - u^i(z^{i*}_t))}{\bar{\lambda}^i} - E\sum_{t=0}^{\tau}[\zeta_t c^{i\prime}_t - \zeta_t z^{i*}_t] + E\sum_{t=0}^{\tau}[\zeta'_t e^{i\prime}_t - \zeta_t e^i_t], \\ &= S^{i,\tau}(p') - S^{i,\tau}(p).\end{aligned}$$

□

Lemma 5. *Fix $\tau < \infty$. Consider arbitrary $p, p' \in \mathcal{P}^\tau$. Equivalent variation in the reduced form economy converges to the quasiliner limit*

$$\lim_{\beta\to 1} \tilde{EV}^i_{p,p'} = \tilde{EV}^{i,Q}_{p,p'}.$$

Proof of Lemma 5: We restrict attention to discount factors from $[\underline{\beta}, 1]$ as defined in Lemma 3. We verify sufficient conditions for the convergence of the equivalent variation for individual agent (Assumptions 2-3 in Weretka (2018b)) within the reduced-form problem. First note that since $\tau < \infty$, and $S < \infty$, a collection of all histories, h_t such that $t \leq \tau$ is finite. Consequently, box $\tilde{X}^{i,b}$ can be reinterpreted as a subset of $\mathbb{R}^N$. The first order condition is uniformly bounded partial derivatives of the budget constraint (18). For any history h_t, one has $\partial\tilde{b}^i_p/\partial c^i_{h_t} = \pi_{h_t}\beta^t\zeta_{h_t}$ and $\partial\tilde{b}^i_p/\partial w^i = 1$. Therefore, $\bar{b} \geq \partial\tilde{b}^i_p/\partial c^i_{h_t} \geq \underline{b}$ and $\bar{b} \geq \partial\tilde{b}^i_p/\partial w^i \geq \underline{b}$, where bounds $\bar{b} \equiv \max(1, \max_{h_t:t\leq\tau} \zeta_{h_t}\pi_{h_t}) < 0$ and $\underline{b} \equiv \min(1, (\underline{\beta}^i)^\tau \min_{h_t:t\leq\tau} \zeta_{h_t}\pi_{h_t}) > 0$ are well-defined since $\tau < \infty$ and $S < \infty$, and $\pi_{h_t} > 0$ for all date-events h_t. Thus, Assumption 2 is satisfied. In the reduced-form representation for each $\beta \in [\underline{\beta}, 1]$, preferences are strictly convex on the respective domains $\tilde{X}^i$. In Step 1, we demonstrated that optimal choice and equivalent variation for the quasilinear model ($\beta = 1$) are well-defined. Fix arbitrary convex box $X^{i,b}$ such that the optimal choice and equivalent variation point with quasilinear preferences are in the interior. For policy p and $\beta \in [\underline{\beta}, 1]$ function $\tilde{b}^i_p(\cdot, \beta)$ is linear in c^i, and hence it is quasi-convex. Finally, by Lemma 5 correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \to X^{i,b}$ is continuous. By Proposition 1 in Weretka (2018b), equivalent variation in the reduced form model satisfies $\lim_{\beta\to 1} \tilde{EV}^i_{p,p'} = EV^{i,Q}_{p,p'}$. □

We now specialize the results to truncations of policies. For a pair of policies $p, p' \in \mathcal{P}$ let $EV^{i,\tau}_{p,p'}$ be the equivalent variation for truncations of the policies to the first $\tau < \infty$ periods, i.e., for periods $t > \tau$ perturbations for both policies are replaced by zero, i.e., the endowments and prices follow the baseline Markov process. Consider x for which $\lim_{\beta\to 1} v^x \equiv \bar{v}^x \in \mathbb{R}_{++}$.

Corollary 2. *Fix $\tau < \infty$ and $p, p' \in \mathcal{P}$. Equivalent variation for truncations of p, p' policies (point-wise, given τ) converges to a finite limit, i.e.,*

$$\lim_{\beta \to 1} EV^{i,\tau}_{p,p'} = EV^{i,Q,\tau}_{p,p'} / \bar{v}^x \in \mathbb{R}$$

where $EV^{i,Q,\tau}_{p,p'}$ is the equivalent variation in the quasilinear problem, for policies p, p' truncated to the first τ periods.

Proof of Corollary 2 : The result follows from Lemma 2 and 5, and the facts that truncations of policies to the first τ periods are in $\mathcal{P}^\tau$.

A.5 Convergence for general policies

Our next lemma shows that the the welfare index derived for the truncated policies approximates well equivalent variation $EV^i_{p,p'}$ when τ is sufficiently high.

Lemma 6. *Fix arbitrary $\underline{\beta} \in (0,1)$. Equivalent variation derived for truncated policies converges, i.e.,*

$$\lim_{\tau \to \infty} EV^{i,\tau}_{p,p'} = EV^i_{p,p'},$$

uniformly for $\beta \in (\underline{\beta}, 1)$.

Proof of Lemma 6: Fix arbitrary $\varepsilon > 0$. Pick τ for which

$$\frac{\Delta^\tau}{1-\Delta} C \left[1 + \frac{\bar{\zeta}}{\underline{\zeta}} \frac{u^{i\prime}(\underline{c})}{u^{i\prime}(\bar{c})} \frac{1}{\underline{\beta}} \right] \frac{\bar{c} + \bar{e} + \bar{\zeta} + C\Delta}{\bar{v}^x} \leq \varepsilon \tag{24}$$

Since $\Delta < 1$, the corresponding τ exists and it does not depend on β. Consider arbitrary $\beta \in (\underline{\beta}, 1)$.

Step 1. By c^i denote a solution to problem (12) for policy p with transfer $EV^i_{p,p'}$, while $c^{i,\tau}$ is a solution for truncation of this policy p^τ with transfer $EV^{i,\tau}_{p,p'}$. Suppose that c^i is weakly preferred to $c^{i,\tau}$ (For the reverse preferences the argument is symmetric.) Under policy p^τ net cost of consumption flow c^i is given by

$$\begin{aligned}
E \sum_{t=0}^{\infty} \beta^t \zeta^\tau_t (c^i_t - e^{i,\tau}_t) &= E \sum_{t=0}^{\infty} \beta^t \zeta_t (c^i_t - e^i_t) \qquad (25) \\
&+ E \sum_{t=\tau+1}^{\infty} \beta^t \zeta^\tau_t (c^i_t - e^{i,\tau}_t) - E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t (c^i_t - e^i_t) \\
&\leq EV^i_{p,p'} v^x + E \sum_{t=\tau+1}^{\infty} \beta^t \left[|\Delta^\zeta_t| c^i_t + |\Delta^\zeta_t| e^i_t + |\Delta^e_t| \zeta_t + |\Delta^e_t \Delta^\zeta_t \right] \\
&\leq EV^i_{p,p'} \bar{v}^x + \frac{C\Delta^\tau}{1-\Delta} (\bar{c} + \bar{e} + \bar{\zeta} + C\Delta),
\end{aligned}$$

For all $\tau' \geq \tau$ flow c^i is affordable given the transfer and by assumption it is preferred to solution to $V^i(p^\tau, EV^{i,\tau}_{p,p'})$. Consequently, by (24) one has

$$EV^{i,\tau}_{p,p'} \leq EV^i_{p,p'} + \varepsilon.$$

Step 2. We next prove the other inequality. Let $c^{i\prime}$ be a solution to (12) for policy p' with no transfer. Using the arguments form Step 1 one can show that under policy $p^{\tau\prime}$ the net cost of the flow cannot exceed

$$E\sum_{t=0}^{\infty}\beta^t\zeta_t^{\tau\prime}(c_t^{i\prime} - e_t^{i\prime\tau}) \leq \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)$$

Consequently $V^i(p'^\tau, \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)/\bar{v}^x) \geq V^i(p', 0)$. The difference in utility

$$\begin{aligned} V^i(p, EV^i_{p,p'}) - \bar{V}(p^\tau, EV^{i,\tau}_{p,p'}) &= V^i(p', 0) - V^i(p'^\tau, 0) \\ &\leq V^i(p'^\tau, \delta(\tau)) - V^i(p'^\tau, 0) \\ &\leq \frac{u^{i\prime}(\underline{c})}{\underline{\zeta}}\frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta) \end{aligned} \tag{26}$$

is bounded, where the first equality follows from (13) the inequality from the previous observation and the last inequality from the fact that the marginal utility of a dollar is bounded from above by $u^{i\prime}(\underline{c})/\underline{\zeta}$. On the other hand, for policy p for any $\gamma > 0$ within the considered range the difference in utility is

$$V^i(p^\tau, EV^{i,\tau}_{p,p'} + \gamma) - V^i(p^\tau, EV^{i,\tau}_{p,p'}) \geq \gamma\frac{\underline{\beta} u^{i\prime}(\bar{c})}{\bar{\zeta}}\bar{v}^x. \tag{27}$$

Equating the two constants on the right hand sides of (26) and (27) gives

$$\gamma(\tau) = \frac{\bar{\zeta}}{\underline{\zeta}}\frac{u^{i'}(\underline{c})}{\underline{\beta} u^{i'}(\bar{c})}\frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)/\bar{v}^x \tag{28}$$

for which $V^i(p^\tau, EV^{i,\tau}_{p,p'} + \gamma(\tau)) \geq V^i(p, EV^i_{p,p'})$. Applying the argument from Step 1 one can show that the solution to problem (12) under policy p^τ and transfer $(EV^{i,\tau}_{p,p'} + \gamma(\tau))\bar{v}^x$ is affordable under policy p with transfer $(EV^{i,\tau}_{p,p'} + \gamma(\tau))\bar{v}^x + \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)$ and it gives higher utility than $\bar{V}(p, EV^i_{p,p'})$. Consequently, by (24) one has

$$EV^i_{p,p'} \leq EV^{i,\tau}_{p,p'} + \varepsilon.$$

The inequalities derived in Steps 1 and 2 imply $|EV^i_{p,p'} - EV^{i,\tau}_{p,p'}| \leq \varepsilon$ for all $\beta \in (\underline{\beta}, 1)$. $\square$

A.6 Concluding argument

The following equality concludes the proof:

$$\lim_{\beta\to 1} EV_{p,p'} \overset{(1)}{=} \sum_i \lim_{\beta\to 1} EV^i_{p,p'} \overset{(2)}{=} \sum_i \lim_{\beta\to 1} \lim_{\tau\to\infty} EV^{i,\tau}_{p,p'} \tag{29}$$

$$\overset{(3)}{=} \sum_i \lim_{\tau\to\infty} \lim_{\beta\to 1} EV^{i,\tau}_{p,p'} \overset{(4)}{=} \sum_i \lim_{\tau\to\infty} \frac{\tilde{EV}^{i,Q,\tau}_{p,p'}}{\bar{v}^x} \tag{30}$$

$$\overset{(5)}{=} \frac{\lim_{\tau\to\infty} \sum_i [S^{i,\tau}(p') - S^{i,\tau}(p)]}{\bar{v}^x} \overset{(6)}{=} \frac{S(\Delta q', \Delta Y') - S(\Delta q, \Delta Y)}{\bar{v}^x} \tag{31}$$

In (29) equality (1) follows from the definition of the aggregate equivalent variation and the sum law for limits, and equality (2) from Lemma 6. In (3) the interchange of limits is justified by Moore-Osgood theorem along with Lemma 6 and Corollary 2. Equation (4) is implied by Corollary 2 while (4) by follows from Lemma Lemmas 5 and the fact that policies p, p' truncated to the first τ periods are in $\mathcal{P}^\tau$. Replacing, prices and endowments from the static model with the recursive counterparts gives gives (6). □