JONES CH. 5 MODEL

(BASED ON ROMER'S MODEL)

BIG POINT: HERE TECHNOLOGICAL CHANGE ARISES WITHIN THE MODEL (ENDOGENOUS GROWTH MODEL EXAMPLE) –

(1) $Y = K^{\alpha} (A L_Y)^{1-\alpha}$ $\quad 0 < \alpha < 1$

(2) $\dot{K} = s_K Y - dK$

(3) $\dot{A} = \delta A^{\phi} L_A^{\lambda}$

$\Rightarrow$

(4) $\hat{A} = \delta \dfrac{L_A^{\lambda}}{A^{1-\phi}}$

NOTATION:

A_t = STOCK OF IDEAS INVENTED UP TO TIME t

L_Y = LABOR USED IN PRODUC

L_A = " " IN R&D

s_K = SAVINGS RATE

d = DEPRECIATION RATE OF PHYSICAL CAPITAL

$\delta > 0$

$0 < \phi < 1, \quad 0 < \lambda < 1$

ASSUME FURTHER: $\hat{L} = n$ (POPUL. GROWTH RATE)

SINCE $L_A + L_Y = L$

DATA $\Longrightarrow$ $\hat{L}_A = \hat{L}_Y = \hat{L} = n$

AND THEREFORE

$\dfrac{L_A}{L} = s_R$ = % WORKERS IN R&D = CONSTANT

& $\dfrac{L_Y}{L} = s_Y = (1 - s_R)$ = % WORKERS IN PROD = CONSTANT

BALANCED GROWTH PATH : $\hat{Y}, \hat{k}, \hat{A}, \ldots$ CONSTANT

(I) GROWTH RATES AT BGP

$$\text{Eq}(2) \Rightarrow \hat{k} = a_k \frac{Y}{k} - d$$

So ALONG BGP $\frac{Y}{k}$ CONSTANT $\Rightarrow \hat{Y} = \hat{k}$

TAKING GROWTH RATES IN (1) :

$$\hat{Y} = \alpha \underbrace{\hat{k}}_{\hat{Y}} + (1-\alpha)\left[\hat{A} + \underbrace{\hat{L}_Y}_{n}\right]$$

$$\Rightarrow \cancel{(1-\alpha)}\,\hat{Y} = \cancel{(1-\alpha)}\left[\hat{A}+n\right] \Rightarrow \boxed{\hat{Y} = \hat{A}+n = \hat{k}}$$

$$\hat{Y}-n = \hat{Y}-\hat{L} = \boxed{\widehat{(Y/L)} \Rightarrow \hat{y} = \hat{A}}$$

OUTPUT PER WORKER AT BGP GROWS AT THE RATE OF TECH. CHANGE $\hat{A}$

WHAT IS $\hat{A}$ AT BGP ?

USING (4) WE HAVE THAT WHEN $\hat{A}$ CONSTANT THEN :

$$\frac{L_A^{\lambda}}{A^{1-\phi}} \quad \text{CONSTANT}$$

$\Rightarrow$ THE NUMERATOR AND DENOMINATOR GROW AT THE SAME RATE:

$$\widehat{\left(L_A^{\lambda}\right)} = \widehat{\left(A^{1-\phi}\right)}$$

$$\Rightarrow \lambda \underbrace{\hat{L}_A}_{= n} = (1-\phi)\,\hat{A}$$

$$\Rightarrow \boxed{\hat{A} = \frac{\lambda n}{1-\phi}} \qquad (5)$$

i.e. GROWTH RATE OF TECHNOLOGY IS A FUNCTION OF THE GROWTH RATE OF POPULATION & PARAMETERS λ & ϕ -

CONCLUSION: ALONG BGP

$$\underbrace{\hat{y}}_{\substack{\text{GROWTH RATE}\\ \text{OF OUTPUT}\\ \text{PER WORKER}}} = \underbrace{\hat{k}}_{\substack{\text{GROWTH RATE OF}\\ \text{CAPITAL}\\ \text{PER WORKER}}} = \hat{A} = \frac{\lambda n}{1-\phi}$$

(II) OUTPUT PER WORKER AT BGP

(DERIVATION IS SIMILAR TO THE ONES IN OTHER MODELS: (i) CHANGE OF VARIABLE (DIVIDE BY AL) (ii) USE FACT THAT ALONG BGP $\widehat{(K/AL)} = 0$)

WILL BE DONE IN APPENDIX (A) BUT NOT REQUIRED FOR EXAM -

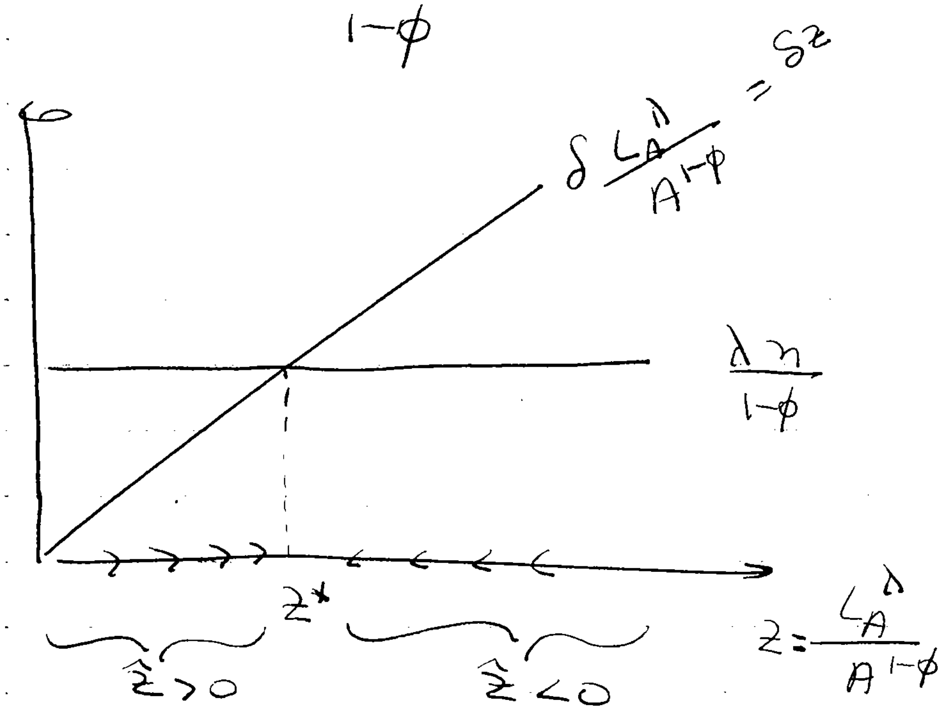
$$(6)\quad \boxed{y_t = \left(\frac{s_K}{d+\hat{A}+n}\right)^{\frac{\alpha}{1-\alpha}} (1-s_R)\, A_t}$$

(III) DIAGRAM THAT SHOWS $\hat{A}$ AT BGP & OUTSIDE

FROM PREVIOUS WORK:

$$(4)\quad \hat{A} = \delta \frac{L_A^{\lambda}}{A^{1-\phi}} \qquad \text{EVERYWHERE}$$

$$(5)\quad \hat{A} = \frac{\lambda n}{1-\phi} \qquad \text{ALONG BGP}$$



NOTATION

TO SIMPLIFY DEFINE A NEW VARIABLE

$$z = \frac{L_A^{\lambda}}{A^{1-\phi}}$$

WE SHOW IN APPENDIX B THAT

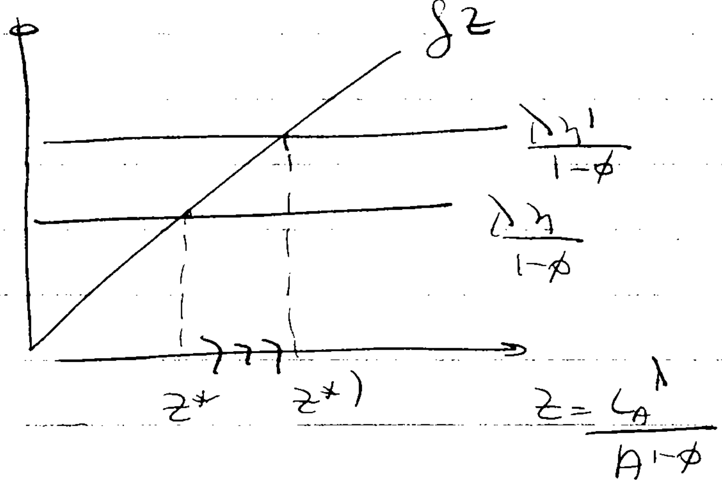
IF $z < z^*$, z IS INCREASING ($\hat{z} > 0$)

& IF $z > z^*$, z IS DECREASING ($\hat{z} < 0$)

COMPARATIVE STATICS

GENERAL FORM: $0 < \lambda < 1$, $0 < \phi < 1$

(I) $\uparrow n$ (INCREASE IN POP. GROWTH RATE)
SUPPOSE WE ARE ALONG BGP & AT TIME
$\bar{t}$, $n \uparrow$ TO n'



SIMPLIFIED VERSION

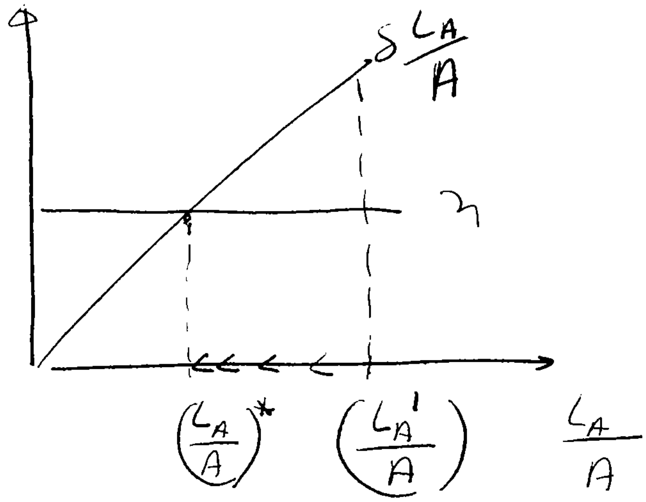
ASSUME $\lambda = 1$, $\phi = 0$
NOW THE GROWTH RATE EQUATIONS
(4) & (5) ARE:

(4') $\hat{A} = \frac{\delta L_A}{A}$ EVERYWHERE

(5') $\hat{A} = n$ AT BGP

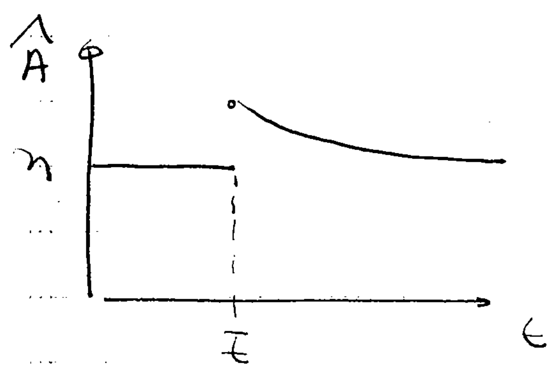
(II) $\uparrow s_m$ (INCREASE IN PERCENTAGE OF WORKERS EMPLOYED IN R&D SECTOR)

$r_K = \frac{L_A}{L}$ THEN $\uparrow r_K \Rightarrow \uparrow L_A$ A TIME $\bar{t}$ $L_A \uparrow$ TO L_A'



SO AT $\bar{t}$, $\hat{A} = \delta \frac{L_A'}{A} > n \Rightarrow \frac{L_A}{A} \downarrow$

UNTIL IT GETS BACK TO $\left(\frac{L_A}{A}\right)^*$



(III) $\uparrow$ or $\downarrow$ in δ (SEE HOMEWORK)

(IV) $\uparrow$ or $\downarrow$ in r_K (SEE HOMEWORK)

APPENDIX A

A DERIVATION OF OUTPUT PER WORKER ALONG BGP

STEP 1: CHANGE OF VARIABLE –

SINCE ALONG BGP $\widehat{(Y/L)} = \hat{y} = \hat{A}$

$\Rightarrow \dfrac{Y}{AL}$ CONSTANT ALONG BGP

THEN NEED TO DIVIDE BY AL

DIVIDING (1) BY AL:

$$\underbrace{\frac{Y}{AL}}_{\tilde{y}} = \frac{K^{\alpha}\,(A L_Y)^{1-\alpha}}{A\,L}$$

$$= \underbrace{\left(\frac{K}{AL}\right)^{\alpha}}_{\tilde{k}} \left(\underbrace{\frac{\cancel{A} L_Y}{\cancel{A} L}}_{a_Y = 1 - a_R}\right)^{1-\alpha}$$

$$\Rightarrow$$

(A1) $$\tilde{y} = \tilde{k}^{\alpha}\,(1-a_R)^{1-\alpha}$$

$$\hat{\tilde{k}} = \widehat{\left(\frac{K}{AL}\right)} = \hat{K} - (\hat{A} + n)$$

USING (2)

$$\hat{\tilde{k}} = s_K \frac{Y}{K} - d - (\hat{A} + n)$$

$$= s_K \frac{Y/AL}{K/AL} - d - (\hat{A} + n)$$

$$= r_K \frac{\tilde{y}}{\tilde{k}} - d - (\hat{A}+n)$$

FROM (A1)

$$\frac{\tilde{y}}{\tilde{k}} = \frac{\tilde{k}^{\alpha}}{\tilde{k}} (1-r_k)^{1-\alpha} = \tilde{k}^{\alpha-1}(1-r_k)^{1-\alpha}$$

THEN

$$\hat{k} = r_K \tilde{k}^{\alpha-1} (1-r_k)^{1-\alpha} - (d+\hat{A}+n)$$

AT BGP $\hat{k}=0$

$$\Longrightarrow$$

$$\frac{r_K}{d+\hat{A}+n} (1-r_k)^{1-\alpha} = \tilde{k}^{1-\alpha}$$

$$\Longrightarrow$$

$$\tilde{k} = \left(\frac{r_K}{d+\hat{A}+n}\right)^{\frac{1}{1-\alpha}} (1-r_k)$$

USING (A.1) AGAIN

$$\tilde{y} = \tilde{k}^{\alpha} (1-r_k)^{1-\alpha} = \left(\frac{r_K}{d+\hat{A}+n}\right)^{\frac{\alpha}{1-\alpha}} (1-r_k)$$

SINCE $y = \tilde{y} \cdot A$

$$\boxed{y_t = \left(\frac{r_K}{d+\hat{A}+n}\right)^{\frac{\alpha}{1-\alpha}} (1-r_k)\, A_t}$$

APPENDIX B

LET $z = \dfrac{L_A^{\lambda}}{A^{1-\phi}}$ (B.1)

$0 < \lambda < 1$

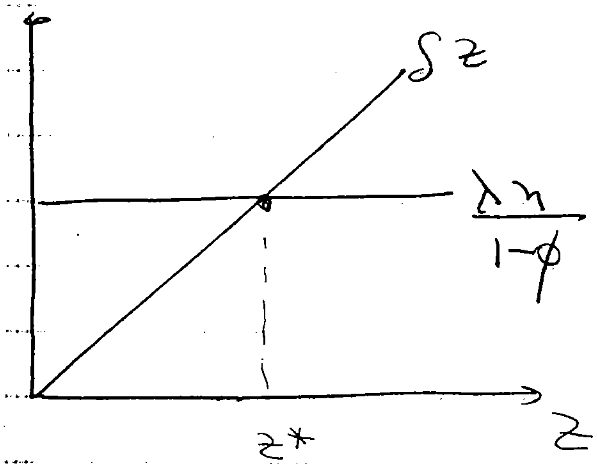
$0 < \phi < 1$

WE WILL SHOW THAT:

(1) THE GROWTH RATE OF z IS ZERO AT z^* (i.e. $\hat{z}^* = 0$, z is constant)

(2) IF $z < z^*$, $\hat{z} > 0$ (ie z is ↑)

IF $z > z^*$, $\hat{z} < 0$ (ie z is ↓)



(1) USING A.2 $\hat{z} = \widehat{\left(L_A^{\lambda}\right)} - \widehat{\left(A^{1-\phi}\right)} =$

$= \lambda \underbrace{\hat{L}_A}_{= n} - (1-\phi)\hat{A} = \lambda n - (1-\phi)\hat{A}$

$\Rightarrow$ (B.2) $\boxed{\hat{z} = \lambda n - (1-\phi)\hat{A}}$

FROM DIAGRAM AT z^* BOTH EQUATIONS (4) & (5)

ARE SATISFIED $\Rightarrow$

$$\hat{A}^* = \delta z^* = \frac{\lambda n}{1-\phi} \quad (B.3)$$

$$\Rightarrow \boxed{z^* = \frac{\lambda n}{(1-\phi)} \frac{1}{\delta}} \quad (B.4)$$

IT IS EASY TO SEE USING (B.2) & (B.3) ~~THAT~~ AT z^*:

$$\hat{z}^* = \lambda n - (1-\phi) \underbrace{\hat{A}^*}_{= \frac{\lambda n}{(1-\phi)}} = 0 \quad (B.5)$$

(2) FROM DIAGRAM:

IF $z < z^* \Rightarrow \hat{A} < \hat{A}^* = \frac{\lambda n}{1-\phi}$

THEN (B.2) & (B.5) $\Rightarrow \hat{z} > 0$

IF $z > z^* \Rightarrow \hat{A} > \hat{A}^* = \frac{\lambda n}{1-\phi}$

THEN (B.2) & (B.5) $\Rightarrow \hat{z} < 0$