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MATCHING WITH COMPLEMENTARY CONTRACTS

MARZENA ROSTEK
Department of Economics, University of Wisconsin-Madison

NATHAN YODER
Department of Economics, Terry College of Business, University of Georgia

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MARZENA ROSTEK
Department of Economics, University of Wisconsin-Madison

NATHAN YODER
Department of Economics, Terry College of Business, University of Georgia

In this paper, we show that stable outcomes exist in matching environments with complementarities, such as social media platforms or markets for patent licenses. Our results apply to both nontransferable and transferable utility settings, and allow for multilateral agreements and those with externalities. In particular, we show that stable outcomes in these settings are characterized by the largest fixed point of a monotone operator, and so can be found using an algorithm; in the nontransferable utility case, this is a *one-sided* deferred acceptance algorithm, rather than a Gale–Shapley algorithm. We also give a monotone comparative statics result as well as a comparative static on the effect of bundling contracts together. These illustrate the impact of design decisions, such as increased privacy protections on social media, or the use of antitrust law to disallow patent pools, on stable outcomes.

KEYWORDS: Complementarities, matching with contracts, stability, contract design.

1. INTRODUCTION

IN MANY SETTINGS OF ECONOMIC INTEREST, agents negotiate sets of discrete agreements with one another. When these agreements are substitutable, the matching literature gives us ample tools to determine which sets of them are *stable*, or robust to renegotiation (e.g., Gale and Shapley (1962), Kelso and Crawford (1982), Roth (1984), or Hatfield and Milgrom (2005)).

However, many of these environments are characterized by complementarities. For instance, complementarities are a defining feature of markets for patent licenses: If a firm needs to secure licenses from multiple rightsholders in order to sell a given product, those licenses are perfect complements (e.g., Shapiro (2000)). Similarly, licenses will be complementary if they lower a firm's marginal cost of production: Acquiring one will cause a firm to produce more units of output, each of which will become cheaper to produce after acquiring subsequent licenses. Complementarities also arise naturally among connections between users of social media platforms like Facebook or LinkedIn. Each connection gives users more frequent opportunities to interact, incentivizing increased use of the platform—and hence causing additional connections to yield more interactions.

In applications like these, each agreement affects the environment in ways that make others more attractive. But accommodating complementary agreements in matching

Marzena Rostek: mrostek@ssc.wisc.edu
Nathan Yoder: nathan.yoder@uga.edu
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models has proven challenging.[1] Positive existence results have been given when there is a continuum of agents (e.g., Azevedo and Hatfield (2018), Che, Kim, and Kojima (2019), Kojima, Pathak, and Roth (2013), Jagadeesan (2017), Scotchmer and Shannon (2019)) or outcomes (Hatfield and Kominers (2015), Bando and Hirai (2019)). Other authors have shown that stable outcomes exist under additional conditions that are compatible with some forms of complementarity. Most notable among these are *pairwise alignment* (Pycia (2012)) between agents' preferences;[2] *acyclic venture structures* (Bando and Hirai (2019)), that is, market structures where contractual relationships form a collection of *trees*;[3] and *unimodular demand types* (Baldwin and Klemperer (2019)) in transferable utility settings with bilateral trades.[4] Existence results have also been given when preferences over agreements satisfy substitutability after the application of a suitable change of basis to the space of outcomes—in particular, under the *gross substitutes and complements* condition of Sun and Yang (2006).

To our knowledge, the literature has yet to provide existence results which accommodate more general forms of complementarities. One reason is its focus on the classical one-to-one and many-to-one matching environments. These settings rule out some complementarities directly: agents on at least one side of the market are limited to a single agreement. If working for Hospital A rules out working for Hospital B, the two jobs cannot be complementary for workers.

Moreover, many-to-one settings rule out other complementarities indirectly by requiring substitutability on one side of the market. In particular, when agents on the *other* side of the market view agreements as complementary, this substitutability can cause stable outcomes to fail to exist. To understand why, consider typical maximal domain results in the literature (e.g., those of Hatfield and Kojima (2008) or Hatfield and Kominers (2017)). These are often interpreted as implying that stable outcomes do not generally exist when agreements are not substitutable. However, a closer examination of the quantifiers used in these results reveals that they do not rule out stability in the presence of complementarities. Each is of the form "if any agent has preferences outside of class $\mathcal{C}$, there exists a profile of preferences in $\mathcal{C}^{N-1}$ for the other agents such that no stable outcome exists." But such statements do not imply that existence is nongeneric when agents' preferences lie in a class $\mathcal{D}$ which does not contain $\mathcal{C}$. This is precisely the case when $\mathcal{D}$ is characterized by complementarities and $\mathcal{C}$ is the class of preferences for which agreements are substitutes. Thus, the message of these converses is more nuanced and powerful than

[1] In particular, these complementarities are different from those discussed in the assortative matching literature following Becker (1973). Assortative matching models consider complementarities *between types*, whereas these settings feature complementarities *between agreements*, which do not arise in a one-to-one setting.

[2] For example, in a two-sided matching market, pairwise alignment requires that workers strictly prefer working with exactly those other workers that their employer prefers hiring alongside them.

[3] Recall that a *tree* is a graph or hypergraph with no cycles (undirected or directed). For instance, in a two-sided matching market, an acyclic venture structure requires that there are never multiple workers who can each be hired by one of the same two firms.

[4] Baldwin and Klemperer (2019) considered the existence of competitive equilibria in quasilinear economies with indivisible goods; by Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013, Theorem 5), these induce stable outcomes in the matching market with contracts consisting of trades for those goods. The authors' result for a different setting (cooperative games) is also related: Theorem 6.7 in Baldwin and Klemperer (2014) shows that by letting goods represent individuals and agents represent coalitions, competitive equilibria induce outcomes in the core (a distinct solution concept from matching-theoretic stability—see footnote 26). For other results from the indivisible goods literature that are compatible with some forms of complementarity, see Bikhchandani and Mamer (1997), Ma (1998), and Candogan, Ozdaglar, and Parrilo (2015), as well as their discussion in Baldwin and Klemperer (2019).

it might appear: if stable outcomes are to generically exist in the presence of arbitrary complementarities, substitutability between agreements must be limited.[5]

This is exactly what we do in this paper: instead of weakening the requirement of substitutability, we abandon it.[6] In its place, we assume complementarity between agreements—which, following Hatfield and Milgrom (2005), we refer to as *contracts*. Our results are readily applicable to market structures studied by the literature, such as supply chains and two-sided many-to-many matching, that (unlike many-to-one environments) do not per se rule out such complementarities. However, our framework is more general: we do not place any assumptions on market structure.[7]

These features are useful in accommodating interesting matching environments with complementarities like the patent licensing and social media examples that we consider throughout the paper. Such settings often have externalities (e.g., patent licenses negatively affect competitors) and may not have a two-sided or acyclic market structure (e.g., social networks need not form a bipartite graph). Moreover, they lack the mutual exclusivity between contracts that rules out complementarity in the many-to-one setting (e.g., the same patent can be licensed to multiple firms at the same time, and restrictions on the number of links agents can form are often unnatural in social networks).

We present two existence results for environments with complementarities. In environments with nontransferable utility, Theorem 1 shows that a unique stable outcome exists when contracts are complementary. Moreover, it characterizes that stable outcome as the largest fixed point of a monotone operator representing the market's aggregate demand for contracts. This operator has not previously appeared in the literature; in particular, it is *not* a Gale–Shapley operator.[8] We show that this fixed point can be found by applying a *one-sided* deferred acceptance algorithm that we introduce, and that it can be thought of as the outcome which clears the market for contracts: there is no excess supply and no excess demand.

Our second main result relaxes Theorem 1's complementarity assumption in order to consider environments with transferable utility. Instead of requiring all contracts to be complements, Theorem 2 only requires that their nonpecuniary elements, which we call *primitive contracts*, are (gross) complements.[9] We then allow the agents involved in each primitive contract to combine it with any set of transfers among them to form a contract. (Transfers are not feasible in a nontransferable utility setting, and so Theorem 1 is not a

[5]That is, there are preferences for which all contracts are substitutes that cannot be accommodated.

[6]In particular, we do not consider a setting isomorphic to an exchange economy with substitutable goods (e.g., Ostrovsky (2008), Hatfield et al. (2013), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2019), or Fleiner, Jagadeesan, Jankó, and Teytelboym (2019)).

[7]Relative to most of the matching literature, allowing for multilateral contracts is an innovation in its own right: Multilateral contracts have so far been investigated by Hatfield and Kominers (2015), who considered divisible agreements, and Teytelboym (2012), who utilized Pycia's (2012) pairwise alignment condition in the contracts context. A larger segment of the literature considers externalities (e.g., Pycia and Yenmez (2019)). In settings with complementarities, this paper shows that accommodating either of these features is possible without imposing the kinds of additional restrictions on preferences that are necessary in environments with substitutability. (We discuss these features in the substitutable context in Rostek and Yoder (2019).)

[8]The monotonicity (and hence convergence) of Gale–Shapley operators such as those used in Hatfield and Milgrom (2005) or Hatfield and Kominers (2012) derives from (full) substitutability between contracts; with complementarities, these operators need not converge, and so do not lead to existence results. In contrast, the monotonicity of our operator derives from complementarity between contracts.

[9]Hatfield and Kominers (2017) used the term *primitive contract* to denote elements that can be combined with one another to form a contract. In our paper, this term plays the same role as *trade* in, for example, Hatfield et al. (2013), or *venture* in Hatfield and Kominers (2015); that is, it denotes an element that can be combined with *transfers* to form a contract.

special case of Theorem 2.) In such environments, Theorem 2 shows that stable outcomes exist, and that the set of primitive contracts they involve is unique.

Theorem 2's characterization of stability relies on an intermediate result, Proposition 3, which gives the welfare theorems that apply in our setting with transferable utility.[10] The social planner's problem in Proposition 3 is *conditional*: Each agent's valuation over primitive contracts is evaluated while holding the primitive contracts that do not involve them fixed. Gross complementarity ensures that the problem's solution correspondence is monotone in the set of primitive contracts on which it conditions. Hence, its fixed points, which we call *conditionally efficient*, are a complete lattice. Proposition 3 shows that these are precisely the sets of primitive contracts that are supported by competitive equilibria. Because of the complementarity in our environment, a specific class of competitive equilibria correspond to stable outcomes, even in the presence of externalities: Theorem 2 shows that outcomes are stable if and only if they correspond to competitive equilibria supporting the largest conditionally efficient set of primitive contracts.

Proposition 3 is not an extension or a special case of the existence result for competitive equilibrium in indivisible goods markets given by Baldwin and Klemperer (2019). While their analysis also allows for some forms of complementarity, the environments we consider are not isomorphic, nor are the sufficient conditions we rely on.[11] In particular, matching settings like those we consider need not be transformable into *any* market for goods, complementary or otherwise.[12]

We also give two comparative statics which apply to both transferable and nontransferable utility settings. The first, Proposition 4, gives a monotone comparative statics result for stability in matching environments with complementarities akin to those for equilibrium in games: With nontransferable (resp., transferable) utility, an increase in a parameter increases the size of the stable outcome (resp., largest conditionally efficient set) when agents' utility from contracts (resp., valuation of primitive contracts) has the single crossing property (resp., increasing differences) in the parameter.

The second concerns the effects of combining, or *bundling*, multiple contracts into a single agreement. For instance, regulators may attempt to block the formation of patent pools or cross-licensing agreements under antitrust law. When contracts are complements or primitive contracts are gross complements, Proposition 5 shows that bundling or unbundling primitive contracts that do not appear in stable outcomes can only cause stable outcomes to include more primitive contracts. Thus, when patent licenses are comple-

[10]Note that, even without externalities, the second welfare theorem does not hold in matching or indivisible goods markets without restrictions on preferences such as the gross substitutes property or (as we show in Proposition 3) the gross complements property. See, for example, Gul and Stacchetti (1999) (in goods markets) or Hatfield et al. (2013) (in bilateral matching markets).

[11]Baldwin and Klemperer (2019) showed that an indivisible goods market always has a competitive equilibrium if and only if the agents have a *unimodular demand type*. Moreover, they showed that every such demand type is a basis change of a demand type which exhibits complementarities, but not every demand type which exhibits complementarities is unimodular. In contrast, we show that competitive equilibria exist in our matching setting whenever agents' demand correspondences satisfy the gross complements condition, but do not give results when agents have a unimodular demand type outside of this class.

[12]The matching settings we consider are not transformable into a market for goods when primitive contracts have externalities. Without externalities, a TU matching environment *can* be transformed into a goods market in a way that preserves gross complementarity and competitive equilibria, but our result does not follow from or imply Baldwin and Klemperer's (2019) unimodularity theorem in this environment—see Examples 1 and 2 in Rostek and Yoder (2020).

mentary, pooling patent licenses that do not transact can never decrease licensing activity.[13]

2. SETTING

There is a finite set I of agents and a set of *contracts* they can sign with one another. Each contract has two parts: a *primitive contract* $\omega \in \Omega$ representing the contract's nonpecuniary elements, and a vector of *transfers* $t^\omega \in T^\omega \subseteq \mathbb{R}^I$ representing the payments to (or from) each agent made as part of the contract.[14] Hence, the set of all contracts is given by $X \equiv \{(\omega, t^\omega) | \omega \in \Omega, t^\omega \in T^\omega\}$. The set of all primitive contracts Ω is finite. Let $\tau : X \to \Omega$ map contracts to their primitive contracts: $\tau((\omega, t^\omega)) = \omega$.

Each primitive contract $\omega \in \Omega$ *names* a set of agents $N(\omega) \subseteq I$ whose agreement is required for contracts involving ω to enter into force. Naturally, we say that each contract (ω, t^ω) names the same agents as its primitive contract ω, and write the set of agents named by $x \in X$ as $N(x) \equiv N(\tau(x))$. For sets of primitive contracts $\Psi \subseteq \Omega$, we write $N(\Psi) \equiv \bigcup_{\omega \in \Psi} N(\omega)$; for sets of contracts $Y \subseteq X$, we write $N(Y) \equiv \bigcup_{x \in Y} N(x)$. For each agent $i \in I$, denote the set of primitive contracts that name i as $\Omega_i \equiv \{\omega \in \Omega | i \in N(\omega)\}$, and define the set of contracts that name i as $X_i \equiv \{x \in X | i \in N(x)\}$. Similarly, let $\Omega_J \equiv \bigcup_{i \in J} \Omega_i$; $X_J \equiv \bigcup_{i \in J} X_i$; $\Omega_{-i} \equiv \Omega \setminus \Omega_i$; and $X_{-i} \equiv X \setminus X_i$. For sets of primitive contracts $\Psi \subseteq \Omega$, write $\Psi_i \equiv \Psi \cap \Omega_i$ and $\Psi_{-i} \equiv \Psi \cap \Omega_{-i}$; likewise, for sets of contracts $Y \subseteq X$, write $Y_i \equiv Y \cap X_i$ and $Y_{-i} \equiv Y \cap X_{-i}$.

Agents have preferences over sets of contracts, or *outcomes*. We allow an agent to have strict preferences over outcomes even when they differ only in the contracts that do not name them; that is, contracts may have externalities. Agent i's preferences are described by the utility function $u_i : 2^X \to \mathbb{R} \cup \{-\infty\}$. (Throughout, we use 2^Y to denote the power set of a set Y.)

A *matching environment* is a tuple $\langle I, \Omega, \{T^\omega\}_{\omega \in \Omega}, N : \Omega \rightrightarrows I, \{u_i\}_{i \in I} \rangle$. We consider two types of matching environments: *transferable utility* (TU) and *nontransferable utility* (NTU).

In a nontransferable utility matching environment, transfers are constrained to zero: $T^\omega = \{0\}$ for each $\omega \in \Omega$. Hence, the set of contracts is finite and isomorphic to the set of primitive contracts; for simplicity, we write $X = \Omega$.

In a transferable utility matching environment, any transfers among the agents named by a primitive contract can be attached to it to form a contract. These transfers must sum to zero: $T^\omega = \{t^\omega \in \mathbb{R}^{N(\omega)} | \sum_{i \in N(\omega)} t_i^\omega = 0\}$. Agents' preferences over sets of contracts are quasilinear in transfers, and preclude them from signing more than one contract associ-

[13]The bundling operation we consider is related to the expressiveness ordering introduced by Hatfield and Kominers (2017) in many-to-many settings with substitutes. However, it is distinct: We consider the effect of bundling contracts together and replacing them with the resulting agreement, while they analyzed the impact of making new bundles available while leaving existing contracts intact. These correspond to separate interventions by a market designer. For instance, our comparative static informs us about the effect of requiring patents to be licensed individually *instead* of being part of a patent pool; theirs, on the other hand, concerns the requirement that patents be available for license individually *in addition* to being available as part of a patent pool.

[14]Primitive contracts play the same role here as *trades* do in, for example, Hatfield et al. (2013), and *ventures* do in Hatfield and Kominers (2015).

ated with the same primitive contract. Formally,

$$u_i(Y) = \begin{cases} v_i(\tau(Y)) - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega, & \text{if } \tau(x) \neq \tau(x') \text{ for each } x, x' \in Y_i, \\ -\infty, & \text{otherwise,} \end{cases} \tag{1}$$

where the *valuation* $v_i : 2^\Omega \to \mathbb{R}$ describes agent i's cardinal preferences over primitive contracts.

2.1. *Choice, Demand, and Complementarity*

Using an agent's preferences, we can derive a *choice correspondence* describing the sets of contracts she might choose to sign when a set of contracts Y is available to her. This choice correspondence has two arguments because we allow for externalities: an agent's preferences over sets of contracts that she might sign—and thus her choices from Y—depend on the set of contracts Z she expects the other agents to sign. Formally, define each agent i's *choice correspondence* $C_i : 2^{X_i} \times 2^{X_{-i}} \rightrightarrows 2^{X_i}$ as follows: $C_i(Y|Z) \equiv \arg\max_S\{u_i(S \cup Z) \text{ s.t. } S \subseteq Y\}$. $C_i(Y|Z)$ gives the sets of contracts that i might choose to sign from the set of available contracts Y when she expects the other agents to sign Z.

In an NTU matching environment, the relevant notion of complementarity pertains to the choice correspondence. We say that *contracts are complements for agent i* if, for all $Y \subseteq Z \subseteq X$, $Y^* \in C_i(Y_i|Y_{-i})$ and $Z^* \in C_i(Z_i|Z_{-i})$ imply $Y^* \cup Z^* \in C_i(Z_i|Z_{-i})$. If contracts are complements for each $i \in I$, we say *contracts are complements*. In words, complementarity between contracts means that an agent never rejects a previously chosen contract when new contracts become available to her or other agents sign new contracts. Instead, either the addition of contracts signed by other agents or the addition of new contracts available to her (or both) may prompt an agent to choose contracts that she previously rejected. When C_i is single-valued, complementarity means that it is monotone (in the usual set order, $\subseteq$) in both the set of contracts available for i to sign and the set of contracts she expects other agents to sign. In general, it is slightly weaker than monotonicity (in the strong set order on $2^{2^X_i}$, $\sqsubseteq$)[15] of C_i in both arguments, because it does not require $Y^* \cap Z^* \in C_i(Y_i|Y_{-i})$. This allows our condition to accommodate some substitutabilities that monotonicity would rule out.[16]

Lemma 1 shows that complementarity between contracts is implied by familiar properties of the utility function u_i.[17] (See Examples 1 and 2 for an illustration.)

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[15]Recall that an order (here, $\subseteq$) on a lattice (here, 2^{X_i}) induces an order $\sqsubseteq$ on the set of subsets of that lattice (here, $2^{2^{X_i}}$). Topkis (1998) referred to $\sqsubseteq$ as the *induced* set order, while Milgrom and Shannon (1994) referred to it as the *strong* set order; we adopt the latter convention. Here, $\sqsubseteq$ is defined as follows: for $\mathcal{Y}, \mathcal{Z} \in 2^{2^{X_i}}$, $\mathcal{Y} \sqsubseteq \mathcal{Z} \Leftrightarrow$ for all $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$, $Y \cap Z \in \mathcal{Y}$ and $Y \cup Z \in \mathcal{Z}$.

[16]Consider, for instance, two contracts x and y that name i and have the same effect on agent i's payoffs in combination as they do individually: $u_i(\{x\}) = u_i(\{y\}) = u_i(\{x, y\}) > u_i(\emptyset)$. Then $C_i(\{x\}|\emptyset) = \{x\}$ and $C_i(\{x, y\}|\emptyset) = \{\{x\}, \{y\}, \{x, y\}\}$. This satisfies our definition of complementary contracts. But $\{x\} \cap \{y\} = \emptyset \notin C_i(\{x\}|\emptyset)$, so C_i is not monotone.

[17]Recall that $u_i(Y)$ is quasisupermodular in Y_i if, for all $Y_i, Y_i' \subseteq X_i$ and all $Y_{-i} \subseteq X_{-i}$, $u_i(Y_i' \cup Y_{-i}) \geq u_i((Y_i' \cap Y_i) \cup Y_{-i})$ implies $u_i((Y_i' \cup Y_i) \cup Y_{-i}) \geq u_i(Y_i \cup Y_{-i})$, and $u_i(Y_i' \cup Y_{-i}) > u_i((Y_i' \cap Y_i) \cup Y_{-i})$ implies $u_i((Y_i' \cup Y_i) \cup Y_{-i}) > u_i(Y_i \cup Y_{-i})$. $u_i(Y)$ has single crossing in (Y_i, Y_{-i}) if, for all $Y_i' \supset Y_i$ and $Y_{-i}' \supset Y_{-i}$, $u_i(Y_i' \cup Y_{-i}) \geq u_i(Y_i \cup Y_{-i})$ implies $u_i(Y_i' \cup Y_{-i}') \geq u_i(Y_i \cup Y_{-i}')$ and $u_i(Y_i' \cup Y_{-i}) > u_i(Y_i \cup Y_{-i})$ implies $u_i(Y_i' \cup Y_{-i}') > u_i(Y_i \cup Y_{-i}')$.

LEMMA 1—Complements, Quasisupermodularity, and Single Crossing: *If $u_i(Y)$ is quasisupermodular in Y_i and has single crossing in (Y_i, Y_{-i}), then contracts are complements for agent i.*

In a TU matching environment, we will also be interested in agent i's *demand correspondence* for primitive contracts. As with her choice correspondence, the presence of externalities requires an agent's demand correspondence to condition on the behavior she expects from the other agents, this time in the form of primitive contracts. Formally, define agent i's demand correspondence $D_i : \mathbb{R}^{\Omega_i} \times 2^{\Omega_{-i}} \rightrightarrows 2^{\Omega_i}$ by $D_i(p_i|\Phi) \equiv \arg\max_{\Psi\subseteq\Omega_i}\{v_i(\Psi \cup \Phi) - \sum_{\omega\in\Psi} p_i^\omega\}$. $D_i(p_i|\Phi)$ specifies the sets of primitive contracts that agent i prefers when the transfers they require from her are given by the *price vector* $p_i \in \mathbb{R}^{\Omega_i}$, given the set of primitive contracts Φ included in other agents' contracts.

The transferable utility setting is incompatible with complementarity between *all* contracts. In particular, two contracts involving the same primitive contract are never complementary, and instead are always substitutes. Consequently, we must employ a different notion of complementarity in TU matching environments; this notion pertains to the demand correspondence, rather than the choice correspondence.

We say that the correspondence D_i satisfies the *gross complements* condition if it is antitone (in the strong set order on $2^{2^{\Omega_i}}$) in price and monotone in other agents' primitive contracts. That is, for any price vectors $p_i \geq q_i \in \mathbb{R}^{\Omega_i}$, any set of primitive contracts $\Phi \subseteq \Phi' \subseteq \Omega_{-i}$, and any $\Psi \in D_i(p_i|\Phi)$, $\Psi' \in D_i(q_i|\Phi')$, we have $\Psi \cap \Psi' \in D_i(p_i|\Phi)$ and $\Psi \cup \Psi' \in D_i(q_i|\Phi')$.[18] If D_i satisfies the gross complements condition for each $i \in I$, we say that *primitive contracts are gross complements*. In words, gross complementarity means that the addition of a primitive contract is more attractive when the price of other primitive contracts is lower, and when other agents sign more contracts.

Lemma 2 shows that gross complements condition on demand is equivalent to supermodularity of the agent's valuation in the contracts that name him and increasing differences in those that do not. (See Example 3 for an illustration.)

LEMMA 2—Gross Complements, Supermodularity, and Increasing Differences: *$v_i(\Psi)$ is supermodular in Ψ_i and has increasing differences in (Ψ_i, Ψ_{-i}) if and only if D_i satisfies the gross complements condition.*

Both our NTU and TU conditions (complementarity between contracts and gross complementarity between primitive contracts, respectively) place restrictions on the externalities created by contracts. These restrictions are entirely about the way contracts that do not name an agent affect his *marginal* utility, and have no implications for the way they affect his utility *level*.[19] In particular, there is no requirement that contracts have *positive* or *negative externalities* (i.e., that $u_i(Y)$ is increasing or decreasing, respectively, in Y_{-i}). All that matters is that externalities are *complementary* with other agents' contracts; that is, when other agents sign more contracts, an agent chooses a (weakly) larger set of primitive contracts from the same set of available contracts (in the NTU setting) or at the same price vector (in the TU setting). This is illustrated in a later example (Example 3), where

[18] Note that this implies complementarity on the single-valued locus of demand; that is, if $\Phi \subseteq \Phi' \subseteq \Omega_{-i}$ and $p_i \geq q_i \in \mathbb{R}^{\Omega_i}$, then for any $\Psi \in D_i(p_i|\Phi)$, there exists $\Psi' \in D_i(q_i|\Phi')$ with $\Psi' \supseteq \Psi$, and for any $\Psi' \in D_i(q_i|\Phi)$, there exists $\Psi \in D_i(p_i|\Phi')$ with $\Psi' \supseteq \Psi$.

[19] Indeed, Lemmas 1 and 2 give sufficient conditions for complementarity/gross complementarity which have no implications for the effect of other agents' contracts on an agent's utility level.

each primitive contract has negative externalities, and in Example A.1 in the Supplemental Material (Rostek and Yoder (2020)), where some primitive contracts have positive externalities and others have negative externalities; in both examples, primitive contracts are gross complements.

2.2. *Remarks on Model Assumptions*

Several assumptions that are common in the matching literature are not present in our setting. In particular, we do not assume that the market has a certain structure (e.g., a two-sided market, an acyclic network, etc.), that agreements are bilateral, or that externalities are absent. As our results show, these assumptions are not necessary to ensure the existence of stable outcomes in NTU matching environments where contracts are complements, or in TU matching environments where primitive contracts are gross complements.

Other authors, such as Hatfield and Milgrom (2005) and Hatfield et al. (2019), have shown that in settings where contracts specify transfers, choice-theoretic and demand-theoretic notions of (full) substitutability are equivalent. We emphasize that the same is *not* true about choice-theoretic and demand-theoretic notions of complementarity, which are fundamentally different. In particular, gross complementarity is defined in transferable utility matching environments, while complementarity between (all) contracts is ruled out by transferable utility. In spite of this, we show throughout the paper that the two complementarity notions lead to parallel conclusions through similar arguments.

We define complementarity in terms of behavior (choice or demand correspondences) rather than preferences (utility or valuation functions) in order to mirror the approach of the literature on (full) substitutability. However, Lemmas 1 and 2 show that a single condition on preferences is sufficient for both complementarity between contracts (in the NTU case) and gross complementarity (in the TU case). In particular, we could impose the same functional form (1) for payoffs in both TU and NTU settings; doing so would be without loss of generality, since the NTU setting constrains transfers to zero. We could then require $v_i(\Psi)$ to be supermodular in Ψ_i and have increasing differences in (Ψ_i, Ψ_{-i}) in both settings.

While this would further unify the treatment of TU and NTU matching environments, it has two distinct disadvantages. First, supermodularity and increasing differences (or even quasisupermodularity and single crossing) are stronger than necessary to ensure complementarity in the NTU setting. In particular, imposing these conditions would rule out more substitutability than necessary to ensure that stable outcomes exist. (See footnote 16.) Second, adopting a quasilinear functional form outside of the TU setting would make the entire model appear to be based on quasilinearity, and thus appear to rule out important applications (such as social media, Example 2) in the NTU setting.

3. SOLUTION CONCEPT

We follow the bulk of the matching literature in adopting *stability* as our primary solution concept. Below, we extend it to accommodate externalities while maintaining its central features. Namely, outcomes are stable if they are robust to unilateral deletion of contracts and multilateral addition of contracts.

DEFINITION—Stability: A set of contracts $Y \subseteq X$ is *stable* if it is

(i) *Individually rational*: $Y_i \in C_i(Y_i|Y_{-i})$ for all $i \in I$.

(ii) *Unblocked*: There does not exist a *block* $Z \subseteq X \setminus Y$ such that for all $i \in N(Z)$, $Z_i \subseteq Y'$ for some $Y' \in C_i((Z \cup Y)_i | (Z \cup Y)_{-i})$.

With single-valued choice and no externalities, this definition is identical to the standard definition in the matching with contracts literature (e.g., Hatfield et al. (2013), Hatfield and Kominers (2012), Fleiner, Jankó, Tamura, and Teytelboym (2016)). Each of these defines a block of Y as a set of contracts Z which, when offered alongside Y, are chosen by each of the agents they name. That is, each contract in Z is among those chosen from $Y \cup Z$ by each agent that the contract names. Which of the contracts in Y those agents choose from $Y \cup Z$ is immaterial. As is well understood, this means that the agents which participate in a block need not agree about the existing contracts in Y that they will keep.[20]

To extend stability to environments with externalities, we must specify the set of contracts among other agents that an agent takes as given when making choices as part of a block.[21] Here, we have agents take as given both the existing contracts among other agents (Y_{-i}) and the contracts among other agents which are part of the blocking set (Z_{-i}). We do so for two reasons.

First, as in the standard stability concept without externalities, when an agent i participates in a block Z of Y, he chooses contracts from the set $(Y \cup Z)_i$ of existing and new contracts that name him. In doing so, he takes as given that those contracts are available to him, and hence that every other agent named by those contracts is willing to sign them. To be consistent, he should take the same thing as given about the contracts in the set $(Y \cup Z)_{-i}$ of existing and new contracts that do not name him. Doing so is equivalent to taking the presence of $(Y \cup Z)_{-i}$ as given, since those contracts do not require his agreement.

In addition, we want the set of contracts that an agent takes as given when making choices as part of a block to be consistent with the set of contracts that other agents actually sign as part of that block. That is, if agent i takes Z' as given when making choices as part of a block Z of Y, it should be the case that each other agent j participating in the block chooses each contract in Z'_j as part of it.

When contracts are complements or primitive contracts are gross complements, our definition of blocking satisfies this criterion for any block that is relevant for stability:

[20]When indifferences are present, and choice may be multi-valued, our definition of stability is a refinement of the definition advanced by Hatfield et al. (2013), because it requires robustness to more blocks. For Z to block Y, Hatfield et al. (2013) required that each contract in Z must be part of *every* set that agents named by contracts in Z might choose from $Z \cup Y$. In contrast, our definition requires only that each contract in Z is part of *some* set that agents named by contracts in Z might choose from $Z \cup Y$.

Hence, our definition allows for blocks involving contracts which some agents are indifferent about, which can be important when contracts are complements. Consider an NTU matching environment with $I = \{1, 2\}$, $X = \{a, b\}$, and $N(a) = N(b) = I$. Agent 1 views the two contracts as perfect complements: $C_1(\{a\}|\cdot) = C_1(\{b\}|\cdot) = \{\emptyset\}$, $C_1(\{a, b\}|\cdot) = \{\{a, b\}\}$. Agent 2 strictly prefers signing at least one of the contracts to signing none of them, but is indifferent among outcomes with at least one contract: $C_2(\{a\}|\cdot) = \{\{a\}\}$, $C_2(\{b\}|\cdot) = \{\{b\}\}$, $C_2(\{a, b\}|\cdot) = \{\{a\}, \{b\}, \{a, b\}\}$. Given its treatment of indifferences, the stability definition of Hatfield et al. (2013) would predict that $\emptyset$ is unblocked (and thus stable), even though both agents prefer $\{a, b\}$ to $\emptyset$ and would choose both a and b when both are offered. In contrast, $\{a, b\}$ is the unique stable outcome under our definition.

Blocks involving indifferences can similarly affect the set of stable outcomes in environments with full substitutability (e.g., Hatfield and Kominers (2012), Hatfield et al. (2013)). In particular, if (in the terminology of Hatfield et al. (2013)) agent 1 is the seller of contract a and the buyer of contract b, then both agents in the example above have fully substitutable preferences.

[21]See also the discussion in Hatfield and Kominers (2015).

Proposition 1 shows that if a set of contracts Y survives the individual rationality prong (i) of our stability criterion, it is blocked by Z if and only if all contracts in $(Y \cup Z)_i$ are chosen by each agent i, given the presence of $(Y \cup Z)_{-i}$. That is, an individually rational Y is blocked by Z if and only if $Z \cup Y$ is also individually rational; that is, if and only if agents choose *more* contracts as part of the block rather than *different* ones. This means that there is never any disagreement among participants in such blocks about the survival of existing contracts that involve them. Instead, taking as given that all existing contracts will continue to be available—as is standard in the literature—is fully consistent with the behavior of other agents.

PROPOSITION 1—Blocking and Individual Rationality: *Suppose that a set of contracts $Y \subseteq X$ is individually rational. If*

(i) *the matching environment is NTU and contracts are complements, or*

(ii) *the matching environment is TU and primitive contracts are gross complements,*

then a set of contracts $Z \subseteq X \setminus Y$ blocks Y if and only if $Y \cup Z$ is individually rational.

Proposition 1 also implies that with complementary contracts or gross complements, an individually rational set of contracts Y is blocked by *some* set of contracts if and only if there is a strictly larger individually rational set of contracts $Y' \supset Y$. This leads to the following characterization of stability in terms of individual rationality.

COROLLARY 1—Stability and Individual Rationality: *If*

(i) *the matching environment is NTU and contracts are complements, or*

(ii) *the matching environment is TU and primitive contracts are gross complements,*

then a set of contracts $Y \subseteq X$ is stable if and only if (a) *Y is individually rational and* (b) *there is no $Y' \supset Y$ that is individually rational.*

One final point about the definition of a block is worth mentioning. Stability—both here, and in the broader matching with contracts literature—does not require every agent that participates in a block to receive higher utility after the block than before it. This is by design: Because agents can sign more than one contract, and in addition, because contracts may have negative externalities, the question whose answer determines whether an agent is willing to participate in a block is not "does participating make me better off than before?" but rather "does agreeing to all of my new contracts as part of the block make me better off than if I only agreed to some or none of them?" This is precisely what is captured by the choice-theoretic notion of stability: Agents choose to create new contracts as part of a block only if they think that, given the behavior they anticipate from the other agents, vetoing some or all of those contracts would not make them better off. That is, stability requires blocks to be *self-enforcing* rather than *payoff-improving*. Moreover, as Proposition 1 shows, with either gross complements or complementary contracts, the behavior that agents anticipate from others is consistent with their actual behavior whenever a block is relevant for stability.

4. EXISTENCE AND CHARACTERIZATION

In this section, we show that stable outcomes exist in NTU matching environments where contracts are complements (Section 4.1), and in TU matching environments where primitive contracts are gross complements (Section 4.2).

4.1. *Stability With Nontransferable Utility*

In Section 3, we showed that in environments with complementarities, stability is characterized by the set of individually rational outcomes (Proposition 1, Corollary 1). In the NTU case, we give a simple characterization of individual rationality (Lemma 4)—and hence of stability (Proposition 2)—in terms of the fixed points of a novel monotone operator. This allows us to prove our existence result for NTU matching environments (Theorem 1).

When choice correspondences can be multi-valued (as we allow), this characterization employs complementarity in an additional way. Lemma 3 shows that when contracts are complements, there is a largest element of $C_i(Y_i|Y_{-i})$ for each $Y \subseteq X$. Hence, checking whether Y is individually rational only requires us to check whether Y_i is the largest element of $C_i(Y_i|Y_{-i})$. By Corollary 1, complementarity between contracts allows us to determine which outcomes are stable merely by knowing which outcomes are individually rational. Consequently, complementarity between contracts also allows all relevant information about an agent's choices from a set Y to be encoded in a single *acceptance set* $A_i(Y)$—the largest set of contracts chosen from Y_i given Y_{-i}—instead of requiring a collection of sets $C_i(Y_i|Y_{-i})$.

More formally, define agent i's *acceptance function* $A_i : 2^X \to 2^{X_i}$ by $A_i(Y) \equiv \bigcup_{Z \in C_i(Y_i|Y_{-i})} Z$. For any $Y \subseteq X$, agent i's acceptance set from Y, $A_i(Y)$, gives the set of contracts in Y which name her and which she is willing to sign, given the presence of the contracts in Y which do not name her. (Equivalently, it is the set of contracts which agent i does not reject from Y_i, given the presence of the contracts in Y_{-i}.)

LEMMA 3—Contracts are Complements ⇒ Acceptance Set is Chosen: *In a nontransferable utility matching environment where contracts are complements for agent i, $A_i(Y) \in C_i(Y_i|Y_{-i})$ for all $Y \subseteq X$.*

These individual acceptance functions generate an *aggregate* acceptance function $A : 2^X \to 2^X$ according to $A(Y) \equiv \bigcap_{i \in I}(A_i(Y) \cup Y_{-i})$. In an NTU matching environment, this aggregate acceptance function can be interpreted as agents' aggregate demand for contracts:[22] Like aggregate demand at a price vector in a goods market, the aggregate acceptance set at Y, $A(Y)$, aggregates individual choices from budget sets based on a dual variable. Instead of a price vector, that dual variable is a set of available contracts: When faced with the set of contracts Y, $A(Y)$ gives the contracts $x \in Y$ that each agent $i \in N(x)$ is willing to sign (given the existence of the contracts in Y_{-i}).

Lemma 4 shows that the set of individually rational outcomes is exactly the set of fixed points of the aggregate acceptance function. Intuitively, when an outcome Y is equal to the aggregate acceptance set at Y, $A(Y)$, no one rejects any contracts from it—and thus it is individually rational.

LEMMA 4—Individually Rational Outcomes as Fixed Points: *In a nontransferable utility matching environment where contracts are complements, $Y \subseteq X$ is individually rational if and only if $A(Y) = Y$.*

This fixed point characterization of individual rationality is useful when contracts are complements because complementarity allows us to characterize stable outcomes using

[22]Note that in contrast to Hatfield and Milgrom (2005), "aggregate demand" refers to aggregation over agents rather than the cardinality of the chosen set of an individual agent.

the set of individually rational outcomes (Corollary 1). Hence, Lemma 4 yields a characterization of stability in terms of the fixed points of A (Proposition 2) which enables our existence result for NTU matching environments (Theorem 1).

PROPOSITION 2—Stable Outcomes as Fixed Points: *In a nontransferable utility matching environment where contracts are complements*, $Y \subseteq X$ *is stable if and only if* (a) $A(Y) = Y$ *and* (b) $A(Y') \neq Y'$ *for all* $Y' \supset Y$.

The two conditions in Proposition 2 correspond to the two parts of the definition of stability. Condition (a) is equivalent to individual rationality (Lemma 4). Given individual rationality and complementarity between contracts, (b) is equivalent to unblockedness (Lemma 4 and Corollary 1). While Proposition 2 requires complementarities, a weaker characterization of stability in terms of aggregate choice holds more generally; see Rostek and Yoder (2019).

These conditions have two interpretations, one of which is economic and the other mathematical. Economically, (a) says that *there is no excess supply of contracts at* Y: all contracts available at Y are accepted by the agents they name. Condition (b) says that *there is no excess demand for contracts at* Y: the agents would not accept a larger set of contracts were it available.[23] We can therefore think of stable outcomes as outcomes which clear the market for contracts.[24]

Mathematically, (a) says that Y *is a fixed point of* A and (b) says that *there is no fixed point of* A *larger than* Y. This is the first of two facts necessary to prove our main result for NTU matching environments, Theorem 1. The other is the monotonicity of the acceptance function when contracts are complements.

LEMMA 5—Complementarity and Monotonicity: *In a nontransferable utility matching environment where contracts are complements*, *the aggregate acceptance function* A *is monotone* (*in the usual set order*, $\subseteq$).

Proposition 2 tells us that a set is stable if and only if it is a fixed point of A and there are no larger fixed points of A. Lemma 5, along with Tarski's fixed point theorem, tells us that A has a largest fixed point. This yields our existence and uniqueness result for nontransferable utility matching environments, Theorem 1.

THEOREM 1—Stability With Complementary Contracts: *In a nontransferable utility matching environment where contracts are complements*, *the aggregate acceptance function* A *has a largest fixed point* X^* *on* 2^X, *which is the unique stable outcome*.

When contracts are substitutes, it is well-known that stable outcomes are precisely those which can result from a two-sided deferred acceptance algorithm (Gale and Shapley (1962), Hatfield and Milgrom (2005), Hatfield and Kominers (2012)). In each stage of

[23]Like its counterpart in an exchange economy, the concept we describe as *no excess supply* refers to an absence of excess quantity (of contracts) supplied at a single value of a variable describing the choices available to the agents (the set of available contracts). In contrast, the concept we describe as *no excess demand* is different, referring to an absence of contracts demanded at *any larger set of available contracts* in excess of the set of available contracts.

[24]Other authors such as Hatfield and Kominers (2012) have given stability a market clearing interpretation, but the results we present here and in Rostek and Yoder (2019) show that stability is equivalent to market clearing at the *aggregate* level, that is, in terms of the market's aggregate acceptance/choice function.

this algorithm, agents on one side of these contracts (e.g., colleges, hospitals, or sellers) may make new offers to agents on the other side (e.g., applicants, doctors, or buyers). From among their new offers and the offers they already hold, the recipients of these offers choose a set to reject and a set to hold until the next stage.[25]

Theorem 1 shows that when contracts are complements, we can continue to think of stable outcomes as the results of an algorithm in which agents defer acceptance of contracts, this time with only one side. The agents start each stage n of this *one-sided deferred acceptance algorithm* with a set of available contracts $Y[n]$. In the first stage, this is the set of all contracts: $Y[1] = X$. Each agent may then reject a set of available contracts that name them. When making this choice, agents take as given the existence of the available contracts that do not name them. Hence, in stage n, each agent i rejects $Y[n] \setminus A_i(Y[n])$. All contracts which have not yet been rejected continue to be available in the next stage. Thus, $Y[n+1] = \bigcap_{i \in I}(A_i(Y[n]) \cup Y[n]_{-i}) = A(Y[n]) \subseteq Y[n]$ for each n. Then, since X is finite, the algorithm must eventually terminate at a fixed point $Y[\bar{n}] = A(Y[\bar{n}])$ for some $\bar{n}$. Since A is monotone, so is its $\bar{n}$-fold composition with itself, $A^{\bar{n}}$; hence, for any fixed point Z of A, $Y[\bar{n}] = A^{\bar{n}}(X) \supseteq A^{\bar{n}}(Z) = Z$. Thus, the algorithm terminates at the largest fixed point of A, which Theorem 1 shows is the unique stable outcome.

Both this result and its transferable utility counterpart (Theorem 2) may seem related to those showing the core is nonempty when the coalitional value function is supermodular (e.g., Sherstyuk (1999)). These nonempty core results consider environments where *agents are complementary for coalitions*, whereas we consider those where *contracts are complementary for agents*. These sets of environments are very different. For instance, the core considers environments where agents can be part of a single coalition; this means that agreements to join a coalition cannot be complementary for the agents. Conversely, our environment accommodates the formation of complementary *overlapping* coalitions.[26]

It is important to note that the unique stable outcome need not be Pareto efficient. This is intuitive in the presence of externalities, but is also true in their absence when an agent can sign multiple contracts at the same time. Recall that stability does not require robustness to each deviation that improves the deviating agents' payoffs. Instead, it requires the absence of blocks which are self-enforcing in the sense that participation is optimal for each of the deviating agents. When agents can sign multiple contracts, these sets of deviations may differ: Participating in a block by signing multiple new contracts may increase an agent's payoff, but shirking by signing only some of them may increase his payoff even more. Example 1 illustrates this.

EXAMPLE 1—Stable Outcomes Need Not Be Pareto Efficient: Suppose there are three contracts $X = \{a, b, c\}$ which each name both of two agents $N = \{1, 2\}$ with utility func-

[25]We note a further difference between our results and those in the literature. Unlike environments with substitutable contracts, environments where contracts are complementary need not satisfy the *irrelevance of rejected contracts (IRC)* condition (Aygün and Sönmez (2013)). While choices result from preference maximization in our model—and thus satisfy IRC—this is only to unify our NTU and TU matching environments, and is not necessary for Theorem 1. All of its arguments would still hold in the absence of IRC, so long as contracts remain complements (which does not imply IRC).

[26]In addition, stability considers different deviations than the core: the core considers only deviations in which the deviating coalition stops interacting with the rest of the agents, while stability allows coalitions to maintain existing relationships when they deviate. This rules out the blocks that are relevant for stability with complementarities, since Proposition 1 shows that agents will never want to delete existing contracts as part of a block of an individually rational outcome. Moreover, unlike with the core, the unique stable outcome need not be Pareto efficient.

tions which satisfy the following inequalities:

$$u_1(\{a,c\}) > u_1(\{a,b,c\}) > u_1(c) > u_1(a) > u_1(\{b,c\}) > u_1(\{a,b\}) > u_1(\emptyset) > u_1(b);$$
$$u_2(\{a,b\}) > u_2(\{a,b,c\}) > u_2(b) > u_2(a) > u_2(\{b,c\}) > u_2(\{a,c\}) > u_2(\emptyset) > u_2(c).$$

These utility functions satisfy quasisupermodularity, so by Lemma 1, Theorem 1 applies. The aggregate acceptance function is given by

$$A(\{a,b,c\}) = A(\{a,b\}) = A(\{a,c\}) = A(a) = a;$$
$$A(\{b,c\}) = A(b) = A(c) = A(\emptyset) = \emptyset.$$

From Theorem 1, the unique stable outcome is a, even though it is not Pareto efficient: moving to $\{a,b,c\}$ would increase the utility of both agents. (The Pareto efficient outcomes are $\{a,c\}$, $\{a,b\}$, and $\{a,b,c\}$.) However, $\{a,b,c\}$ does not block a: agent 1 cannot commit to signing b, whereas agent 2 cannot commit to signing c.

Example 2 applies our results in the context of the social media industry.

EXAMPLE 2—Network Formation in Social Media: Consider the following simple model of behavior on a social media website such as Facebook. The rich heterogeneity among users in the model allows us to illustrate the scope of our existence and characterization result for NTU matching environments (Theorem 1) as well as the comparative statics we give in Section 5.1.

There are n users: $I = \{1,\dots,n\}$. When a pair of users i, j is linked (e.g., are "friends" on Facebook), user i receives posts (e.g., they appear in her "feed" on Facebook) from agent j at rate $\lambda_{ij} > 0$, whereas user j receives posts from agent i at rate $\lambda_{ji} > 0$. The total number of posts that user i expects to receive from each user she is linked to depends on the time $s_i \geq 0$ that user i chooses to spend on social media. Specifically, when she chooses to browse social media for time s_i, she expects to receive $s_i\lambda_{ij}$ posts from each user j she is linked to. Spending time on social media is (quadratically) costly: When user i browses social media for time s_i, her payoffs are reduced by $c_i s_i^2/2$, where $c_i > 0$.

When one user receives a post from another, both receive a payoff: The reader receives R_{ij}, while the poster receives P_{ij}. Since social media websites like Facebook allow users to restrict the audience of their posts as well as hide posts made by a particular user, we assume $R_{ij} \geq 0$ and $P_{ij} \geq 0$ for each i, j.

Finally, a link between users i and j makes some of the personal information of each visible to the other, imposing fixed privacy costs of $K_{ij} \geq 0$ on agent i and $K_{ji} \geq 0$ on agent j.

Let ω_{ij} denote a link between agents i and $j > i$. If Y is the set of links in the social network, agent i's payoff from the action profile $\{s_j\}_{j\in I}$ is given by

$$s_i\tilde{R}_i(Y) + \left(\sum_{j=1}^{n} \hat{P}_{ij}(Y)s_j\right) - c_i s_i^2/2 - \tilde{K}_i(Y),$$

where

$$\tilde{R}_i(Y) = \sum_{j:\omega_{ij}\in Y \text{ or } \omega_{ji}\in Y} R_{ij}\lambda_{ij}; \qquad \tilde{K}_i(Y) = \sum_{j:\omega_{ij}\in Y \text{ or } \omega_{ji}\in Y} K_{ij}\lambda_{ij};$$

$$\hat{P}_{ij}(Y)=\begin{cases}P_{ij}\lambda_{ji}, & \omega_{ij}\in Y \text{ or } \omega_{ji}\in Y,\\ 0, & \text{otherwise.}\end{cases}$$

Then, given the social network Y, each user chooses $s_i^*(Y)\equiv\frac{\tilde{R}_i(Y)}{c_i}$. Hence, each user's payoffs from the network Y are given by

$$u_i(Y)=\frac{1}{2c_i}\big(\tilde{R}_i(Y)\big)^2+\left(\sum_{j=1}^{n}\frac{\hat{P}_{ij}(Y)\tilde{R}_j(Y)}{c_j}\right)-\tilde{K}_i(Y).$$

Now consider a matching environment in which users form links with one another. Specifically, let the primitive contracts represent links in the social network, that is, $\Omega=\{\omega_{ij}|i,j\in I;\, j>i\}$, and let each link name the two agents it connects: $N(\omega_{ij})=\{i,j\}$ for each ω_{ij}.

Contracts are complements in this environment: Observe that $\tilde{R}_i(Y)$, $\hat{P}_{ij}(Y)$, and $\tilde{K}_i(Y)$ are each increasing, modular functions of Y. The first $n+1$ terms of u_i are products of these functions and so they are each supermodular. Then u_i is the sum of supermodular functions, and so is itself supermodular. By Lemma 1, this implies that contracts are complements, since supermodularity implies quasisupermodularity and the single crossing property.

Consequently, Theorem 1 characterizes the unique stable social network in this setting: Among the (potentially many) individually rational networks, one must contain a larger set of links than any of the others. This is the unique stable outcome; for the reasons discussed in Example 1, it is not necessarily efficient.

Example 2 describes an important application in which contracts are complements. Since these contracts are links in a network, it also shows how our results can be applied in the context of network formation.

The classical solution concept in this literature is *pairwise stability* (Jackson and Wolinsky (1996)). This concept considers blocks (i.e., deviations involving new links) which consist of the addition of a single link. However, in many network formation settings, there are other deviations involving new links which may be important to consider. For instance, deviations in which agents both add a link and remove others may be relevant when links are substitutes, while deviations in which they add multiple links at the same time may be relevant when links are complements.[27] Hence, the literature also considers stronger solution concepts that consider larger sets of deviations.[28] Because they require robustness to more deviations that involve new links, existence conditions for these concepts are more demanding than the relatively weak conditions which ensure that pairwise stable networks exist.[29]

[27]In Example 2, for instance, adding many links might improve a user's payoff from using social media when adding a single link would not. Hence, users might choose to form multiple links simultaneously that would not be chosen individually. Indeed, deviations involving multiple links are common in practice: When joining Facebook, for instance, users often form links with several friends or acquaintances simultaneously.

[28]For instance, *strong stability* (Jackson and van den Nouweland (2005)) considers deviations which are profitable for each member of a coalition that can feasibly implement them, whether or not they are individually rational.

[29]Pairwise stable networks can be derived from the proper equilibria (which always exist) of a network formation game as long as agents' preferences are strict in their own links (Calvó-Armengol and İlkılıç (2009)).

In this context, Theorem 1 contributes a sharp prediction to the network formation literature: When links are complementary, no additional conditions are necessary to ensure that a network exists which is robust to deviations that may involve multiple links and are self-enforcing (in the sense that participation in them is individually rational). Moreover, that network is unique, and we can find it through the use of our one-sided deferred acceptance algorithm.

4.2. *Stability With Transferable Utility*

Just as we used a market-clearing characterization of stability (Proposition 2) as an intermediate step in our existence result for NTU matching environments, we establish a connection to the concept of *competitive equilibrium* to show that stable outcomes exist in TU matching environments.

We follow the definition of competitive equilibrium for matching markets with multilateral contracts and externalities introduced by Hatfield and Kominers (2015), adapted to our discrete setting. Formally, a set of primitive contracts Ψ and a set of price vectors $\{p_i\}_{i\in I}$ is a competitive equilibrium if it clears the market: $\Psi_i \in D_i(p_i|\Psi_{-i})$ for each $i \in I$ and $\sum_{i\in N(\omega)} p_i^\omega = 0$ for each $\omega \in \Omega$.

To characterize the set of competitive equilibria and show that it is nonempty, we use a social planner's problem with externalities similar to the one employed by Hatfield and Kominers (2015). Given a transferable utility matching environment, define its *conditional welfare function* $W : 2^\Omega \times 2^\Omega \to \mathbb{R}$ and its *conditional optimizer correspondence* $F : 2^\Omega \rightrightarrows 2^\Omega$ as follows:

$$W(\Phi|\Psi) \equiv \sum_{i\in I} \tilde{v}_i(\Phi|\Psi), \qquad F(\Psi) \equiv \arg\max_{\Phi\subseteq\Omega} W(\Phi|\Psi),$$

where agent i's *conditional valuation* $\tilde{v}_i : 2^\Omega \times 2^\Omega \to \mathbb{R}$ is defined as $\tilde{v}_i(\Phi|\Psi) \equiv v_i(\Phi_i \cup \Psi_{-i})$.

If Ψ solves the social planner's problem holding the primitive contracts which do not name each agent i fixed at Ψ_{-i}, that is, if Ψ is a fixed point of F, then we say it is *conditionally efficient*. Conditionally efficient sets of primitive contracts are precisely those where there are no gains from recontracting for agents that take as given the set of primitive contracts that do not name them.

LEMMA 6—Gross Complementarity, Supermodularity, and Monotonicity: *If agent i's demand correspondence satisfies the gross complements property, then $\tilde{v}_i(\Phi|\Psi)$ is supermodular in Φ and has increasing differences in (Φ, Ψ). If primitive contracts are gross complements, then the conditional welfare function $W(\Phi|\Psi)$ is supermodular in Φ and has increasing differences in (Φ, Ψ), $F(\Psi)$ is a complete lattice for each Ψ, and the conditional optimizer correspondence F is increasing* (*in the strong set order*).

The observations of Lemma 6 give rise to a conditional version of the welfare theorems, and a lattice characterization of conditional efficiency.

While pairwise stability does not consider deviations involving the removal of multiple links at the same time, Calvó-Armengol and İlkılıç (2009) showed that the networks found this way are robust to these deviations. Hence, we focus on deviations involving new links as the crucial difference between pairwise and matching-theoretic stability.

The existence conditions for strong stability given by Jackson and van den Nouweland (2005), on the other hand, are stronger. In particular, they require the coalitional value of a network to be anonymous (i.e., invariant to permutations of agents) and its average value to be highest for the grand coalition.

PROPOSITION 3—Competitive Equilibria With Gross Complementarity: *In a transferable utility matching environment where primitive contracts are gross complements*,

(i) Ψ *is conditionally efficient if and only if there exists a set of price vectors* $\{p_i\}_{i\in I}$ *such that* $(\Psi, \{p_i\}_{i\in I})$ *is a competitive equilibrium*.

(ii) *The collection of conditionally efficient sets of primitive contracts is a nonempty complete lattice*.

Proposition 3 states that when primitive contracts are gross complements, any conditionally efficient set of them can be combined with a suitable set of price vectors to form a competitive equilibrium. This result relies on a Fenchel-type min-max duality theorem for supermodular functions on a lattice given by Fujishige (1984). We use this result to show that a social planner's conditional primal and dual problems are equivalent in our setting, so long as agents' demand correspondences satisfy the gross complements condition.[30]

Like Baldwin and Klemperer (2019), our results are achieved without transforming the setting in question into a Kelso and Crawford (1982) matching market with gross substitutes. This is a departure from the many papers in the indivisible goods and matching literatures that use this technique to establish the existence of competitive equilibria. We instead use discrete convex duality results, while Baldwin and Klemperer (2019) used results from tropical geometry.[31]

Next, we show that stable outcomes are precisely those formed by combining the largest conditionally efficient set of primitive contracts with a competitive equilibrium price vector.[32]

THEOREM 2—Stability With Gross Complementarity: *In a transferable utility matching environment where primitive contracts are gross complements*:

(i) *There is a largest conditionally efficient set of primitive contracts* Ω^*.

(ii) *A set of contracts* $Y \subseteq X$ *is stable if and only if* $\tau(Y) = \Omega^*$ *and there is a competitive equilibrium* $(\Omega^*, \{p_i\}_{i\in I})$ *such that* $t_i^\omega = p_i^\omega$ *for each* $(\omega, t^\omega) \in Y$ *and* $i \in I$.

With gross complementarity, Theorem 2 tells us that the willingness of agents to make coordinated deviations can be captured by their optimization against a vector of latent prices for the enactment of primitive contracts. In other words, just as Proposition 2 showed in the NTU context, stability corresponds to *no excess demand and no excess supply*—this time in the classical demand-theoretic sense. For deviations which only involve

[30]Proposition 3 resembles Hatfield and Kominers (2015, Theorem 10), which shows that competitive equilibria exist in multilateral matching environments with externalities and a continuum of outcomes. Unlike in Hatfield and Kominers (2015), the set of outcomes is not convex in our environment, and so we cannot rely on concave valuations and Kakutani's fixed point theorem the way they do. Instead, we must rely on supermodular valuations, a *discrete* convex duality result from Fujishige (1984), and Tarski's fixed point theorem. Importantly, the complementarity in our environment allows us to additionally show that some competitive equilibria (the largest ones) correspond to stable outcomes (Theorem 2).

[31]When duality results have appeared in the matching and indivisible goods literatures, they have generally been used to show that the set of competitive equilibrium price vectors is a lattice (e.g., Gul and Stacchetti (1999), Hatfield et al. (2013)). An exception is recent work by Candogan, Epitropou, and Vohra (2016), which uses discrete duality results for $M^\natural$-concave functions to show that stable outcomes exist in the context of substitutable contracts. Our results on existence with gross complements, in contrast, rely on duality results for supermodular functions.

[32]This contrasts with Hatfield et al. (2013) who showed that all competitive equilibria are stable. This is due to the fact that we allow for externalities and the slight difference (see footnote 20) in the way our solution concepts treat indifferences.

the elimination of contracts, this is intuitive: agents demand a primitive contract at the current transfer level if and only if they do not want to veto it. Consequently, an outcome is individually rational if and only if there is no excess supply. For deviations involving signing new contracts, it is more subtle. Theorem 2 shows that the existence of a set of prices for which no one wants to make an individual deviation (i.e., for which there is no excess demand) implies the nonexistence of any transfers that would allow a joint deviation (i.e., a block).

We can find the largest conditionally efficient set of primitive contracts in a TU matching environment (i.e., the largest fixed point of F) similarly to the way our one-sided deferred acceptance algorithm finds the largest fixed point of A in an NTU matching environment. By Lemma 6, for each $\Psi \subseteq \Omega$, $F(\Psi)$ is a complete lattice, and so has a largest element $\bigcup_{\Phi\in F(\Psi)} \Phi$. Then we can define the maximal selection of F, $F_\vee : 2^\Omega \to 2^\Omega$, by $F_\vee(\Psi) = \bigcup_{\Phi\in F(\Psi)} \Phi$. Since F is monotone, so is its maximal selection $F_\vee$ (Topkis (1998, Theorem 2.4.3)). It follows that we can find the largest fixed point of the selection by starting at the set of all primitive contracts and repeatedly applying $F_\vee$, the same way we showed that the largest fixed point of A could be found following our NTU existence result. Lemma 7 shows that the set of primitive contracts produced by this algorithm is also the largest fixed point of the conditional optimizer correspondence F.

LEMMA 7—Algorithm for Stable Outcomes in TU Matching Environments: *The set of fixed points of $F_\vee$ is a complete lattice, and the largest fixed point of $F_\vee$ is also the largest conditionally efficient set of primitive contracts.*

At each set of primitive contracts Ψ, $F_\vee$ realizes the gains from trade among the agents when they each take as given the primitive contracts in Ψ that do not name them. Each round of the fixed point algorithm updates what they take as given to match the results of the previous round. We use this algorithm to illustrate the scope of Theorem 2 in the context of patent licensing between competing firms.

EXAMPLE 3—Patent Licensing Among Competing Firms: Consider a general Bertrand–Nash model of differentiated product competition with linear demand. There are n firms: $I = \{1, 2, \dots, n\}$. Each firm $i \in I$ sells a single product. Demand for the firms' products is linear, and given by $Q(p) = a + Sp$. Demand is downward sloping: S is negative definite. Each firm has constant marginal cost c_i, and sets prices to maximize profits $(p_i - c_i)Q(p)_i$ given the pricing decisions of the other firms. Then firm i's first-order pricing condition is $Q(p)_i + S_{ii}(p_i - c_i) = 0$. Let $\bar{S} \equiv \text{diag}(S_{11}, S_{22}, \dots, S_{nn})$. Aggregating among firms yields

$$a + Sp + \bar{S}p - \bar{S}c = 0 \quad \Rightarrow \quad p^* = -(S + \bar{S})^{-1}(a - \bar{S}c).$$

Thus, given the vector of marginal costs c, firm i's equilibrium profit is given by (where e_i is a vector with 1 in the ith position and zeroes elsewhere)

$$\underbrace{\big((p^* - c)'e_i\big)}_{\text{firm } i \text{ markup}} \underbrace{\big(e_i'(a + Sp^*)\big)}_{\text{firm } i \text{ demand}} = \underbrace{(a + Sc)'(S + \bar{S})^{-1}}_{-(p^*-c)'} e_i \underbrace{(-S_{ii})e_i'(S + \bar{S})^{-1}(a + Sc)}_{-e_i'(a+Sp^*)}.$$

(See the Supplemental Material for a derivation.) Now suppose that $n - 1$ of these firms $i \in \{1, 2, \dots, n-1\}$ are *incumbents* who own patents on technologies that would lower the marginal cost of firm n, an *entrant*, if it were to adopt them. In particular, each primitive contract in Ω_i represents the license of a patent owned by firm $i < n$ to firm n, and each license $\omega \in \Omega_n = \Omega$ reduces firm n's costs by θ_ω. Denote firm n's marginal cost when it does not license any patents by c_n^0. Then, in a transferable utility matching environment, we have

$$v_i(\Psi) = \big(a + Sc(\Psi)\big)'(S+\bar{S})^{-1}e_i(-S_{ii})e_i'(S+\bar{S})^{-1}\big(a+Sc(\Psi)\big),$$
$$\text{where } c(\Psi) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n^0 - \sum_{\omega\in\Psi}\theta_\omega \end{bmatrix}'.$$

Now since S is negative definite, $S_{ii} < 0$; thus $e_i(-S_{ii})e_i'$ is positive semidefinite, and hence so is $\hat{S}_i \equiv 2S(S+\bar{S})^{-1}e_i(-S_{ii})e_i'(S+\bar{S})^{-1}S$. Then, since $\hat{S}_i$ is its Hessian matrix, $(a+Sc)'(S+\bar{S})^{-1}e_i(-S_{ii})e_i'(S+\bar{S})^{-1}(a+Sc)$ is convex in c, and thus in $-c_n$. Since patent licenses only lower the cost of firm n, and since convex functions of sums are supermodular in the set of summands (Topkis (1998, Lemma 2.6.2(a))), it follows that v_i is supermodular. Then, by Lemma 2, primitive contracts are gross complements, and we can apply Theorem 2.

For concreteness, we consider two of these matching environments and use Lemma 7 to solve for the set of patent licenses that transact in stable outcomes. First, let $n = 4$ and suppose each incumbent firm $j \in \{1, 2, 3\}$ has a single patent they can license to the entrant. We label these licenses as $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $N(\omega_j) = \{j, 4\}$. Now let

$$\begin{matrix} c_1 = c_2 = c_3 = 45, & c_4^0 = 75, \\ \theta_1 = 10, \quad \theta_2 = 2, & \theta_3 = 5, \end{matrix} \qquad a = \begin{bmatrix}100\\100\\100\\100\end{bmatrix}, \qquad S = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix}.$$

We have

$$F_\vee(\Omega) = F_\vee\big(\{\omega_1,\omega_2,\omega_3\}\big) = \{\omega_2,\omega_3\} \quad F_\vee\big(\{\omega_2,\omega_3\}\big) = \{\omega_3\} \quad F_\vee\big(\{\omega_3\}\big) = \{\omega_3\}$$
$$\Rightarrow \quad \Omega^* = \{\omega_3\}.$$

However, Ω^* does not maximize the firms' total profits. In fact, these would be highest if the entrant did not license any patents at all. The license for firm 3's patent transacts in spite of this fact because firm 3 does not internalize the license's negative externality on the other incumbents.

Alternatively, consider the same setting, but with more concentrated intellectual property rights. In particular, suppose that firm 1's patent was owned by firm 3 instead: $N(\omega_1) = \{3, 4\}$. Then we have $F_\vee(\Omega) = \{\omega_1, \omega_2, \omega_3\}$; hence, our algorithm converges immediately, and all licenses transact in any stable outcome. This lowers firm profits even further: $W(\Omega|\Omega) < W(\{\omega_3\}|\{\omega_3\})$. However, it benefits consumers in the product market: Since the products are substitutes, the equilibrium price of each one falls as a result of the decrease in the entrant's marginal cost.

The difference in the stable outcomes of these two environments illustrates a "patent thicket" (Shapiro (2000))—a dispersion in the ownership of patents which prevents some of them from being licensed. But the reason for this effect is very different from the Cournot complements problem discussed in Shapiro (2000).[33] In fact, such problems can-

[33]The *Cournot complements problem* refers to the fact that when the producers of complementary goods compete monopolistically, they charge higher prices and receive lower profits than if they were controlled by a single firm.

not occur in a stable outcome, because stable outcomes are robust to deviations by groups of agents, such as a licensee and multiple patent holders. Instead, the Cournot complements problem results from a specific noncooperative bargaining protocol (simultaneous price posting) rather than being inherent to environments with complementarities.[34] In our example, licensing expands not because intellectual property rights are *more concentrated*, but rather because they are allocated to a *different* firm that is less affected by increased competition from the entrant.

We pause to consider the features of the model in Example 3 that give rise to complementarities between patent licenses. First, note that patent licenses to the same firm will always be complementary from the perspective of any firm in the model, for any linear, downward sloping specification of market demand. Intuitively, lowering a firm's marginal cost with a patent license causes it to produce more units of output. This increases the number of inframarginal units whose cost is reduced by the license of an additional patent—and hence its marginal benefit to the licensee. Second, while there is only a single entrant in Example 3, the model can be extended to multiple potential licensees when they produce complementary products; we do this in the Supplemental Material.[35] Finally, we rely on the fact that technologies have independent effects on firm n's cost to establish the supermodularity of the firms' valuations; if these effects were to interact, we would not necessarily obtain the gross complements condition.

5. COMPARATIVE STATICS

The characterization of stable outcomes we offer in Theorems 1 and 2 allows us to provide two sets of comparative statics. Section 5.1 discusses the way stable outcomes change when primitive contracts become more complementary; Section 5.2 shows how they change when primitive contracts are bundled together.

5.1. *Monotone Comparative Statics*

Our first comparative statics exercise considers how stable outcomes change when preferences change. When primitive contracts are gross complements (in the TU case) or when preferences satisfy Lemma 1's sufficient conditions for contracts to be complements (in the NTU case), we show that when the characteristics of the environment change so that primitive contracts are more valuable to the agents they name, more primitive contracts will appear in stable outcomes. This need not hold more generally: Without complementarity, increasing the value of some primitive contracts can cause agents to substitute away from others.

[34]This is easiest to see by applying Theorem 2 to a patent licensing environment where firms do not compete in the product market: In that case, patent licenses would not have externalities, and efficiency and conditional efficiency would coincide. Hence, stable outcomes would maximize total profits, regardless of which firm controlled which patents.

[35]Licensing a patent will cause a firm to increase output. If its product is complementary to that of another firm, the second firm's marginal revenue will increase, causing it to produce more output as well. This once again means that there are more units of output for a patent license to the second firm to lower the cost of producing. In contrast, if the two firms produced substitutes, this intuition would be reversed: a patent license to either firm would *lower* the output of the other.

PROPOSITION 4—Monotone Comparative Statics: *Let Θ be a partially ordered set, and $\{M(\theta)\}_{\theta\in\Theta}$ be a parameterized collection of matching environments with $M(\theta) \equiv \langle I, \Omega, \{T^\omega\}_{\omega\in\Omega}, N, \{u_i(\cdot,\theta)\}_{i\in I}\rangle$.*

(i) *If $\{M(\theta)\}_{\theta\in\Theta}$ are nontransferable utility matching environments, and each $u_i(Y,\theta)$ is quasisupermodular in Y_i and has the single crossing property in (Y_i, Y_{-i}) and in (Y_i, θ), then $X^*(\theta)$, the unique stable outcome in $M(\theta)$, is increasing (in the usual set order, $\subseteq$) in θ.*

(ii) *If $\{M(\theta)\}_{\theta\in\Theta}$ are transferable utility matching environments with valuations $\{v_i(\cdot,\theta)\}_{i\in I}$, and each $v_i(\Psi,\theta)$ is supermodular in Ψ_i and has increasing differences in (Ψ_i, Ψ_{-i}) and in (Ψ_i, θ), then $\Omega^*(\theta)$, the largest conditionally efficient set of primitive contracts in $M(\theta)$, is increasing (in the usual set order, $\subseteq$) in θ.*[36]

As we show in Rostek and Yoder (2020), part (i) of this result, for NTU matching environments, is the matching-theoretic counterpart of Milgrom and Shannon's (1994) comparative static for Nash equilibria of games with strategic complementarities.

EXAMPLE 2—Revisited: Recall that in Example 2, agent i's payoff from the set of social connections Y was given by

$$u_i(Y) = \frac{1}{2c_i}\big(\tilde{R}_i(Y)\big)^2 + \left(\sum_{j=1}^{n} \frac{\hat{P}_{ij}(Y)\tilde{R}_j(Y)}{c_j}\right) - \tilde{K}_i(Y).$$

We can see from their definitions that $\tilde{R}(Y)_i$ is increasing and supermodular in $(Y, \{R_{ij}\}_{j=1}^n, \{\lambda_{ij}\}_{j=1}^n)$; that $\hat{P}(Y)_{ij}$ is increasing and supermodular in $(Y, \{P_{ij}\}_{j=1}^n, \{\lambda_{ji}\}_{j=1}^n)$; and that $-\tilde{K}(Y)_i$ is increasing and supermodular in $(Y, \{-K_{ij}\}_{j=1}^n)$. Moreover, each $1/c_j$ is nonnegative and decreasing in c_j. Each term of u_i is a product of these functions and is thus supermodular in $(Y, \{R_{ij}\}_{i,j\in I}, \{P_{ij}\}_{i,j\in I}, \{\lambda_{ij}\}_{i,j\in I}, \{-K_{ij}\}_{i,j\in I}, \{-c_i\}_{i\in I})$; consequently, u_i is as well. Since supermodularity implies the single crossing property, Proposition 4 shows that the stable social network will expand with an increase in users' payoffs $\{R_{ij}\}_{i,j\in I}$ from reading another user's posts, the rates $\{\lambda_{ij}\}_{i,j\in I}$ at which they receive posts from other users, their payoffs $\{P_{ij}\}_{i,j\in I}$ from having their posts read, and will become smaller with an increase in the costs of forming links, $\{K_{ij}\}_{i,j\in I}$, or of spending time on the platform, $\{c_i\}_{i\in I}$.

Among the implications of this example is that a social media platform which profits from user engagement has a financial incentive to institute privacy protections for its users: Decreasing the cost of forming new connections will prompt users to form more connections with one another—leading them to spend more time on the platform.

5.2. *Bundling and Contract Design*

Our second comparative static concerns the effects of *bundling*, that is, replacing two or more independent primitive contracts with a single new primitive contract that duplicates the effects of each. Bundling is prevalent in many matching environments. For instance, licenses for related patents held by different firms—such as those associated with a technical standard—can be combined into *patent pools*; likewise, two firms can simultaneously make their intellectual property portfolios available to one another through the use of

[36]The reason we use increasing differences in the parameter instead of the single crossing property in the TU result is that the former aggregates well whereas the latter may not: see Quah and Strulovici (2012).

a *cross-license*. In such environments, a designer can exert substantial control over outcomes by affecting the way that primitive contracts are bundled—for instance, by using antitrust law to block the formation of patent pools.

We say that a matching environment $\langle I, \Omega, \{T^\omega\}_{\omega\in\Omega}, N, \{u_i\}_{i\in I}\rangle$ is *more bundled* than $\langle I, \hat{\Omega}, \{\hat{T}^\omega\}_{\omega\in\hat{\Omega}}, \hat{N}, \{\hat{u}_i\}_{i\in\hat{I}}\rangle$ if there is a surjective *bundling map* $\alpha : \hat{\Omega} \to \Omega$ such that:

(i) when α bundles a set of primitive contracts together, it preserves the agents they name: $N(\omega) = \hat{N}(\alpha^{-1}(\omega))$ for all $\omega \in \Omega$; and

(ii) when α bundles a set of primitive contracts together, agents' payoffs are affected by the resulting primitive contract in the same way as the original primitive contracts: Either

(a) both environments are NTU, and for each $i \in I$ and $\Psi \subseteq \Omega$, $u_i(\Psi) = \hat{u}_i(\alpha^{-1}(\Psi))$; or

(b) both environments are TU with valuations $\{v_i\}_{i\in I}$ and $\{\hat{v}_i\}_{i\in I}$, respectively, and for each $i \in I$ and $\Psi \subseteq \Omega$, $v_i(\Psi) = \hat{v}_i(\alpha^{-1}(\Psi))$.

Our comparative static describes the effects of bundling primitive contracts together (e.g., combining several bilateral agreements into a multilateral one), and of unbundling them into their component parts (e.g., the reverse). In particular, suppose we have an NTU matching environment where contracts are complements, or a TU matching environment where primitive contracts are gross complements. Then, if we perform either unbundling (Proposition 5(a)) or bundling (Proposition 5(b)) on primitive contracts which do not appear in the stable outcome, the new stable outcome will be weakly larger than the old one.

PROPOSITION 5—Effects of Bundling Contracts Not Signed in the Stable Outcome: *Suppose that* $M = \langle I, \Omega, \{T^\omega\}_{\omega\in\Omega}, N, \{u_i\}_{i\in I}\rangle$ *is more bundled than* $\hat{M} = \langle I, \hat{\Omega}, \{\hat{T}^\omega\}_{\omega\in\hat{\Omega}}, \hat{N}, \{\hat{u}_i\}_{i\in I}\rangle$ *with bundling map* α, *and either*

(i) *both* M *and* $\hat{M}$ *are NTU matching environments where contracts are complements, and their unique stable outcomes are given by* Ω^* *and* $\hat{\Omega}^*$, *respectively*; *or*

(ii) *both* M *and* $\hat{M}$ *are TU matching environments where primitive contracts are gross complements, and their largest conditionally efficient sets of primitive contracts are given by* Ω^* *and* $\hat{\Omega}^*$, *respectively.*

It follows that:

(a) (*Unbundling case*) *If* $\alpha^{-1}(\alpha(\omega)) = \omega$ *for all* $\omega \in \alpha^{-1}(\Omega^*)$, *then* $\alpha^{-1}(\Omega^*) \subseteq \hat{\Omega}^*$.

(b) (*Bundling case*) *If* $\alpha^{-1}(\alpha(\omega)) = \omega$ *for all* $\omega \in \hat{\Omega}^*$, *then* $\Omega^* \supseteq \alpha(\hat{\Omega}^*)$.

Proposition 5 shows that with complementarities, bundling or unbundling primitive contracts can overcome obstacles to their implementation, and will never impose new hurdles. When agents are unable to form a new multilateral agreement, allowing its negotiation as a set of bilateral agreements instead may cause some of its benefits to be realized. Similarly, if bilateral negotiation fails, organizations which facilitate multilateral negotiation (such as patent pools) may be helpful.

In the NTU case (Proposition 5(i)), bundling or unbundling contracts that are not part of the largest fixed point of the aggregate acceptance function A (and thus are not part of any fixed point of A) cannot affect existing fixed points: whether $A(Y) = Y$ cannot depend on contracts not in Y. In contrast, in the TU case (Proposition 5(ii)), whether $F(\Psi) = \Psi$ *does* depend on primitive contracts not in Ψ. Hence, bundling or unbundling primitive contracts that are not part of the largest fixed point of the conditional optimizer function F may remove its existing fixed points. However, we show that it also ensures that there is a (possibly new) fixed point that is weakly larger than all of the old ones. Because of this, bundling or unbundling primitive contracts that are not part of the stable outcome increases the size of the largest fixed point of the aggregate acceptance function A in the NTU case, and the conditional optimizer correspondence F in the TU case. This corresponds to the unique stable outcome (Theorem 1) or unique set of primitive contracts involved in stable outcomes (Theorem 2).

EXAMPLE 4—Bundling in NTU Matching Environments: Recall that in the NTU matching environment $M = \langle I, \Omega, \{\{0\}\}_{\omega\in\Omega}, N, \{u_i\}_{i\in I}\rangle$ from Example 1 with $\Omega = \{a, b, c\}$ and $I = \{1, 2\}$, the unique stable outcome was $\{a\}$ despite the fact that $\{a, b, c\}$ gave both agents higher utility. Suppose now that we define another, more bundled NTU matching environment $\tilde{M} = \langle I, \tilde{\Omega}, \{\{0\}\}_{\omega\in\tilde{\Omega}}, \tilde{N}, \{\tilde{u}_i\}_{i\in I}\rangle$ with primitive contracts $\tilde{\Omega} = \{\tilde{a}, \widetilde{bc}\}$ according to the bundling map $\alpha : \Omega \to \tilde{\Omega}$ given by

$$\alpha(a) = \tilde{a}, \qquad \alpha(b) = \alpha(c) = \widetilde{bc}.$$

That is, let $\tilde{u}_i = u_i \circ \alpha^{-1}$ and $\tilde{N} = N \circ \alpha^{-1}$. α bundles together b and c, which are not signed in the stable outcome of $\tilde{M}$, and does not bundle a, the only contract in that stable outcome, with any other contracts: $\alpha^{-1}(\alpha(a)) = \alpha^{-1}(\tilde{a}) = a$. We have

$$\tilde{u}_1(\{\tilde{a}, \widetilde{bc}\}) > \tilde{u}_1(\{\tilde{a}\}) > \tilde{u}_1(\{\widetilde{bc}\}) > \tilde{u}_1(\emptyset); \qquad \tilde{u}_2(\{\tilde{a}, \widetilde{bc}\}) > \tilde{u}_2(\{\tilde{a}\}) > \tilde{u}_2(\{\widetilde{bc}\}) > \tilde{u}_2(\emptyset).$$

The $\{\tilde{u}_i\}_{i\in I}$ inherit quasisupermodularity from the $\{u_i\}_{i\in I}$ described in Example 1. (In fact, bundling preserves complementarity more generally; Lemma B.1 in the Supplemental Material shows this formally.) Then Theorem 1 applies once again. This time, the aggregate acceptance function is just the identity: $A(\Psi) = \Psi$ for all $\Psi \subseteq \tilde{\Omega}$. Then the unique stable outcome of $\tilde{M}$ is $\tilde{X}^* = \{\tilde{a}, \widetilde{bc}\}$. This is consistent with the prediction of Proposition 5(b): $\tilde{X}^* \supseteq \alpha(\{a\}) = \{\tilde{a}\}$.

Here, bundling changes the stable outcome by allowing the two agents to overcome the commitment problems that prevented them from realizing the Pareto-improving deviation from $\{a\}$ to $\{a, b, c\}$.

6. CONCLUSION

This paper introduces a framework for analyzing settings where agents form complementary agreements with one another. This opens new possibilities for future research. In particular, understanding stability in environments characterized by complementarity may prove useful in the analysis of environments that feature both complementarity and substitutability.

Additionally, there is a growing literature on the structural estimation of matching environments. (For a survey, see Chiappori and Salanié (2016).) The results from this paper suggest there might be new possibilities for the use of matching models in applied work on environments characterized by complementarities.

Finally, we note that there is a formal connection between our results for matching environments with complementary contracts and results from the literature on normal-form games with strategic complementarities (e.g., Milgrom and Shannon (1994)). We explore this connection in Rostek and Yoder (2020).

APPENDIX A: PROOFS

It will occasionally be convenient to abuse notation and represent a set of primitive contracts $\Psi \subseteq \Omega$ as an indicator vector $\Psi \in \{0,1\}^{\Omega}$, or $\Psi_i \subseteq \Omega_i$ as an indicator vector $\Psi_i \in \{0,1\}^{\Omega_i}$.

PROOF OF LEMMA 1 (COMPLEMENTS, QUASISUPERMODULARITY, AND SINGLE CROSSING): By Milgrom and Shannon (1994, Theorem 4), C_i is monotone in both arguments; it follows that contracts are complements for i. *Q.E.D.*

PROOF OF LEMMA 2 (GROSS COMPLEMENTS, SUPERMODULARITY, AND INCREASING DIFFERENCES): We can write $v_i(\Psi) - \sum_{\omega\in\Psi_i} p_i^{\omega}$ as $V_i(\Psi_i, \Psi_{-i}, -p_i)$, where $V_i : \{0,1\}^{\Omega_i} \times 2^{\Omega_{-i}} \times \mathbb{R}^{\Omega_i}$ is defined by $V_i(Y, \Phi, q) = v_i(Y \cup \Phi) + q \cdot Y$. By Milgrom and Shannon (1994, Theorem 4), D_i satisfies the gross complements condition if and only if $V_i(Y, \Phi, q)$ is quasisupermodular in Y and has single crossing in $(Y, (\Phi, q))$. Single crossing in $(Y, (\Phi, q))$ is equivalent to the combination of single crossing in (Y, Φ) and (Y, q): that the former implies the latter is obvious. To see that the latter implies the former, suppose $Y'' \geq Y' \in \{0,1\}^{\Omega_i}$, $\Phi'' \geq \Phi' \in 2^{\Omega_{-i}}$, $q'' \geq q' \in \mathbb{R}^n$. Then if $V_i(Y, \Phi, q)$ has single crossing in (Y, Φ) and (Y, q), $V_i(Y'', \Phi', q') \geq V_i(Y', \Phi', q') \Rightarrow V_i(Y'', \Phi'', q') \geq V_i(Y', \Phi'', q') \Rightarrow V_i(Y'', \Phi'', q'') \geq V_i(Y', \Phi'', q'')$.

By definition, $V_i(Y, \Phi, q)$ has increasing differences, and thus single crossing, in (Y, q): for $Y'' \geq Y' \in \{0,1\}^{\Omega_i}$, $V_i(Y'', \Phi, q) - V_i(Y', \Phi, q) = v_i(Y'' \cup \Phi) - v_i(Y' \cup \Phi) + q \cdot (Y'' - Y')$ is increasing in q. From Topkis (1998, Theorem 2.6.6),[37] $V_i(Y, \Phi, q)$ has single crossing in (Y, Φ) if and only if $v_i(\Psi)$ has increasing differences in (Ψ_i, Ψ_{-i}). Hence, $V_i(Y, \Phi, q)$ has single crossing in $(Y, (\Phi, q))$ if and only if $v_i(\Psi)$ has increasing differences in (Ψ_i, Ψ_{-i}).

Moreover, from Topkis (1998, Theorem 2.6.5), $V_i(Y, \Phi, q)$ is quasisupermodular in Y if and only if $v_i(\Psi)$ is supermodular in Ψ_i. The result follows. *Q.E.D.*

To prove our TU characterization of blocking and stability (Proposition 1(ii), Corollary 1(ii)), we define a price vector for agent i, $\gamma_i(Y)$, at which the primitive contracts demanded by agent i are exactly those which are part of the contracts he chooses when the set of available contracts is Y. Define $\gamma_i : 2^X \to \mathbb{R}^{\Omega_i}$ by the rule

$$\gamma_i^{\omega}(Y) \equiv \begin{cases} \min\{K, \min\{t_i^{\omega} | (\omega, t^{\omega}) \in Y\}\}, & \omega \in \tau(Y), \\ K, & \omega \notin \tau(Y), \end{cases}$$

for a constant $K > \max_{i\in I, \Psi\subseteq\Omega} v_i(\Psi) - \min_{i\in I, \Psi'\subseteq\Omega} v_i(\Psi')$.

LEMMA 8: *For each agent i, $D_i(\gamma_i(Y) | \tau(Y)_{-i}) = \{\tau(Z) | Z \in C_i(Y_i | Y_{-i})\}$.*

PROOF: By definition,

$$\begin{aligned} &D_i(\gamma_i(Y) | \tau(Y)_{-i}) \\ &\quad = \arg\max_{\Psi\subseteq\Omega_i} \{v_i(\Psi \cup \tau(Y)_{-i}) - \Psi \cdot \gamma_i(Y)\} \\ &\quad = \arg\max_{\Psi\subseteq\Omega_i} \left\{ v_i(\Psi \cup \tau(Y)_{-i}) - \sum_{\omega\in\Psi\cap\tau(Y)} \min\{K, \min\{t_i^{\omega} | (\omega, t^{\omega}) \in Y\}\} - \sum_{\omega\in\Psi\setminus\tau(Y)} K \right\}. \end{aligned}$$

[37]The statement of Topkis (1998, Theorem 2.6.6) requires that the parameter space be a chain, but the proof goes through as long as it is a partially ordered set.

Since no primitive contract priced at K or higher can be part of a solution to this problem,

$$\begin{aligned}
&D_i\big(\gamma_i(Y)|\tau(Y)_{-i}\big)\\
&\quad= \arg\max_{\Psi\subseteq\Omega_i}\Big\{v_i\big(\Psi\cup\tau(Y)_{-i}\big) - \sum_{\omega\in\Psi\cap\tau(Y)}\min\big\{t_i^\omega|\big(\omega,t^\omega\big)\in Y\big\} - \sum_{\omega\in\Psi\setminus\tau(Y)} K\Big\}\\
&\quad= \arg\max_{\Psi\subseteq\tau(Y_i)}\Big\{v_i\big(\Psi\cup\tau(Y)_{-i}\big) - \sum_{\omega\in\Psi}\min\big\{t_i^\omega|\big(\omega,t^\omega\big)\in Y\big\}\Big\}\\
&\quad= \tau\bigg(\arg\max_{S\subseteq Y_i}\Big\{v_i\big(\tau(S)\cup\tau(Y)_{-i}\big) - \sum_{(\omega,t^\omega)\in S} t_i^\omega\Big\}\bigg)\\
&\quad= \big\{\tau(Z)|Z\in C_i(Y_i|Y_{-i})\big\}.
\end{aligned}$$

Q.E.D.

This also yields a useful corollary concerning the monotonicity of the choice correspondence in terms of primitive contracts:

LEMMA 9: *Suppose that agent i's demand satisfies the gross complements property. Then* $\{\tau(Z)|Z\in C_i(Y_i|Y_{-i})\}$ *is monotone (in the strong set order) in* Y.

PROOF: Observe that $\gamma_i(Y)$ is weakly decreasing in Y, and $\tau(Y)_{-i}$ is weakly increasing in Y. The statement follows from Lemma 8 and the gross complements property. *Q.E.D.*

PROOF OF PROPOSITION 1 (BLOCKING AND INDIVIDUAL RATIONALITY): ($\Rightarrow$) Part (i) (NTU): Since Y is individually rational, $Y_i\in C_i(Y_i|Y_{-i})$ for each $i\in I$. Suppose Z blocks Y. Then for each i, there exists $Y_i'\in C_i(Z_i\cup Y_i|Z_{-i}\cup Y_{-i})$ such that $Z_i\subseteq Y_i'$. Since contracts are complements, $Y_i'\cup Y_i\in C_i(Z_i\cup Y_i|Z_{-i}\cup Y_{-i})$; then $Z_i\cup Y_i\subseteq Y_i'\cup Y_i\subseteq Z_i\cup Y_i\Rightarrow Z_i\cup Y_i=Y_i'\cup Y_i\in C_i(Z_i\cup Y_i|Z_{-i}\cup Y_{-i})$. Hence, $Y\cup Z$ is individually rational.

Part (ii) (TU): Suppose first that Y is blocked by Z with $\tau(Z)\cap\tau(Y)\neq\emptyset$. Then there exists $\omega\in\Omega$ such that $(\omega,\hat{t}^\omega)\in Z$ and $(\omega,t^\omega)\in Y$. For agents $i\in N(\omega)$ to choose a set of contracts containing $(\omega,\hat{t}^\omega)$ when (ω,t^ω) is available, it must be that $\hat{t}_i^\omega\leq t_i^\omega$. Since $t^\omega\in T^\omega$, summing across $i\in N(\omega)$ yields $\sum_{i\in N(\omega)}\hat{t}_i^\omega\leq 0$, which holds strictly if $\hat{t}_i^\omega\neq t_i^\omega$ for any $i\in N(\omega)$. Then since $\hat{t}^\omega\in T^\omega$, we must have $(\omega,\hat{t}^\omega)\in Y$, a contradiction since Z blocks Y and thus $Z\subseteq X\setminus Y$.

Now suppose Y is blocked by Z with $\tau(Z)\cap\tau(Y)=\emptyset$. First, note that Z cannot contain distinct contracts (ω,t^ω) and $(\omega,\hat{t}^\omega)$, since no agent in $N(\omega)$ will choose both. For each $i\in N(Z)$, since Z blocks Y, $Z_i\cup Y'\in C_i((Y\cup Z)_i|(Y\cup Z)_{-i})$ for some $Y'\subseteq Y_i$. Then $\tau(Z_i)\cup\tau(Y')\in\{\tau(Z')|Z'\in C_i((Y\cup Z)_i|(Y\cup Z)_{-i})\}$. Since Y is individually rational, $\tau(Y_i)\in\{\tau(Z')|Z'\in C_i(Y_i|Y_{-i})\}$. Then, by Lemma 9, $\tau(Y_i)\subseteq\tau(Z_i)\cup\tau(Y')$, and since $Y'\subseteq Y_i$, $\tau(Z_i)\cup\tau(Y_i)\in\{\tau(Z')|Z'\in C_i((Y\cup Z)_i|(Y\cup Z)_{-i})\}$. For each $i\notin N(Z)$, since Y is individually rational, $\tau(Y_i)\in\{\tau(Z')|Z'\in C_i(Y_i|Y_{-i})\}$. Then, by Lemma 9, $\tau(Z_i)\cup\tau(Y_i)=\tau(Y_i)\in\{\tau(Z')|Z'\in C_i(Y_i|(Y\cup Z)_{-i})\}=\{\tau(Z')|Z'\in C_i((Y\cup Z)_i|(Y\cup Z)_{-i})\}$. Since $\tau(Z_i)\cap\tau(Y_i)=\emptyset$, it follows that $(Z\cup Y)_i\in C_i((Y\cup Z)_i|(Y\cup Z)_{-i})$ for each $i\in I$.

($\Leftarrow$) If $Y\cup Z$ is individually rational, then, for each i, $Z_i\subseteq Y_i\cup Z_i\in C_i(Z_i\cup Y_i|Z_{-i}\cup Y_{-i})$ and so Z blocks Y. *Q.E.D.*

PROOF OF COROLLARY 1 (STABILITY AND INDIVIDUAL RATIONALITY): ($\Rightarrow$) (a) is true by definition. Suppose that (b) fails, and there exists an individually rational $Y'\supset Y$.

Then, by Proposition 1, $Y' \setminus Y$ blocks Y, a contradiction. ($\Leftarrow$) Y is individually rational by (a). Suppose Z blocks Y. Then, by Proposition 1, $Y \cup Z \supset Y$ is individually rational, contradicting (b). *Q.E.D.*

PROOF OF LEMMA 3 (CONTRACTS ARE COMPLEMENTS $\Rightarrow$ ACCEPTANCE SET IS CHOSEN): Choose $Z = Y$ in the definition of complementary contracts. Then, for any $Y^*, Z^* \in C_i(Y_i|Y_{-i})$, $Y^* \cup Z^* \in C_i(Y_i|Y_{-i})$. By induction, $A_i(Y) \equiv \bigcup_{Y' \in C_i(Y_i|Y_{-i})} Y' \in C_i(Y_i|Y_{-i})$. *Q.E.D.*

PROOF OF LEMMA 4 (INDIVIDUALLY RATIONAL OUTCOMES AS FIXED POINTS): ($\Rightarrow$) Suppose Y is individually rational. Then, for all i, $Y_i \in C_i(Y_i|Y_{-i}) \Rightarrow Y_i = A_i(Y) \Rightarrow Y = A_i(Y) \cup Y_{-i}$. Thus, $Y = A(Y)$. ($\Leftarrow$) Suppose $A(Y) = Y$. Then, for each i, $Y = A(Y) \subseteq A_i(Y) \cup Y_{-i} \subseteq Y$ by definition of A. It follows that $A_i(Y) = Y_i$. Then, by Lemma 3, $Y_i \in C_i(Y_i|Y_{-i})$ for each i and Y is individually rational. *Q.E.D.*

PROOF OF PROPOSITION 2 (STABLE OUTCOMES AS FIXED POINTS): Follows directly from Lemma 4 and Corollary 1. *Q.E.D.*

PROOF OF LEMMA 5 (COMPLEMENTARITY AND MONOTONICITY): Let $Y \subseteq Z \subseteq X$. From Lemma 3, for all $i \in I$, $A_i(Y) \in C_i(Y_i|Y_{-i})$ and $A_i(Z) \in C_i(Z_i|Z_{-i})$. By the definition of complements, $A_i(Y) \cup A_i(Z) \in C_i(Z_i|Z_{-i})$. Then $A_i(Y) \cup A_i(Z) \subseteq A_i(Z) \Leftrightarrow A_i(Y) \subseteq A_i(Z) \Rightarrow A_i(Y) \cup Y_{-i} \subseteq A_i(Z) \cup Z_{-i}$. Then $A(Y) \subseteq A(Z)$, as desired. *Q.E.D.*

PROOF OF THEOREM 1 (STABILITY WITH COMPLEMENTARY CONTRACTS): By Lemma 5, A is monotone. Then, by Tarski's fixed point theorem, its set of fixed points is a complete lattice, and so has a largest element X^*. The theorem follows from Proposition 2. *Q.E.D.*

PROOF OF LEMMA 6 (GROSS COMPLEMENTARITY, SUPERMODULARITY, AND MONOTONICITY): Supermodularity and increasing differences of $\tilde{v}_i$ follow directly from Lemma 2. Supermodularity and increasing differences of W follow since $W(\Phi|\Psi) = \sum_{i \in I} \tilde{v}_i(\Phi|\Psi)$. Then, by Topkis (1998, Theorem 2.8.1), F is increasing (in the strong set order). *Q.E.D.*

To prove our result characterizing the set of competitive equilibria, we employ a discrete convex duality result from Fujishige (1984). Define the convex and concave conjugate functions of $f : 2^\Omega \to \mathbb{R}$ as follows:

$$f^\circ(p) \equiv \min_{\Phi \subseteq \Omega} \{p \cdot \Phi - f(\Phi)\}, \qquad f^\bullet(p) \equiv \max_{\Phi \subseteq \Omega} \{p \cdot \Phi - f(\Phi)\}.$$

Note that

$$f^\circ(p) = -\max_{\Phi \subseteq \Omega} \{f(\Phi) - p \cdot \Phi\} = -(-f)^\bullet(-p),$$

$$f^\bullet(p) = -\min_{\Phi \subseteq \Omega} \{f(\Phi) - p \cdot \Phi\} = -(-f)^\circ(-p).$$

LEMMA 10—Fujishige (1984, Theorem 3.3): *For a supermodular function* $g : 2^\Omega \to \mathbb{R}$ *and a submodular function* $f : 2^\Omega \to \mathbb{R}$,

$$\min_{\Phi \subseteq \Omega} \{f(\Phi) - g(\Phi)\} = \max_{p \in \mathbb{R}^\Omega} \{g^\circ(p) - f^\bullet(p)\}.$$

COROLLARY 2: *For two supermodular functions* $f, g : 2^\Omega \to \mathbb{R}$,

$$\max_{\Phi\subseteq\Omega}\{f(\Phi)+g(\Phi)\} = \min_{p\in\mathbb{R}^\Omega}\{-g^\circ(p) - f^\circ(-p)\}.$$

PROOF: Since f is supermodular, $-f$ is submodular. Then, by Lemma 10, $\max_{\Phi\subseteq\Omega}\{f(\Phi)+g(\Phi)\} = -\min_{\Phi\subseteq\Omega}\{-f(\Phi)-g(\Phi)\} = -\max_{p\in\mathbb{R}^\Omega}\{g^\circ(p)-(-f)^\bullet(p)\} = \min_{p\in\mathbb{R}^\Omega}\{-g^\circ(p)+(-f)^\bullet(p)\} = \min_{p\in\mathbb{R}^\Omega}\{-g^\circ(p)-f^\circ(-p)\}$, as desired. *Q.E.D.*

Noting that the class of supermodular functions is closed under affine transformations yields the following more general version:

COROLLARY 3: *For two supermodular functions* $f, g : 2^\Omega \to \mathbb{R}$,

$$-(f+g)^\circ(q) = \min_{p\in\mathbb{R}^\Omega}\{-g^\circ(p+q) - f^\circ(-p)\}.$$

PROOF: For each $q\in\mathbb{R}^\Omega$, let $g_q : 2^\Omega\to\mathbb{R}$ be defined by $g_q(\Phi) = g(\Phi) - q\cdot\Phi$. Since g is supermodular, and the class of supermodular functions is closed under affine transformations, so is g_q. Then, by Corollary 2, $\max_{\Phi\subseteq\Omega}\{f(\Phi)+g_q(\Phi)\} = \min_{p\in\mathbb{R}^\Omega}\{-g_q^\circ(p) - f^\circ(-p)\}$. Furthermore, $g_q^\circ(p) = \min_{\Phi\subseteq\Omega}\{p\cdot\Phi - g(\Phi) + q\cdot\Phi\} = g^\circ(p+q)$. Then we have $-(f+g)^\circ(q) = -\min_{\Phi\subseteq\Omega}\{q\cdot\Phi - f(\Phi) - g(\Phi)\} = \max_{\Phi\subseteq\Omega}\{f(\Phi)+g(\Phi)-q\cdot\Phi\} = \max_{\Phi\subseteq\Omega}\{f(\Phi)+g_q(\Phi)\} = \min_{p\in\mathbb{R}^\Omega}\{-g_q^\circ(p)-f^\circ(-p)\} = \min_{p\in\mathbb{R}^\Omega}\{-g^\circ(p+q)-f^\circ(-p)\}$, as desired. *Q.E.D.*

Now define agent i's *conditional profit function* $\pi_i : \mathbb{R}^\Omega\times 2^\Omega$ as $\pi_i(p|\Psi) \equiv \max_{\Phi\subseteq\Omega}\{\tilde{v}_i(\Phi|\Psi) - \Phi\cdot p\}$, and note that $\pi_i(p|\Psi) = -(\tilde{v}_i(\cdot|\Psi))^\circ(p) = (-\tilde{v}_i(\cdot|\Psi))^\bullet(-p)$ for each $\Psi\subseteq\Omega$. Inductively applying Corollary 3 to sums of the agents' conditional value functions yields the following:

LEMMA 11—Duality in the Conditional Social Planner's Problem: *If primitive contracts are gross complements, then, for each* $\Psi\subseteq\Omega$,

$$\max_{\Phi\subseteq\Omega} W(\Phi|\Psi) = \min_{\{r_i\}_{i\in I}\in\mathbb{R}^{\Omega\times I}}\left\{\sum_{i\in I}\pi_i(r_i|\Psi) \textit{ s.t. } \sum_{i\in I} r_i = 0\right\}. \quad (2)$$

PROOF: Label the agents in I as $i = 1, 2, \ldots, |I|$ and note that by Lemma 9, $\tilde{v}_i(\Phi|\Psi)$ is supermodular in Φ for each i. We proceed by induction. For our initial step, we have from Corollary 3 that for each $q\in\mathbb{R}^\Omega$,

$$\begin{aligned}-\big(\tilde{v}_1(\cdot|\Psi)+\tilde{v}_2(\cdot|\Psi)\big)^\circ(q) &= \min_{p\in\mathbb{R}^\Omega}\big\{-\big(\tilde{v}_1(\cdot|\Psi)\big)^\circ(p+q) - \big(\tilde{v}_2(\cdot|\Psi)\big)^\circ(-p)\big\} \\ &= \min_{r_1,r_2\in\mathbb{R}^\Omega}\big\{\pi_1(r_1|\Psi)+\pi_2(r_2|\Psi) \text{ s.t. } r_1+r_2 = q\big\}.\end{aligned}$$

For our induction step, suppose that for each $q\in\mathbb{R}^\Omega$ and some $2\le k<|I|$,

$$-\left(\sum_{i=1}^k \tilde{v}_i(\cdot|\Psi)\right)^\circ(q) = \min_{\{r_i\}_{i=1}^k\in\mathbb{R}^{\Omega k}}\left\{\sum_{i=1}^k \pi_i(r_i|\Psi) \text{ s.t. } \sum_{i=1}^k r_i = q\right\}. \quad (3)$$

Now since sums of supermodular functions are supermodular, $\sum_{i=1}^{k} \tilde{v}_i(\cdot|\Psi)$ is supermodular; then from Corollary 3,

$$-\left(\sum_{i=1}^{k+1} \tilde{v}_i(\cdot|\Psi)\right)^{\circ}(q) = \min_{p\in\mathbb{R}^{\Omega}}\left\{-\left(\sum_{i=1}^{k} \tilde{v}_i(\cdot|\Psi)\right)^{\circ}(p+q) - \left(\tilde{v}_{k+1}(\cdot|\Psi)\right)^{\circ}(-p)\right\}$$
$$= \min_{p\in\mathbb{R}^{\Omega}}\left\{\min_{\{r_i\}_{i=1}^{k}\in\mathbb{R}^{\Omega k}}\left\{\sum_{i=1}^{k}\pi_i(r_i|\Psi) \text{ s.t. } \sum_{i=1}^{k} r_i = p+q\right\} + \pi_{k+1}(-p|\Psi)\right\}$$
$$= \min_{\{r_i\}_{i=1}^{k+1}\in\mathbb{R}^{\Omega k+1}}\left\{\sum_{i=1}^{k+1}\pi_i(r_i|\Psi) \text{ s.t. } \sum_{i=1}^{k+1} r_i = q\right\}.$$

Then (3) holds for each $2 \le k \le |I|$ and each $q \in \mathbb{R}^{\Omega}$. Choosing $k = |I|$ and $q = 0$ yields $\max_{\Phi\subseteq\Omega} W(\Phi|\Psi) = \max_{\Phi\subseteq\Omega}\{\sum_{i\in I}\tilde{v}_i(\Phi|\Psi)\} = -\min_{\Phi\subseteq\Omega}\{-\sum_{i\in I}\tilde{v}_i(\Phi|\Psi)\} = -(\sum_{i\in I}\tilde{v}_i(\cdot|\Psi))^{\circ}(0) = \min_{\{r_i\}_{i\in I}\in\mathbb{R}^{\Omega\times I}}\{\sum_{i\in I}\pi_i(r_i|\Psi) \text{ s.t. } \sum_{i\in I} r_i = 0\}$, as desired. *Q.E.D.*

In keeping with the literature, we call the left-hand side of (2) the social planner's Ψ-conditional *primal* problem (the set of solutions to which is precisely $F(\Psi)$) and the right-hand side the social planner's Ψ-conditional *dual* problem.

LEMMA 12—Primal Solutions Maximize Conditional Profits: *Suppose that primitive contracts are gross complements. If $r^* = \{r_i^*\}_{i\in I}$ is a solution to the Ψ-conditional dual problem and $\Phi^* \in F(\Psi)$, then $\tilde{v}_i(\Phi^*|\Psi) - r_i^* \cdot \Phi^* = \pi_i(r_i^*|\Psi)$ for each $i \in I$.*

PROOF: By definition, for each i we have $\tilde{v}_i(\Phi^*|\Psi) - r_i^* \cdot \Phi^* \le \pi_i(r_i^*|\Psi)$. Since the r_i^* sum to zero, summing over the i yields $\sum_{i\in I}\tilde{v}_i(\Phi^*|\Psi) \le \sum_{i\in I}\pi_i(r_i^*|\Psi)$. We know from Lemma 11 that this holds with equality, which requires $\tilde{v}_i(\Phi^*|\Psi) - r_i^* \cdot \Phi^* = \pi_i(r_i^*|\Psi)$ for each i. *Q.E.D.*

We need to show that the dual problem has a solution which is a lifting of a set of price vectors, that is, such that $r_i^{\omega} = 0$ for all $\omega \in \Omega$ and all $i \notin N(\omega)$. To this end, define the map $\rho : \mathbb{R}^{\Omega\times I} \to \mathbb{R}^{\Omega\times I}$:

$$\rho[r]_i^{\omega} \equiv \begin{cases} 0, & i \notin N(\omega), \\ r_i^{\omega} + \dfrac{1}{|N(\omega)|}\displaystyle\sum_{j\notin N(\omega)} r_j^{\omega}, & i \in N(\omega). \end{cases}$$

Observe that $\sum_{i\in I}\rho[r]_i = 0$ for any $r = \{r_i\}_{i\in I}$ with $\sum_{i\in I} r_i = 0$.

LEMMA 13—Dual Solutions are Closed Under ρ: *Suppose that primitive contracts are gross complements. If $r = \{r_i\}_{i\in I}$ solves the social planner's Ψ-conditional dual problem, then so does $\rho[r] = \{\rho[r]_i\}_{i\in I}$.*

PROOF: First, choose $\Phi^* \in F(\Psi)$, and note that by Lemma 12, for each $\omega \in \Phi^*$ and each $j \notin N(\omega)$, we have $\pi_j(r_j|\Psi) \ge \tilde{v}_j(\Phi^* \setminus \{\omega\}|\Psi) - (\Phi^* \setminus \{\omega\}) \cdot r_j = \tilde{v}_j(\Phi^*|\Psi) + r_j^{\omega} - \Phi^* \cdot r_j = \pi_j(r_j|\Psi) + r_j^{\omega}$ and so $r_j^{\omega} \le 0$; and for each $\omega \notin \Phi^*$ and each $j \notin N(\omega)$, we have $\pi_j(r_j|\Psi) \ge \tilde{v}_j(\Phi^* \cup \{\omega\}|\Psi) - (\Phi^* \cup \{\omega\}) \cdot r_j = \tilde{v}_j(\Phi^*|\Psi) - r_j^{\omega} - \Phi^* \cdot r_j = \pi_j(r_j|\Psi) - r_j^{\omega}$ and so $r_j^\omega \geq 0$. It follows that for each $\omega \in \Omega$ and each $i \in N(\omega)$, $\rho[r]_i^\omega \leq r_i^\omega$ if $\omega \in \Phi^*$ and $\rho[r]_i^\omega \geq r_i^\omega$ if $\omega \notin \Phi^*$.

Now, by Lemma 12, for all $i \in I$ and all $\Phi \subseteq \Omega$ we have

$$\begin{aligned}
&\tilde{v}_i(\Phi^*|\Psi) - \Phi^* \cdot r_i \geq \tilde{v}_i(\Phi_i \cup \Phi^*_{-i}|\Psi) - (\Phi_i \cup \Phi^*_{-i}) \cdot r_i \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - \Phi^* \cdot r_i \geq \tilde{v}_i(\Phi|\Psi) - (\Phi_i \cup \Phi^*_{-i}) \cdot r_i \quad (\tilde{v}_i(\Phi_i \cup \Phi^*_{-i}|\Psi) = \tilde{v}_i(\Phi|\Psi)) \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - \Phi^*_i \cdot r_i \geq \tilde{v}_i(\Phi|\Psi) - \Phi_i \cdot r_i \\
&\qquad\qquad (\text{subtract } \Phi^*_{-i} \cdot r_i \text{ from both sides}) \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - (\Phi^* \cap \Phi)_i \cdot r_i - (\Phi^* \setminus \Phi)_i \cdot r_i \\
&\qquad\qquad \geq \tilde{v}_i(\Phi|\Psi) - (\Phi \cap \Phi^*)_i \cdot r_i - (\Phi \setminus \Phi^*)_i \cdot r_i \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - (\Phi^* \cap \Phi)_i \cdot \rho[r]_i - (\Phi^* \setminus \Phi)_i \cdot r_i \\
&\qquad\qquad \geq \tilde{v}_i(\Phi|\Psi) - (\Phi \cap \Phi^*)_i \cdot \rho[r]_i - (\Phi \setminus \Phi^*)_i \cdot r_i \\
&\qquad\qquad (\text{add } (\Phi \cap \Phi^*)_i \cdot (r_i - \rho[r]_i) \text{ to both sides}) \\
&\quad \Rightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - (\Phi^* \cap \Phi)_i \cdot \rho[r]_i - (\Phi^* \setminus \Phi)_i \cdot r_i \\
&\qquad\qquad \geq \tilde{v}_i(\Phi|\Psi) - (\Phi \cap \Phi^*)_i \cdot \rho[r]_i - (\Phi \setminus \Phi^*)_i \cdot \rho[r]_i \\
&\qquad\qquad (\text{since } \rho[r]_i^\omega \geq r_i^\omega \text{ for } \omega \notin \Phi^* \text{ and } i \in N(\omega)) \\
&\quad \Rightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - (\Phi^* \cap \Phi)_i \cdot \rho[r]_i - (\Phi^* \setminus \Phi)_i \cdot \rho[r]_i \\
&\qquad\qquad \geq \tilde{v}_i(\Phi|\Psi) - (\Phi \cap \Phi^*)_i \cdot \rho[r]_i - (\Phi \setminus \Phi^*)_i \cdot \rho[r]_i \\
&\qquad\qquad (\text{since } \rho[r]_i^\omega \leq r_i^\omega \text{ for } \omega \in \Phi^* \text{ and } i \in N(\omega)) \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - \Phi^*_i \cdot \rho[r]_i \geq \tilde{v}_i(\Phi|\Psi) - \Phi_i \cdot \rho[r]_i \\
&\quad \Leftrightarrow \quad \tilde{v}_i(\Phi^*|\Psi) - \Phi^* \cdot \rho[r]_i \geq \tilde{v}_i(\Phi|\Psi) - \Phi \cdot \rho[r]_i,
\end{aligned}$$

where the last line follows since $\rho[r]_i^\omega = 0$ for all $\omega \notin \Omega_i$. It follows that $\tilde{v}_i(\Phi^*|\Psi) - \Phi^* \cdot \rho[r]_i = \pi_i(\rho[r]_i|\Psi)$ for each $i \in I$. Summing over i yields $W(\Phi^*|\Psi) = \sum_{i \in I} \pi_i(\rho[r]_i|\Psi)$; it follows from Lemma 11 that $\rho[r] = \{\rho[r]_i\}_{i \in I}$ solves the social planner's Ψ-conditional dual problem. *Q.E.D.*

PROOF OF PROPOSITION 3 (COMPETITIVE EQUILIBRIA WITH GROSS COMPLEMENTARITY): Part (i). ($\Rightarrow$) Choose a solution $r^* = \{r_i^*\}_{i \in I}$ to the Ψ-conditional dual problem. By Lemma 13, $\rho[r^*]$ also solves the Ψ-conditional dual problem. For each $i \in I$, let $p_i = \rho[r^*]_i^{\Omega_i}$. By definition of ρ, $\sum_{i \in N(\omega)} p_i^\omega = \sum_{i \in N(\omega)} \rho[r^*]_i^\omega = 0$ for each $\omega \in \Omega$. By Lemma 12 and by definition of $\tilde{v}_i$, for all $i \in I$ and $\Phi \subseteq \Omega$, $v_i(\Psi) - \Psi \cdot \rho[r^*]_i \geq \tilde{v}_i(\Phi_i \cup \Psi_{-i}) - \Phi \cdot \rho[r^*]_i$. Since $\rho[r^*]_i^\omega = 0$ for all $\omega \in \Omega_{-i}$, this implies $v_i(\Psi) - \Psi_i \cdot \rho[r^*]_i \geq \tilde{v}_i(\Phi_i \cup \Psi_{-i}) - \Phi_i \cdot \rho[r^*]_i$, or equivalently, $v_i(\Psi) - \Psi_i \cdot p_i \geq \tilde{v}_i(\Phi_i \cup \Psi_{-i}) - \Phi_i \cdot p_i$. Then $\Psi_i \in D_i(p_i|\Psi_{-i})$ for all $i \in I$, and $(\Psi, \{p_i\}_{i \in I})$ is a competitive equilibrium.

($\Leftarrow$) Suppose $(\Psi, \{p'_i\}_{i \in I})$ is a competitive equilibrium but Ψ is not conditionally efficient. For all $i \in I$, since $\Psi_i \in D_i(p'_i|\Psi_{-i})$, we have $\pi_i(p'_i \oplus 0_{\Omega_{-i}}|\Psi) = v_i(\Psi) - p'_i \cdot \Psi_i$ for each i. Since $\sum_{i\in I} p'_i \oplus 0_{\Omega_{-i}} = 0$ and Ψ is not conditionally efficient, we have

$$\sum_{i\in I} \pi_i\big(p'_i \oplus 0_{\Omega_{-i}}|\Psi\big) = \sum_{i\in I} v_i(\Psi) < \max_{\Phi\subseteq\Omega} W(\Phi|\Psi)$$
$$= \min_{\{r_i\}_{i\in I}\in\mathbb{R}^{\Omega\times I}} \left\{\sum_{i\in I} \pi_i(r_i|\Psi) \text{ s.t. } \sum_{i\in I} r_i = 0\right\},$$

a contradiction.

Part (ii). By Lemma 6, F is increasing (in the strong set order). Then, by Topkis (1998, Theorem 2.5.1), the set of fixed points of F is a nonempty complete lattice. *Q.E.D.*

LEMMA 14: *Suppose Y is a complete lattice with partial order $\succeq$, that $G : Y \rightrightarrows Y$ is an increasing correspondence, and that $G(x)$ is a complete lattice for all $x \in Y$. If $z \succeq y$ for some $y \in Y$ and $z \in G(y)$, then G has a fixed point $y^* \succeq y$.*

PROOF: For each $x \in Y$, $G(x)$ has a $\succeq$-largest element since it is a complete lattice. Denote this element $G_\vee(x)$. Then $G_\vee(y) \succeq z \succeq y$. Since G is monotone, so is $G_\vee : Y \to Y$ (Topkis (1998, Theorem 2.4.3)). Then, for all $x \succeq y$, $G_\vee(x) \succeq G_\vee(y) \succeq y$. Then $G_\vee$ maps $\{x \in Y|x \succeq y\}$ into itself. Since $\{x \in Y|x \succeq y\}$ is a subcomplete sublattice of Y (see Topkis (1998, Example 2.2.5(e))), it follows from Tarski's fixed point theorem, applied to the restriction of $G_\vee$ to $\{x \in Y|x \succeq y\}$, that $G_\vee$ has a fixed point in $\{x \in Y|x \succeq y\}$—which is then also a fixed point of G. *Q.E.D.*

COROLLARY 4: *Suppose that $G : 2^\Omega \rightrightarrows 2^\Omega$ is an increasing correspondence, and that $G(\Phi)$ is a complete lattice for all $\Phi \subseteq \Omega$. If $Y \supseteq \Psi$ for some $\Psi \subseteq \Omega$ and $Y \in G(\Psi)$, then G has a fixed point $\Psi^* \supseteq \Psi$.*

LEMMA 15: *In a transferable utility matching environment where primitive contracts are gross complements, if $Y \subseteq X$ is individually rational, then there is a conditionally efficient set of primitive contracts Ψ^* such that $\Psi^* \supseteq \tau(Y)$.*

PROOF: Since Y is individually rational, by definition, $Y_i \in C_i(Y_i|Y_{-i})$ for all $i \in I$. This can only be true if there is no $x, x' \in Y_i$ with $\tau(x) = \tau(x')$ for any i.

Then we have

$$v_i\big(\tau(Y)\big) - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega \geq v_i\big(\Phi_i \cup \tau(Y)_{-i}\big) - \sum_{(\omega,t^\omega)\in Y_i,\omega\in\Phi} t_i^\omega \quad \big(\forall\Phi \subseteq \tau(Y), i \in I\big)$$
$$\Rightarrow \quad W\big(\tau(Y)|\tau(Y)\big) = \sum_{i\in I} v_i\big(\tau(Y)\big) \geq \sum_{i\in I} v_i\big(\Phi_i \cup \tau(Y)_{-i}\big) = W\big(\Phi|\tau(Y)\big)$$
$$\big(\forall\Phi \subseteq \tau(Y)\big)$$
$$\Leftrightarrow \quad W\big(\tau(Y)|\tau(Y)\big) \geq W\big(\Phi \cap \tau(Y)|\tau(Y)\big) \quad (\forall\Phi \subseteq \Omega)$$
$$\Rightarrow \quad W\big(\tau(Y) \cup \Phi|\tau(Y)\big) \geq W\big(\Phi|\tau(Y)\big) \quad (\forall\Phi \subseteq \Omega)$$

by supermodularity of W in its first argument (Lemma 6). It follows that for any $\Psi \in F(\tau(Y))$, $\Psi \cup \tau(Y) \in F(\tau(Y))$. By Lemma 6, F is increasing and for each $\Phi \subseteq \Omega$, $F(\Phi)$ is a complete lattice. Then by Corollary 4, F has a fixed point $\Psi^* \supseteq \tau(Y)$. *Q.E.D.*

LEMMA 16—Transfers From Individually Rational Outcomes Can Be Swapped With Competitive Equilibrium Prices: *In a transferable utility matching environment where primitive contracts are gross complements, if Y is individually rational and $(\Psi, \{p_i\}_{i\in I})$ is a competitive equilibrium for $\Psi \supseteq \tau(Y)$, then $(\Psi, \{q_i\}_{i\in I})$ is also a competitive equilibrium for $\{q_i\}_{i\in I}$ defined by*

$$q_i^\omega = \begin{cases} t_i^\omega, & (\omega, t^\omega) \in Y, \\ p_i^\omega, & \omega \in \Omega_i \setminus \tau(Y)_i. \end{cases}$$

PROOF: Since Y is individually rational, we cannot have $x \neq x' \in Y$ with $\tau(x) = \tau(x')$, since no agent in $N(x) = N(x')$ will ever choose both simultaneously. It follows that $\{q_i\}_{i\in I}$ is well-defined.

For $\omega \in \tau(Y)$, we have $\sum_{i\in N(\omega)} q_i^\omega = \sum_{i\in N(\omega)} t_i^\omega = 0$, since $t^\omega \in T^\omega$. For $\omega \notin \tau(Y)$, we have $\sum_{i\in N(\omega)} q_i^\omega = \sum_{i\in N(\omega)} p_i^\omega = 0$, since $(\Psi, \{p_i\}_{i\in I})$ is a competitive equilibrium. Then $\sum_{i\in N(\omega)} q_i^\omega = 0$ for each $\omega \in \Omega$.

Since $(\Psi, \{p_i\}_{i\in I})$ is a competitive equilibrium, for each $i \in I$ and all $\Phi \subseteq \Omega_i$,

$$\begin{aligned} & v_i(\Psi) - \sum_{\omega\in\Psi_i} p_i^\omega \geq v_i\big(\Phi \cup \tau(Y)_i \cup \Psi_{-i}\big) - \sum_{\omega\in\Phi\cup\tau(Y)_i} p_i^\omega \\ \Leftrightarrow \quad & v_i(\Psi) - \sum_{\omega\in\Psi_i\setminus\tau(Y)_i} p_i^\omega \geq v_i\big(\Phi \cup \tau(Y)_i \cup \Psi_{-i}\big) - \sum_{\omega\in\Phi\setminus\tau(Y)_i} p_i^\omega \\ \Leftrightarrow \quad & v_i(\Psi) - \sum_{\omega\in\Psi_i\setminus\tau(Y)_i} p_i^\omega - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega \\ & \quad \geq v_i\big(\Phi \cup \tau(Y)_i \cup \Psi_{-i}\big) - \sum_{\omega\in\Phi\setminus\tau(Y)_i} p_i^\omega - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega. \end{aligned} \tag{4}$$

Since Y is individually rational, $u_i(Y) \geq u_i(\{(\omega, t^\omega) \in Y | \omega \in \Phi\} \cup Y_{-i})$, that is,

$$v_i\big(\tau(Y)\big) - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega \geq v_i\big(\big(\Phi \cap \tau(Y)_i\big) \cup \tau(Y)_{-i}\big) - \sum_{(\omega,t^\omega)\in Y_i, \omega\in\Phi} t_i^\omega.$$

By Lemma 2, since $\tau(Y)_{-i} = \tau(Y) \cap \Omega_{-i} \subseteq \Psi \cap \Omega_{-i} = \Psi_{-i}$,

$$\begin{aligned} & v_i\big(\tau(Y)_i \cup \Psi_{-i}\big) - v_i\big(\tau(Y)\big) \geq v_i\big(\big(\Phi \cap \tau(Y)_i\big) \cup \Psi_{-i}\big) - v_i\big(\big(\Phi \cap \tau(Y)_i\big) \cup \tau(Y)_{-i}\big) \\ & v_i\big(\tau(Y)_i \cup \Phi \cup \Psi_{-i}\big) - v_i\big(\tau(Y)_i \cup \Psi_{-i}\big) \geq v_i(\Phi \cup \Psi_{-i}) - v_i\big(\big(\Phi \cap \tau(Y)_i\big) \cup \Psi_{-i}\big) \\ & \quad \Rightarrow \quad v_i\big(\tau(Y)_i \cup \Phi \cup \Psi_{-i}\big) - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega \geq v_i(\Phi \cup \Psi_{-i}) - \sum_{(\omega,t^\omega)\in Y_i, \omega\in\Phi} t_i^\omega. \end{aligned}$$

From (4),

$$\begin{aligned} & v_i(\Psi) - \sum_{\omega\in\Psi_i\setminus\tau(Y)_i} p_i^\omega - \sum_{(\omega,t^\omega)\in Y_i} t_i^\omega \geq v_i(\Phi \cup \Psi_{-i}) - \sum_{\omega\in\Phi\setminus\tau(Y)_i} p_i^\omega - \sum_{(\omega,t^\omega)\in Y_i, \omega\in\Phi} t_i^\omega \\ & \quad \Leftrightarrow \quad v_i(\Psi) - \Psi_i \cdot q_i \geq v_i(\Phi \cup \Psi_{-i}) - \Phi \cdot q_i. \end{aligned}$$

Hence, $\Psi_i \in D_i(q_i | \Psi_{-i})$. *Q.E.D.*

LEMMA 17—Competitive Equilibria Form Individually Rational Outcomes: *Suppose* $(\Psi, \{p_i\}_{i\in I})$ *is a competitive equilibrium in a transferable utility matching environment. Then* $Y = \{(\omega, t^\omega)|\omega \in \Psi, t_i^\omega = p_i^\omega\ \forall i \in N(\omega)\}$ *is individually rational.*

PROOF: Since $(\Psi, \{p_i\}_{i\in I})$ is a competitive equilibrium, for each $i \in I$,

$$\Psi_i \in \arg\max_{\Phi\subseteq\Omega_i}\left\{v_i(\Phi \cup \Psi_{-i}) - \sum_{\omega\in\Phi} p_i^\omega\right\} \Rightarrow \Psi_i \in \arg\max_{\Phi\subseteq\Psi_i}\left\{v_i(\Phi \cup \Psi_{-i}) - \sum_{\omega\in\Phi} p_i^\omega\right\}$$

$$\Leftrightarrow \quad Y_i \in \arg\max_{Z\subseteq Y_i}\left\{v_i\big(\tau(Z) \cup \Psi_{-i}\big) - \sum_{(\omega,t^\omega)\in Z} t_i^\omega\right\} = C_i(Y_i|Y_{-i}).$$

It follows that Y is individually rational. *Q.E.D.*

PROOF OF THEOREM 2 (STABILITY WITH GROSS COMPLEMENTARITY): Part (i). Follows from part (ii) of Proposition 3.

Part (ii). ($\Leftarrow$) Suppose $\tau(Y) = \Omega^*$ and $(\Omega^*, \{p_i\}_{i\in I})$ is a competitive equilibrium such that $t_i^\omega = p_i^\omega$ for each $(\omega, t^\omega) \in Y$ and $i \in I$, but Y is not stable. By Lemma 17, Y is individually rational. Then, by Corollary 1(ii), there must be some individually rational $Z \supset Y$. Then, by Lemma 15, there exists a conditionally efficient set of primitive contracts Ψ^* such that $\Psi^* \supseteq \tau(Z) \supset \tau(Y) = \Omega^*$, a contradiction since Ω^* is the largest conditionally efficient set of primitive contracts.

($\Rightarrow$) Suppose Y is stable. Then Y is individually rational, and by Lemma 15 there exists a conditionally efficient set of primitive contracts Ψ^* such that $\Psi^* \supseteq \tau(Y)$. Then $\tau(Y) \subseteq \Omega^*$. By Proposition 3, there exists a competitive equilibrium $(\Omega^*, \{p_i'\}_{i\in I})$. By Lemma 16, there exists a competitive equilibrium $(\Omega^*, \{q_i\}_{i\in I})$ such that $q_i^\omega = t_i^\omega$ for each $(\omega, t^\omega) \in Y$ and $i \in N(\omega)$. It follows from Lemma 17 that $Y' = Y \cup \{(\omega, t^\omega)|\omega \in \Omega^* \setminus \tau(Y), t_i^\omega = q_i^\omega\ \forall i \in N(\omega)\}$ is individually rational. Since $Y' \supseteq Y$, and Y is stable, it follows from Corollary 1(ii) that $Y' = Y$. Then $\{(\omega, t^\omega)|\omega \in \Omega^* \setminus \tau(Y), t_i^\omega = q_i^\omega\ \forall i \in N(\omega)\}$ is empty, implying $\Omega^* = \tau(Y)$. *Q.E.D.*

PROOF OF LEMMA 7 (ALGORITHM FOR STABLE OUTCOMES IN TU MATCHING ENVIRONMENTS): By Topkis (1998, Theorem 2.4.3), $F_\vee$ is monotone. Then, by Tarski's fixed point theorem, its set of fixed points is a complete lattice, and so has a largest element Φ^*. By Proposition 3, the collection of fixed points of F is a complete lattice, and so has a largest element Ψ^*. Suppose $\Psi^* \neq \Phi^*$. Since $F_\vee$ is a selection from F, Φ^* is also a fixed point of F. Then $\Phi^* \subset \Psi^*$. Since Φ^* is the largest fixed point of $F_\vee$, it follows that $F_\vee(\Psi^*) \neq \Psi^*$. Since $\Psi^* \in F(\Psi^*)$, it must be that $F_\vee(\Psi^*) \supset \Psi^*$. Then, by Corollary 4, $F_\vee$ has a fixed point $\Phi' \supseteq \Psi^* \supset \Phi^*$, a contradiction. *Q.E.D.*

PROOF OF PROPOSITION 4 (MONOTONE COMPARATIVE STATICS): (i) By Lemma 1, contracts are complements in each $M(\theta)$. Denote each agent's choice correspondence in $M(\theta)$ by $C_i(Y_i|Y_{-i}, \theta) = \arg\max_{S\subseteq Y} u_i(S \cup Y_{-i}, \theta)$ and her acceptance function by $A_i(Y, \theta) = \bigcup_{Z\in C_i(Y_i|Y_{-i},\theta)} Z$. Denote the aggregate acceptance function in $M(\theta)$ by $A(Y, \theta) = \bigcap_{i\in I}(A_i(Y, \theta) \cup Y_{-i})$; by Theorem 1, $X^*(\theta)$ is the largest fixed point of $A(\cdot, \theta)$.

By Theorem 4 in Milgrom and Shannon (1994), $C_i(Y_i|Y_{-i}, \theta)$ is increasing in θ and Y. By Lemma 3, $A_i(Y, \theta)$ is the largest element of $C_i(Y_i|Y_{-i}, \theta)$ and so is also increasing in θ and Y. Then so is $A(Y, \theta)$. The result follows from Topkis (1998, Theorem 2.5.2(b)).

(ii) If each $v_i(\Psi, \theta)$ is supermodular in Ψ_i and has increasing differences in (Ψ_i, Ψ_{-i}) and in (Ψ_i, θ), then each $\tilde{v}_i(\Psi|\Phi, \theta)$ is supermodular in Ψ and has increasing differences in (Ψ, Φ) and in (Ψ, θ). Then the same is true of $W(\Psi|\Phi, \theta) = \sum_{i\in I} \tilde{v}_i(\Psi|\Phi, \theta)$. Then, by Topkis (1998, Theorem 2.8.1), the conditional optimizer correspondence $F(\Phi, \theta) = \arg\max_{\Psi\subseteq\Omega} W(\Psi|\Phi, \theta)$ is increasing in both arguments (in the strong set order) and for each Φ, θ, $F(\Phi, \theta)$ is a complete lattice. The result follows from Topkis (1998, Theorem 2.5.2(b)). *Q.E.D.*

PROOF OF PROPOSITION 5 (EFFECTS OF BUNDLING CONTRACTS NOT SIGNED IN THE STABLE OUTCOME): The proof relies on Lemma B.2 in the Supplemental Material. Part (i) (NTU): Let A and $\hat{A}$ be the aggregate acceptance functions for M and $\hat{M}$, respectively. (a) By Theorem 1, $\Omega^* = A(\Omega^*)$. By Lemma B.2(i), $\Omega^* = A(\alpha(\alpha^{-1}(\Omega^*)))$. By Lemma B.2(viii), $\Omega^* = \alpha(\hat{A}(\alpha^{-1}(\Omega^*))) \Rightarrow \alpha^{-1}(\Omega^*) = \alpha^{-1}(\alpha(\hat{A}(\alpha^{-1}(\Omega^*))))$; by Lemma B.2(i), $\alpha^{-1}(\Omega^*) = \hat{A}(\alpha^{-1}(\Omega^*))$. By Theorem 1, $\hat{\Omega}^*$ is the largest fixed point of $\hat{A}$, so it must be that $\alpha^{-1}(X^*) \subseteq \hat{\Omega}^*$. (b) By Theorem 1, $\hat{\Omega}^* = \hat{A}(\hat{\Omega}^*)$, hence $\alpha(\hat{\Omega}^*) = \alpha(\hat{A}(\hat{\Omega}^*)) = A(\alpha(\hat{\Omega}^*))$ (by Lemma B.2(viii)). By Theorem 1, Ω^* is the largest fixed point of A, so it must be that $\alpha(\hat{\Omega}^*) \subseteq \Omega^*$.

Part (ii) (TU): Let F and $\hat{F}$ be the conditional optimizer correspondences for M and $\hat{M}$, respectively. (a) First note that by Lemma B.2(i), $\alpha(\alpha^{-1}(\Omega^*)) = \Omega^*$. Since Ω^* is conditionally efficient, by definition $\Omega^* \in F(\Omega^*) \Leftrightarrow \alpha(\alpha^{-1}(\Omega^*)) \in F(\alpha(\alpha^{-1}(\Omega^*)))$. By Lemma B.2(x), there exists $\hat{\Phi} \in \hat{F}(\alpha^{-1}(\Omega^*))$ with $\hat{\Phi} \supseteq \alpha^{-1}(\Omega^*)$. By Lemma 6, $\hat{F}$ is increasing, and for each $\Psi \subseteq \hat{\Omega}$, $\hat{F}(\Psi)$ is a complete lattice. Then by Corollary 4, $\hat{F}$ has a fixed point $\Psi^* \supseteq \alpha^{-1}(\Omega^*)$. Since $\hat{\Omega}^*$ is the largest conditionally efficient set of primitive contracts in $\hat{M}$, it follows that $\hat{\Omega}^* \supseteq \Psi^* \supseteq \alpha^{-1}(\Omega^*)$. (b) Since $\hat{\Omega}^*$ is conditionally efficient, by definition $\hat{\Omega}^* \in \hat{F}(\hat{\Omega}^*)$. By Lemma B.2(xi), there exists $\Phi \in F(\alpha(\hat{\Omega}^*))$ with $\Phi \supseteq \alpha(\hat{\Omega}^*)$. By Lemma 6, F is increasing, and for each $\Psi \subseteq \Omega$, $F(\Psi)$ is a complete lattice. Then, by Corollary 4, F has a fixed point $\Psi^* \supseteq \alpha(\hat{\Omega}^*)$. Since Ω^* is the largest conditionally efficient set of primitive contracts in M, it follows that $\Omega^* \supseteq \Psi^* \supseteq \alpha(\hat{\Omega}^*)$. *Q.E.D.*

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