The Marriage Model with Search Frictions

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Consider a heterogeneous agent matching model in which the payoff of each matched individual is a fixed function of both partners' types. In a 1973 article, Becker showed that assortative matching arises in a frictionless setting simply if everyone prefers higher partners. This paper shows that if finding partners requires time-consuming search and individuals are impatient, then productive interaction matters. Matching is positively assortative—higher types match with higher sets of types—when the proportionate gains from having better partners rise in one's type. With multiplicatively separable payoffs, these proportionate gains are constant in one's type, and "block segregation" arises, a common finding of the literature.

I. Introduction

Our understanding of the economics of social mating or other partnerships owes to Becker (1973): Assume that production arises from pairwise interaction and depends solely on underlying types, with competitive wages allocating output. If individual types are strategic complements—also known as *supermodularity*, namely, that higher types enjoy higher match payoff gains as their match partner rises—then the resulting matching is *positively assortative*. Shimer and Smith (2000) revis-

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ited Becker's model with matching instead preceded by a time-consuming random search process. Assuming the Nash bargaining solution—where everyone earns his or her outside options plus an equal share of the joint match surplus over these outside options—they find that matching is assortative only when production obeys stronger complementarity conditions.

In this paper I focus on matching markets with exogenously specified match payoffs. Becker (1973) briefly considers this paradigm for social matches, such as marriage proper. Here, wages are not available to equilibrate matches, and payoffs are neither transferable nor quasi-linear in a transferable resource. Stealing a term from cooperative game theory, Smith (1992) called this *nontransferable utility* (NTU), in contrast to the *transferable utility* (TU) world of Shimer and Smith (2000). The NTU setting is obviously a polar case and ignores intramatch transfers such as doing the dishes or deciding where to live. But in defense of the NTU model, disagreements about matching in social settings are not uncommon. And whenever we observe a potential match or split desired by one party but not the other, utility is obviously not fully transferable. For the total match surplus either is positive or is not, and there can be no disagreement.

Assortative matching arises in this matching world whenever everyone prefers higher partners. For without prices, the currency of the social matching market is quite indivisible—oneself; the more prized mates with more to "spend" will then "buy" each other. Perfectly assortative matching follows as the market clears top to bottom.

This paper studies NTU matching in which everyone faces a time-consuming and random search for partners, just as in Shimer and Smith's article. For if such frictions matter *anywhere*, they do so on the social matching scene.[1] I find that Becker's simple proviso that everyone prefer higher partners no longer delivers assortative matching. After accounting for the value of time spent searching, I show that a stronger condition than even complementary production is needed for assorting: complementarity of log payoffs. I then show that in the knife-edge transitional case with multiplicatively separable production, an interval of "highest" individuals match only with each other, the next highest match only with each other, and so on; there is no intermingling. This yields a new development of this well-known *block segregation* result,[2] where it is now seen as a special case of a coherent bigger matching picture.

Let me summarize the framework. First, individuals are heterogeneous, having one-dimensional scalar types. Second, the payoff that

[1] Bergstrom and Bagnoli (1993) study marriage from a search perspective.

[2] It has been found by McNamara and Collins (1990), Smith (1992), Bloch and Ryder (1994), Eeckhout (1996), Burdett and Coles (1997), and Chade (2001).

anyone earns increases in the type of his partner. Thus everyone has identical *ordinal* preferences over partners; namely, higher is better. While this is a pure quality world with agreement on relative beauty, individuals might well disagree on absolutes, since their payoffs also depend on their own types. Third, the only cost of search is time, and the only decision margin is with whom to match. Of course, both parties must first approve a potential match.

Search frictions not only are a somewhat realistic assumption of the "real-world" marriage market but also negate one unrealistic aspect of the Walrasian context. For intuitively, individuals must accept a range of possible matches, some more preferred than others; they need not match with their ideal partner. Since higher partners are preferred in an NTU search setting, anyone will agree to matches with all types above some *marginal partner*. This corresponds to the reservation wage in the theory of wage search. Since matching sets are no longer singletons, assortative matching must be reformulated. As in Shimer and Smith (2000), I call matching (strictly) assortative when the marginal partner is (strictly) increasing in one's type. This says that higher types are choosier.

I seek a theorem that depends only on productive interaction. Namely, the threshold partner is monotonic in one's type *for any* (atomless) type distribution and any level of search frictions. Morgan (1995) establishes that with a constant search cost, if higher types derive a higher marginal gain from "matching up," then they entertain higher standards for their match partners. This leads once more to Becker's TU condition that types be productively complementary, which gives higher individuals a greater benefit from holding out for better partners. But higher types have higher expected continuation payoffs and must be compensated for a greater value of time.

One's threshold partner depends on the search cost. With endogenous opportunity time costs, higher types are choosier only if they enjoy more lucrative rewards from holding out than even supermodularity provides: My main finding here is that strictly assortative matching obtains when the production function $f(x, y)$ is *strictly log supermodular*: The simplest statement of this is the inequality $f(x_2, y_2)f(x_1, y_1) > f(x_2, y_1)f(x_1, y_2)$ for any types $x_2 > x_1$ and $y_2 > y_1$. I show how this ensures that the opportunity costs of using any matching set rise proportionately faster in one's own type than the benefits, thereby elevating their threshold partners.

Decision making in a search model always entails comparing a certain prize with an uncertain one (the outside option, which is an expectation). I have assumed that everyone has increasing preferences across partners. But in a world of uncertainty, such as search, this does not mean that everyone behaves alike, even given identical opportunity sets.

Individuals with the same opportunity sets have identical preferences across gambles and thus make identical choices, if and only if they entertain identical cardinal preferences. So motivated, imagine the match payoffs $f(x, y)$ as the cardinal utility function over partners y of an individual of type x. Using the classic (1944) result of von Neumann and Morgenstern, I prove that cardinal preferences coincide exactly when $f(x_2, y_2)f(x_1, y_1) \equiv f(x_2, y_1)f(x_1, y_2)$. This is the knife-edge case of log supermodularity. Typical payoff functions in the literature such as $f(x, y) = xy$ or $f(x, y) = y$ have exploited this property. This forces block segregation, for types in each interval have the same opportunity set (those willing to match with them) and thus make the same matching decisions given the same cardinal preferences.

In the search literature, this segregation result was discovered repeatedly in the 1990s, and its history is recounted in Section V*B*. In this paper, it serves as a springboard to motivate my log supermodularity condition for strict assorting.

As in Shimer and Smith's article, I prove existence of a search equilibrium in value function space. But the proof style must radically differ in this NTU context since value functions need no longer be continuous; this owes to the unexploited matching rents. In fact, since I allow that output might fall in one's own type for a fixed partner, values need not even be weakly increasing. The existence proof instead exploits the fact that the best-reply dynamics produce value functions of bounded variation. That value functions may be discontinuous is my key point of departure from Shimer and Smith, and this is a critical distinction between the existence proof methodology of TU and NTU search models.

After the model, I present the transition from block segregation to monotonic matching sets and the linked passage from identical von Neumann–Morgenstern preferences to log supermodularity. Simulated economies depict this result. I conclude with a big-picture overview of the assortative matching literature and a novel existence proof.

II. The Frictionless Matching Model

There is an atomless unit mass of agents who are indexed by their exogenous and publicly observable productivity type $x \in [0, 1]$. The fraction of agents with type at most y is denoted $L(y)$. I assume throughout that L is differentiable, with Borel measurable type density l. For technical reasons later on, I also require that l be positive and boundedly finite: $0 < \underline{l} < l(x) < \bar{l} < \infty$ for all x.

Consider the following coordination game played by these individuals. Everyone chooses another type to match with, and a match is formed if both parties choose each other. Each type x earns payoff $f(x, y) > 0$ in a match with type y, and zero if unmatched. Output need not be symmetric in a match; in fact, asymmetry plays a key role in the analysis. For instance, $f(x, y) = y$ means that everyone cares only about her partner's type. For technical reasons, I assume that production f is continuous and continuously differentiable, and so bounded above by $\bar{f} < \infty$ on $[0, 1]^2$.

Assumption 1 (Monotonicity). Preferences over partners are increasing; that is, $f(x, y)$ rises in y.

Given assumption 1, there is a unique Nash equilibrium in which everyone matches, and in it, every type x matches with the same type x. I do not prove this since it intuitively extrapolates on the finite agent result mentioned in Becker (1973). But it does speak to the extremely weak and natural condition for matching to be assortative: namely, that higher is better. I next add search frictions and find a less obvious condition.

III. Matching with Search Frictions

I now develop a continuous-time, infinite-horizon matching model in which meeting other agents is time-consuming and haphazard because of search frictions.

At any instant in continuous time, an agent is either *matched* or *unmatched*. Only the unmatched engage in (costless) search for a new partner. When two unmatched agents meet, either may veto the proposed match; it is consummated if both accept. Type x earns a flow payoff $f(x, y)$ when matched with y. Each agent maximizes her expected present value of payoffs, discounted at the interest rate $r > 0$. Since a match that is profitable to accept is profitable to sustain in a steady-state environment, I simplify the notation by ignoring the possibility of quits.

To fix a steady-state population of unmatched agents, assume exogenous match dissolutions. Nature randomly destroys any match with a constant flow probability (Poisson rate) $\delta > 0$; that is, it lasts an elapse time of t or more with chance $e^{-\delta t}$. At the moment the match is destroyed, both agents reenter the search pool.

Unmatched individuals periodically meet others drawn at random from the unmatched pool. One meets any $y \in Y \subseteq [0, 1]$ at an exponential rate proportional to the mass of those unmatched in Y: $\rho \int_Y u(y)dy$. Here, $u \leq l$ is the *unmatched density function*; that is, $\int_Y u(x)dx$ is the mass of unmatched agents with types $x \in Y \subseteq [0, 1]$. As in Shimer and Smith (2000), the descriptive theory extends to any anonymous search technology, but this quadratic search technology[3] is needed in the existence proof.

[3] Namely, the chance that one meets anyone in the search pool is independent of the number of other potential partners. Diamond and Maskin (1979) introduced this term.

A steady-state (pure) strategy for a type x agent is a time-invariant set[4] $A(x)$ of agents with whom x is willing to match. The *opportunity set* $\Omega(x) = \{y|x \in A(y)\}$ is an inverse set consisting of agents willing to match with x. Type x's matching set is $\mathcal{M}(x) \equiv A(x) \cap \Omega(x)$, where (x, y) is mutually agreeable iff $y \in \mathcal{M}(x)$. Let $\mathcal{M} : [0, 1] \rightrightarrows [0, 1]$ be the match correspondence and α the match indicator function: This means that $\alpha(x, y) = 1$ if $y \in \mathcal{M}(x)$ and zero otherwise.

In a steady state, the flow creation and flow destruction of matches for every type of agent must exactly balance. The density of matched agents $x \in [0, 1]$ is the difference $l(x) - u(x)$; these agents' matches exogenously dissolve with flow probability δ. The flow of matches created by unmatched type x agents is $\rho u(x) \int_{\mathcal{M}(x)} u(y)dy$. Putting this together, in a steady state for all types $x \in [0, 1]$, we have

$$\delta[l(x) - u(x)] = \rho u(x) \int_{\mathcal{M}(x)} u(y)dy = \rho u(x) \int_0^1 \alpha(x, y)u(y)dy. \quad (1)$$

IV. Values and Search Equilibrium

Individuals must trade off the immediate rewards of agreeing to match against the option value of remaining unmatched. To this end, let $\mathcal{V}(x)$ denote the average present value to type x of the unmatched status, presuming an optimal steady-state strategy, and $\mathcal{V}(x|y)$ her average present value[5] when matched to type y. While unmatched, type x earns nothing; but at flow rate $\rho \int_{\mathcal{M}(x)} u(y)dy$, she meets and matches with some $y \in \mathcal{M}(x)$, enjoying a capital gain $[\mathcal{V}(x|y) - \mathcal{V}(x)]/r$. Consequently, we have

$$\begin{aligned}\mathcal{V}(x) &= \frac{\rho}{r}\int_{\mathcal{M}(x)} [\mathcal{V}(x|y) - \mathcal{V}(x)]u(y)dy \\ &= \frac{\rho}{r}\int_{\Omega(x)} \max\langle \mathcal{V}(x|y) - \mathcal{V}(x), 0\rangle u(y)dy. \quad (2)\end{aligned}$$

Also, x enjoys a flow payoff $f(x, y)$ when matched with y. Her match is destroyed with flow chance δ, and she suffers a capital loss $[\mathcal{V}(x|y) - \mathcal{V}(x)]/r$. Hence,

[4] Sets are Borel measurable. Stationary acceptance sets in a stationary world are assumed without loss of generality: As no one affects the future of the economy, any acceptance set optimal at time s remains so at time $t > s$. That agents of the same type use the same strategy is also not a restriction.

[5] The average present value is the product of the present value and the interest rate.

$$\mathcal{V}(x|y) = f(x, y) - \frac{\delta[\mathcal{V}(x|y) - \mathcal{V}(x)]}{r} = \frac{rf(x, y) + \delta\mathcal{V}(x)}{r + \delta}. \quad (3)$$

Altogether, when (2) and (3) are combined, her unmatched value is

$$\mathcal{V}(x) = \frac{\rho\int_{\mathcal{M}(x)} f(x, y)u(y)dy}{(r+\delta) + \rho\int_{\mathcal{M}(x)} u(y)dy} = \frac{\int_{\mathcal{M}(x)} f(x, y)u(y)dy}{\psi + \int_{\mathcal{M}(x)} u(y)dy}, \quad (4)$$

where $\psi = (r + \delta)/\rho$ is a simple measure of search frictions. This is less than the expected output of a match because the match does not start immediately.

In equilibrium, everyone maximizes her present discounted payoffs, taking all other opportunity sets as given. For simplicity, a match is consummated iff both parties find it weakly agreeable. Finally, I ask that all dynamics be in a steady state.

A search equilibrium is a triple $(\mathcal{V}, \mathcal{M}, u)$ characterized by three properties:

- Opportunities via values: Given the matching sets and unmatched rates, the average present unmatched values $\mathcal{V}(x)$ are properly calibrated, that is, satisfy (4).
- Optimal matching: Matching sets $\mathcal{M}(x)$ are optimally chosen; namely, $y \in \mathcal{M}(x)$ if and only if $f(x, y) \geq \mathcal{V}(x)$ and $f(y, x) \geq \mathcal{V}(y)$.
- Pointwise steady state: Given (1), the unmatched density $u(x)$ obeys the balanced flow equation

$$u(x) = \frac{\delta l(x)}{\delta + \rho\int_{\mathcal{M}(x)} u(y)dy}. \quad (5)$$

Simply, the flow into the unmatched pool balances the flow out at every type.

To avoid getting sidetracked here, the next key result is proved in the Appendix.

Proposition 1 (Existence). Given assumption 1, a search equilibrium exists, and the average present value function $\mathcal{V}$ is almost everywhere differentiable.

The proof must produce a triple $(\alpha, u, \mathcal{V})$ of match indicator functions α, unmatched densities u, and value functions $\mathcal{V}$. The proof first requires that the map $\mathcal{V} \mapsto \alpha \mapsto u$ be continuous. In contrast to the TU model of Shimer and Smith (2000), value functions need not be continuous. In a scenario that may emerge, imagine that a mass of the best types matches with every $y \geq \frac{2}{3}$. When their acceptance rule is shifted to $y \geq \frac{2}{3} - \epsilon$, the value function dramatically jumps in $(\frac{2}{3} - \epsilon, \frac{2}{3})$. Because of this continuity failure in the sup norm, existence has been stymied. For this reason, existence proofs in this literature are all constructive. The general existence argument here uses the weak-* topology for value functions—the standard topology for probability distributions. Namely, $\mathcal{V}_n$ tends to $\mathcal{V}$ if the 0-1 indicator functions $\mathbb{I}_{f(x,y)\geq \mathcal{V}_n(x)}$ converge pointwise to $\mathbb{I}_{f(x,y)\geq \mathcal{V}(x)}$ for any y. *This topology is not defined by any metric.* Loosely, it treats value functions as if they were cumulative distribution functions for a probability measure.

Why are values only almost everywhere differentiable? The value function slope jumps precisely where the matching set jumps, since (4) otherwise yields a formula for $\mathcal{V}'$ upon differentiation. Such jumps are not impossible and occur in Section V*B*, for instance.

I first focus on a given individual and consider how she compares potential matches. Since type x matches with any type y providing output $f(x, y) \geq \mathcal{V}(x)$, preferences over match partners are monotone.

Lemma 1 (Monotonic preferences). Given assumption 1, in a search equilibrium, if type x accepts y, then she is strictly willing to accept any $z > y$.

That preferences are monotonic already buys us one useful conclusion. Since anyone willing to match with x is willing to match with $y > x$, it follows that opportunity sets are monotone.

Lemma 2 (Increasing opportunities). Given assumption 1, in a search equilibrium, higher types have weakly larger opportunity sets: $\Omega(x) \subseteq \Omega(y)$ for $y > x$.

This monotonicity result is a key property. In particular, everyone matches.

Lemma 3 (Everyone matches). Given assumption 1, for all types $x > 0$, the matching set $\mathcal{M}(x)$ has positive measure in any search equilibrium, and $\mathcal{M}(0)$ is nonempty.

Proof. If $\mathcal{M}(x)$ has zero measure, then $\mathcal{V}(x) = 0$ by (4). Then $\Omega(x)$ has zero measure by (2) and (3). From lemma 2, $\Omega(y) \subseteq \Omega(x)$ has zero measure for $y \leq x$. So $\mathcal{V}(y) = 0$ and $\mathcal{A}(y) = [0, 1]$ for all $y \leq x$, contrary to $\Omega(x)$ zero measure. Finally, $\mathcal{M}(0) = \varnothing$ implies $\mathcal{V}(0) = 0$, and $0 \in \mathcal{A}(x)$, a contradiction. QED

All individuals have reservation partners for matching, and the resulting matching set consists of those types for whom there is a double coincidence of wants.

Lemma 4 (Marginal partners). In a search equilibrium, we have $\mathcal{A}(x) = [a(x), 1]$ and $\Omega(x) = \{y : y \geq a(x)\}$, and therefore $\mathcal{M}(x) = \{y : y \geq a(x), x \geq a(y)\}$. Also, the marginally acceptable type $a(x)$ for type x obeys $f(x, a(x)) \equiv \mathcal{V}(x)$.

Proof. First, $f(x, 1) \geq \mathcal{V}(x)$. Since $f(x, y)$ is continuous in y, if ever $f(x, y) < \mathcal{V}(x)$, then x matches with types above an indifference partner $a(x) > y$. QED

Lemma 5 (The threshold partner). Assume assumption 1. In a search equilibrium, the threshold partner $a(\cdot)$ and value function $\mathcal{V}(\cdot)$

are each continuous exactly where they are differentiable. Further, any jump points of these two functions coincide.

Proof. Because $f(x, a(x)) = \mathcal{V}(x)$ by lemma 4 and the partials f_1 and f_2 exist, whereas $\mathcal{V}'$ exists almost everywhere by propositon 1, we have $f_1(x, a(x)) + f_2(x, a(x))a'(x) = \mathcal{V}'(x)$. So $a(\cdot)$ must be differentiable whenever $\mathcal{V}(\cdot)$ is. By the fundamental theorem of calculus applied to (4) and from the simple structure of $\mathcal{M}(x)$ deduced in lemma 4, $\mathcal{V}(\cdot)$ is differentiable whenever $a(\cdot)$ is continuous—and so certainly when it is differentiable. Thus points of continuity and differentiability of these functions precisely match. QED

V. Assortative Matching

A. *What Is Assortative Matching?*

Shimer and Smith (2000) developed a simple set-valued generalization of assortative matching. Matching is positively assortative if, when any two agreeable matches are severed, both the greater two and lesser two types can be agreeably rematched. This definition has many nice properties and reduces to the frictionless notion for single-valued matches: Everyone is willing to match with her own type. For if x will not match with her own type, then no higher types will either. But then $\Omega(x) \cap \mathcal{A}(x) = \emptyset$, which would contradict lemma 3.

Since matching sets for any type $x > 0$ have positive measure, if matching is positively assortative, then there exists a weakly monotonic increasing marginal partner $a(\cdot)$ for $\mathcal{A}$, whereas the inverse opportunity set assumes the form $\Omega(x) = [0, b(x))$ for an increasing (and possibly set-valued) upper bound $b(\cdot)$.[6] So $\mathcal{M}(x) = [a(x), b(x))$.

Shimer and Smith also explore negatively assortative matching, with the analogous definition: If $x_1 < x_2$ and $y_1 < y_2$, then $y_1 \in \mathcal{M}(x_1)$ and $y_2 \in \mathcal{M}(x_2)$ imply $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$. Shimer and Smith show how this reduces to $\mathcal{M}(x) = [a(x), b(x))$, where $a(\cdot)$ and $b(\cdot)$ are decreasing. This produces a decreasing opportunity set, contrary to lemma 2. So we have the following lemma.

LEMMA 6 (Not negatively assortative matching). For no productive interaction obeying assumption 1 does negatively assortative matching arise in a search equilibrium of this NTU model.

[6] Shimer and Smith prove this in two steps. First, if matching sets are nonempty, then they must be convex. For if not, then some x matches with both y_1 and y_3 where $y_1 < y_3$ but not the middling y_2. If y_2 matches with $x' < x$, then (x', y_2) and (x, y_1) match, and thus so does (x, y_2). The case $x' > x$ is similar. But convex matching sets have lower and upper bounds, $a(x)$ and $b(x)$. If $a(x)$ or $b(x)$ is not monotonic, then some y has a nonconvex matching set. Finally, if $a(x)$ is decreasing, then so is $b(x)$, and the claim becomes trivial.

B. Block Segregation

Toward understanding who matches with whom, let us assume first that no one's type affects her preferences over gambles across prospective partners and being unmatched. To wit, all types enjoy identical von Neumann–Morgenstern (cardinal) preferences over each other and the unmatched state. For instance, $f(x, y) = y$ or $f(x, y) = xy^2$ enjoy this property, whereas $f(x, y) = x + y$ does not. Any type $x > 0$ with the latter payoff function is discontinuously hurt by the unmatched status, since even a match with $y = 0$ pays him $x > 0$. That multiplicative separability is common to the first two examples and not present in the third is no accident, as the Appendix proves.[7]

LEMMA 7 (Identical cardinal preferences). Individuals have the same cardinal (von Neumann–Morgenstern) preferences over matches iff $f(x, y) = \gamma_1(x)\gamma_2(y)$, for functions γ_1 and γ_2 with $\gamma_1 > 0$.

It is a truism from consumer theory that any individuals with the same preferences and identical opportunities will make the same choices. I can now exploit this fact alone to deduce a crucial phenomenon about the cross-sectional matching.

PROPOSITION 2 (The folk NTU matching result). Assume that everyone has the same von Neumann–Morgenstern preferences over matches. Then the type space $[0, 1]$ partitions into disjoint classes $[\theta_1, 1] \cup [\theta_2, \theta_1] \cup \cdots$, with $1 > \theta_1 > \theta_2 > \cdots$, such that $y \in \mathcal{M}(x)$ iff x and y are in the same class. There are infinitely many classes iff $\gamma_2(0) = 0$.

Proof. In light of (4), given search frictions and impatient individuals, large enough types x will be accepted by one and all. So there is a universally "prized set" $\chi \neq \emptyset$ of high enough types $x \in [0, 1]$ for whom $\Omega(x) = [0, 1]$. By lemma 7, $\mathcal{A}(x) = \mathcal{A}(y)$ for any two types $x, y \in \chi$, and in fact $\mathcal{A}(x) = \mathcal{A}(y) = \chi$. By lemma 1, $\chi = [\theta_1, 1]$.

Next, $\Omega(x) \subseteq [0, \theta_1)$ for all $x \in [0, \theta_1)$, and so we may repeat the above argument on this restricted domain. Finally, when $\gamma_2(0) = 0$, it is never optimal to set $\theta_k = 0$, by lemma 4. In this case, there are infinitely many classes. QED

This block segregation phenomenon has proved a popular result. In a distinctly applied math paper, McNamara and Collins (1990) happen on a version of it for the specific case of employer-employee job search.[8]

[7] Smith (1992) pointed this out and also noted that if there are type-dependent explicit search costs $c(x)$, identical von Neumann–Morgenstern preferences will obtain also if costs are proportional to $c(x) = \bar{c}\gamma_1(x)$. The Appendix actually establishes this more general result.

[8] Careful inspection reveals that their model is nonstationary, since they do not replenish their supply of unmatched agents. But the result cannot possibly hold out of the steady state, as seen in Smith (1992).

I then rediscovered it (with optional search costs) in Smith (1992).[9] It was subsequently found yet again in Bloch and Ryder (1994), then the working paper prequel to Burdett and Coles (1997),[10] and then a prequel to Chade (2001)[11] and Eeckhout (1996). With the exception of Smith (1992), these papers do not comment on the identical von Neumann–Morgenstern preference interpretation, but just study analytically convenient functions such as $f(x, y) = y$ or $f(x, y) = xy$. They did not allow one's own type to influence output, from which this paper derives its richness.

Block segregation is a distinctly NTU phenomenon. In the transferable utility world of Shimer and Smith in which all positive surplus matches are consummated, any such matching set discontinuity is inconsistent with a continuous payoff function. But in an NTU context, there is no bargaining, and some matching rents are left unexploited. Block segregation highlights the dramatic fashion in which this "inefficiency" plays out.

C. *Strictly Assortative Matching*

Block segregation is pathological in two senses. First, discontinuities in a continuous model usually command skepticism. Second, the result just does not ring true of the "real world." To my knowledge, there are no documented cases of block segregation.

Here block segretation suggests how productive interaction might lead to strictly assortative matching. For we can escape block segregation insofar as von Neumann–Morgenstern preferences diverge. Since the implied marginal partner is weakly monotonic increasing, block segregation is a nonstrict assortative matching. Namely, call matching strictly positively assortative if $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$ imply $y_1 \in \text{int}\mathcal{M}(x_1)$ *and* $y_2 \in \text{int}\mathcal{M}(x_2)$ for $x_1 < x_2$ and $y_1 < y_2$. The upper and lower bounds of the matching graph then strictly rise except at zero or one, as in figure 1.

As it turns out, identical von Neumann–Morgenstern preferences are more precisely the condition for constant acceptance sets and yield only block segregation as a consequence. It is in fact the knife-edge case of an increasing marginal partner. To firm up this intuition, observe how the function $f(x, y) \equiv \gamma_1(x)\gamma_2(y)$ in lemma 7 holds iff $f(x_2, y_2)/f(x_2, y_1) = f(x_1, y_2)/f(x_1, y_1)$ for all $x_2 > x_1$, $y_2 > y_1$. This is my key observation,

[9] That paper studied a nonstationary version of the class of models identified as having identical von Neumann–Morgenstern preferences and specialized some of the analysis to the steady state.

[10] They also added a very nice uniqueness theorem for atomless-type distributions.

[11] Chade showed that it also arises with fixed search costs and additive payoffs obeying $f_{12} = 0$.

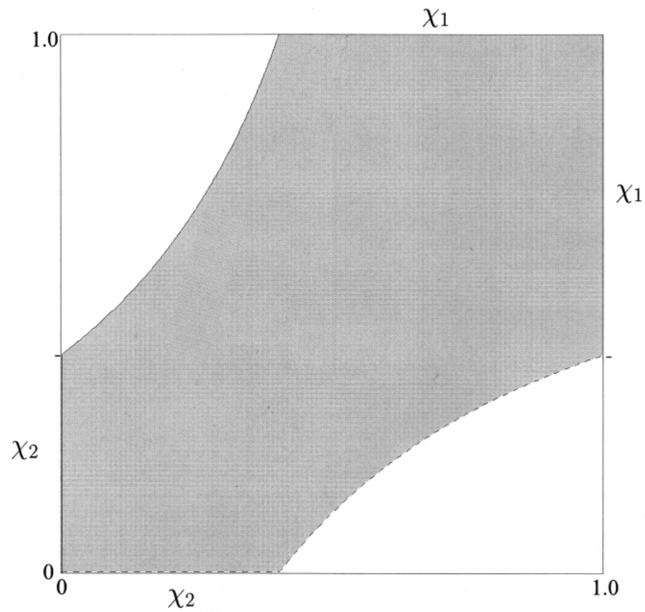

FIG. 1.—Example of supermodular matching. This figure depicts the graph of the matching for the log supermodular payoff function $f(x, y) = e^{xy}$. Assume a uniform type distribution on [0, 1] and parameters $r = 0.3$, $\delta = .1$, and $\rho = 30$. The dashed line and solid lines are the lower threshold $a(\cdot)$ and its reflection in the diagonal, respectively; in the first case, this reduces to $b(\cdot)$. Observe that matching is positively assortative since higher types are choosier. The sets χ_1 and χ_2 from the proof of proposition 3 are also depicted.

since it suggests looking at log supermodular functions with a strict inequality.[12]

ASSUMPTION 2 (Log supermodularity). Productive interaction is strictly log supermodular: match output f obeys $f_1(x, y_2)/f(x, y_2) > f_1(x, y_1)/f(x, y_1)$ for all x and $y_2 > y_1$.[13]

The flip side to this scenario is captured by strict log submodularity, with the less than inequality. Thus *identical von Neumann–Morgenstern preferences emerge as the knife-edge as one passes from log supermodular to log submodular payoff functions!*

It is easy to see how under the monotonicity assumption 1, log supermodularity is stronger than the more typically used supermodularity assumption that the marginal product $f_1(x, y)$ rises in y. With the pro-

[12] For probability density functions, this is also known as the monotone likelihood ratio property.

[13] This implies the discrete version $f(x_2, y_2)/f(x_2, y_1) > f(x_1, y_2)/f(x_1, y_1)$ for $x_2 > x_1$ and $y_2 > y_1$.

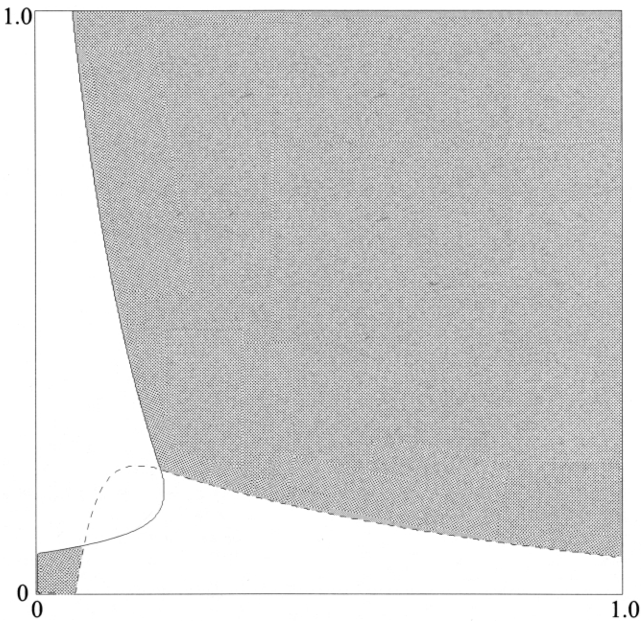

FIG. 2.—Example of submodular matching. This figure depicts the graph of the matching for the payoff function $f(x, y) = xy + x + y$, i.e., supermodular and yet log submodular. The parameters in fig. 1 are assumed, except $\rho = 3$. The matching fails to be positively assortative since most types have a falling lower threshold partner. This shows how the premise assumption 2 of proposition 3 is needed.

duction function $f(x, y)$ increasing in y, we see that the marginal product $f_1(x, y)$ must rise as a proportion of $f(x, y)$.

The next main result asserts that our assumptions guarantee that higher types are choosier since their marginal partner is increasing. The gain from imposing higher matching standards rises faster in one's type than the opportunity time cost of delayed matching. Figures 1 and 2 illustrate the necessity of log supermodularity for this result.

PROPOSITION 3 (Assortative matching). If output $f(x, y)$ is strictly increasing in y and log supermodular (i.e., assumptions 1 and 2), then matching is strictly positively assortative.

Proof. As in the proof of proposition 2, since search is costly, all high enough types $\chi_1 \subseteq [0, 1]$ are accepted by everyone; namely, $\Omega(x) = [0, 1]$ accepts χ_1.[14] I focus on these types χ_1 who choose their matching set, that is, $\mathcal{M}(x) = \mathcal{A}(x) = [a(x), 1)$. I show that log super-

[14] For no one x can do better than if she is certain to meet the maximal type 1. So if $a(x_k) \to 1$ for some (without loss of generality, convergent) sequence of types $x_k \to \bar{x}$ in [0, 1], then x_k close to $\bar{x}$ have values close to zero by (4), but a boundedly positive marginal partner, a contradiction.

modularity forces average match payoffs to rise proportionately faster than the lowest match payoff. To maintain optimality, higher types must have greater reservation partners. This effect is then reinforced for lower individuals below χ_1, for their matching set grows upward, as more individuals accept them.

Step 1: The changing value of time. By proposition 1, we may almost everywhere differentiate the value function $\mathcal{V}(x)$ in (4). In fact, by an envelope theorem, $\mathcal{V}$ is everywhere differentiable on χ_1 since $\Omega(x) = [0, 1]$ is constant there. This yields a formula for the marginal change in the value of time as one's type rises:

$$\mathcal{V}'(x) = \frac{\int_{a(x)}^{1} f_1(x, y)u(y)dy}{\psi + \int_{a(x)}^{1} u(y)dy} \quad (6)$$

for $x \in \chi_1$. Observe that a second term owing to the effect of changing x on $a(x)$ vanishes by the envelope theorem since the threshold $a^*(x)$ maximizes (4).

Step 2: The log supermodular inequality. With the quotient of (6) and (4), the proportionate change of the value function obeys

$$\frac{\mathcal{V}'(x)}{\mathcal{V}(x)} = \frac{\int_{a(x)}^{1} f_1(x, y)u(y)dy}{\int_{a(x)}^{1} f(x, y)u(y)dy} > \frac{f_1(x, a(x))}{f(x, a(x))}. \quad (7)$$

Intuitively, for any positive density u, the left-hand fraction will exceed the right-hand one as long as f_1/f is increasing in y; that is, f is log supermodular.[15]

Step 3: A rising lower threshold for χ_1. Turning from opportunity costs of time to optimality considerations, we now differentiate the optimality equation $\mathcal{V}(x) \equiv f(x, a(x))$ to get $\mathcal{V}'(x) = f_1(x, a(x)) + f_2(x, a(x))a'(x)$. Hence

$$\frac{f_1(x, a(x)) + f_2(x, a(x))a'(x)}{f(x, a(x))} > \frac{f_1(x, a(x))}{f(x, a(x))}$$

by inequality (7). Because $f_2 > 0$, we must have $a'(x) > 0$.

Step 4: A recursive argument. For all types $x \in \chi_1$, I have shown $a'(x) > 0$, whereas $b(x) \equiv 1$. Now consider the "next tier down" χ_2, namely, those types x for whom $\Omega(x) \cap \chi_1 \neq \emptyset$.[16] Since $\mathcal{A}$ and Ω are inverses, we have $b'(x) > 0$ for $x \in \chi_2$. Types in χ_2 have an additional

[15] This distribution-free inequality is a special case of a continuous variable generalization of inequality 3.3.15 in Mitrinović (1970). In a discrete world, $\frac{4}{5} > \frac{3}{4} \Rightarrow (4 + 3)/(5 + 4) > \frac{3}{4}$.

[16] The set χ_1 of globally acceptable types is indicated on the right and top of the box in fig. 1. The set χ_2 is the image of χ_1 in the lower threshold, and in this case, we happen to have $\chi_1 \cup \chi_2 = [0, 1]$. With less search frictions, the matching set would shrink, and more iteration steps would be needed.

incentive to raise their marginal partner, since their growing opportunity set further raises the option value of their time in the unmatched pool and makes them even choosier. To see this formally, see that an extra term appears on the right side of (6) whenever it is differentiable:

$$b'(x)\frac{\partial}{\partial b(x)}\frac{\int_{a(x)}^{b(x)} f(x,\ y)u(y)dy}{\psi + \int_{a(x)}^{b(x)} u(y)dy}.$$

Now, $b'(x) > 0$ for $x \in \chi_2$, and the other factor equals

$$\frac{f(x,\ b)u(b)}{\psi + \int_a^b u(y)dy} - \frac{u(b)\int_a^b f(x,\ y)u(y)dy}{[\psi + \int_a^b u(y)dy]^2} = \frac{u(b)f(x,\ b)}{\psi + \int_a^b u(y)dy} - \frac{f(x,\ a)u(b)}{\psi + \int_a^b u(y)dy} > 0$$

using the optimality relation and $f(x, b) > f(x, a)$. This extra term inflates the left side of (7) and reinforces the logic of step 3. Proceed recursively for $\chi_3, \chi_4, \ldots$, deducing that $a(\cdot)$ and hence $b(\cdot)$ are monotonic increasing until $a(\cdot)$ hits zero. QED

VI. Assortative Matching Overview

How search frictions are modeled is critical. Morgan (1995) explored a discrete time model of market segmentation with an explicit search cost $c > 0$. This makes more sense when search obtains swiftly, hours or days as opposed to months, so that the appropriate measure of search costs is money rather than time. Opportunity time costs of search are endogenous to the equilibrium and matter when search is time-consuming. They capture marriage or long-term partner search.

Assume a reservation partner rule $a(x)$ and an upper threshold one for simplicity. With explicit search costs, the value solves $\int_{a(x)}^{1} [f(x,\ y) - \mathcal{V}(x)]u(y)dy = c$ rather than $\int_{a(x)}^{1} [f(x,\ y) - \mathcal{V}(x)]u(y)dy = \psi\mathcal{V}(x)$ for opportunity time costs, as found in (4). Since the same optimality equation $V(x) = f(x,\ a(x))$ applies in both cases, we have

$$\text{explicit search costs: } c = \int_{a(x)}^{1} [f(x,\ y) - f(x,\ a(x))]u(y)dy,$$

$$\text{opportunity time costs: } \psi = \int_{a(x)}^{1} \left[\frac{f(x,\ y)}{f(x,\ a(x))} - 1\right]u(y)dy.$$

In the first case, supermodularity assumption 1 asserts that the bracketed difference rises in x for fixed $a(x) < y$. If $f_2 > 0$, a rising threshold $a(x)$ is needed to maintain equality. The same finding obtains in the second case given log supermodularity assumption 2 and $f_2 > 0$. See table 1 for the summary big picture of the literature.

TABLE 1
Summary of the Assortative Matching Literature

	No Search	Fixed Cost Search	Opportunity Time Cost Search
NTU	$f_2 > 0$	$f_2 > 0,\ f_{12} > 0$	$f_2 > 0,\ (\log f)_{12} > 0$
TU	$f_{12} > 0$	$f_{12} > 0$	$f_{12} > 0,\ (\log f_1)_{12} > 0,\ (\log f_{12})_{12} > 0$

Note.—As in Becker (1973), TU means that wages are competitively set. The top middle entry is found in Morgan (1995) and the middle bottom entry in Atakan (2006). The right bottom entry is in Shimer and Smith (2000). This paper derives the top right entry.

VII. Concluding Remarks

I have analyzed matching with search frictions in which match payoffs are exogenously specified (NTU). I have characterized a simple new log supermodularity condition for assortative matching and provided a novel and nontrivial existence theorem. This NTU world aptly captures social relationships more than productive ones and, more generally, any contexts in which we observe disagreements on the mutual desirability of matches (see Mortensen 1988).

The theory extends with no significant new insights (but additional complexity) to a world with distinguished sides. For instance, let x index men and y women, and let payoffs $f(x, y)$ and $g(y, x)$ accrue to x and y, respectively. With preferences monotonic increasing over matches ($f_2 > 0$ and $g_2 > 0$), woman x matches with any man $y \geq a(x)$, whereas man x matches with any woman $y \geq b(x)$. In this case, provided that f and g each satisfy assumption 2, then matching is strictly assortative.[17]

I have focused on the "pure quality" matching paradigm in which everyone enjoys identical ordinal preferences over partners. Introducing "match chemistry" effects is the natural next step and intuitively produces assortative matching with a probabilistic flavor if preferences retain a common quality component.

Appendix

Omitted Proofs

A. *Identical Cardinal Preferences: Proof of Lemma 7*

Define the utility set of type x as

$$\mathcal{U}_x \equiv \{\langle f(x, z),\ 0 \leq z \leq 1\rangle,\ f(x, \varnothing) = -c(x)\}.$$

By the von Neumann–Morgenstern expected utility theorem, types x and y have identical von Neumann–Morgenstern preferences exactly when their respective utility sets $\mathcal{U}_x$ and $\mathcal{U}_y$ are positive affine transformations of one another. So the

[17] But unlike the analysis so far, the lowest types on one of the distinguished sides to the market might be shunned in all matches.

linear map of utilities obeys $u_x = \alpha(x, y)u_y + \beta(x, y)$ for all $u_x \in \mathcal{U}_x$ and $u_y \in \mathcal{U}_y$. Considering the respective matches with z, we have $f(x, z) \equiv \alpha(x, y)f(y, z) + \beta(x, y)$ for all z. For this to be an identity in z, necessarily $f(x, z)/f(y, z) = \alpha(x, y)$ and $\beta(x, y) \equiv 0$. So $f(x, z)/f(y, z) = \alpha(x, y)$ is independent of z, which implies $f(x, y) = \gamma_1(x)\gamma_2(y)$, as required. Substituting the unmatched values yields $\beta(x, y) = \alpha(x, y)c(y) - c(x) \equiv 0$ for all x and y. QED

B. *Existence Overview: Proof of Proposition 1*

Let α and u have $\mathcal{L}^1$ norms $\|\alpha\|_{\mathcal{L}^1} = \int_0^1\int_0^1 |\alpha(x, y)|dxdy$ and $\|u\|_{\mathcal{L}^1} = \int_0^1 |u(x)|dx$.

Step 1: The family $\mathcal{F}$: I need a compact subset $\mathcal{F}$ of value function space for which a closed and continuous best-response map T exists. Let $\mathcal{F}$ be the measurable functions $\mathcal{V}$ on $[0, 1]$ with $0 \le \mathcal{V} \le \bar{f}$, also of uniformly bounded variation $\le B$, for a $B < \infty$. As any $\mathcal{V} \in \mathcal{F}$ is integrable, $\mathcal{F}$ is weak-* compact by Alaoglu's theorem.

Lemma 8 (Continuity). (*a*) Posit assumption 1. Any Borel measurable map $\mathcal{V} \mapsto \alpha_{\mathcal{V}}$ from value functions in $\mathcal{F}$ to match indicator functions is continuous. (*b*) The map $\alpha \mapsto u_\alpha$ from match indicator functions to the steady-state unmatched density solving (1) is both well defined and continuous.

The proof of part *a* is totally different from that in Shimer and Smith (2000) and is given below. Shimer and Smith have proved part *b* of lemma 8, their fundamental matching lemma.

Step 2: The best-response value: The value equations (2)–(3) imply

$$\mathcal{V}(x) = \psi \int_{\Omega_{\mathcal{V}}(x)} \max\langle 0, f(x, y) - \mathcal{V}(x)\rangle u_{\mathcal{V}}(y)dy, \tag{A1}$$

where the opportunity set $\Omega_{\mathcal{V}}$ satisfies $\Omega_{\mathcal{V}}(x) = \{y | f(y, x) \ge \mathcal{V}(y)\}$, and the unmatched density $u_{\mathcal{V}}$ corresponds to $\mathcal{V}$, given by lemma 8. For a best-response map T to be closed and continuous on $\mathcal{F}$, I cannot simply let $T\mathcal{V}(x)$ equal the right-hand side of (A1). Rather, adding the expectation of $\mathcal{V}$ to each side of (A1) yields

$$T\mathcal{V}(x) = \frac{\int_{\Omega_{\mathcal{V}}(x)} \max\langle f(x, y), \mathcal{V}(x)\rangle u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x)} u_{\mathcal{V}}(y)dy},$$

and $u_{\mathcal{V}}$ solves (1) with the matching correspondence arising from the value $\mathcal{V}$. Then a fixed point of $T\mathcal{V} = \mathcal{V}$ is a value function for a search equilibrium $\mathcal{V}$.

Then $0 \le \mathcal{V} \le \bar{f}$ implies $0 \le T\mathcal{V} \le \bar{f}$, whereas T clearly preserves measurability, since $\mathcal{A}$ and hence its inverse Ω are Borel measurable. Finally, to show that $T(\mathcal{F}) \subseteq \mathcal{F}$, it suffices to prove that the total variation of any element of $T(\mathcal{F})$ is also bounded above by B. Section D of this appendix proves this inclusion for large B.

Step 3: Continuity of T: I need T to be a continuous operator on $\mathcal{F}$ in the right topology; namely, for any subset $X \subseteq [0, 1]$, for all $\eta > 0$, there exists $\epsilon > 0$ such that $|\int_X [T\mathcal{V}_1(x) - T\mathcal{V}_2(x)]dx| < \eta$ if $\int_X [\mathcal{V}_1(x) - \mathcal{V}_2(x)]dx < \epsilon$. Given lemma 8, this follows in Section E by iterated application of the triangle inequality.

Schauder's fixed-point theorem (see Istratescu 1981, theorem 5.1.3) now yields a fixed point $T\mathcal{V} = \mathcal{V} \in \mathcal{F}$. QED

Since $\mathcal{V}$ is of bounded variation by proposition 1, it is the difference of monotonic functions and therefore is differentiable almost everywhere.

C. Continuity: Proof of Part a *of Lemma 8*

Let $\mathcal{V}_n \to \mathcal{V}$ in the weak-* topology. As $\langle \mathcal{V}_n \rangle$ have uniformly bounded variation[18] by Helly's theorem,[19] there exists a pointwise convergent subsequence $\mathcal{V}_{n_k} \to \mathcal{V}$.

The Lebesgue measure of $\{(x, y) \in S \subseteq [0, 1]^2 | f(x, y) \geq \mathcal{V}(x)\}$ is $\mu_{\mathcal{V}}(S) \equiv \int_S \mathbb{I}_{f \geq \mathcal{V}} dxdy$, where the 0-1 indicator functions $\mathbb{I}_{f(x,y) \geq \mathcal{V}_n(x)}$ converge pointwise to $\mathbb{I}_{f(x,y) \geq \mathcal{V}(x)}$ for any y with $f(x, y) \neq \mathcal{V}(x)$. But $f(x, y) = \mathcal{V}(x)$ occurs for at most one y given $f_2 > 0$, and so $\mu_{\mathcal{V}_n} \to \mu_{\mathcal{V}}$ pointwise converges almost everywhere. But $|\mathbb{I}| \leq 1$, and so by Lebesgue's dominated convergence theorem, convergence is in $\mathcal{L}^1$. So if $\hat{\alpha}$ is the acceptance indicator ($\hat{\alpha}(x, y) = 1$ if $y \in \mathcal{A}(x)$), then $\hat{\alpha}_{\mathcal{V}_n}$ converges to $\hat{\alpha}_{\mathcal{V}}$ in $\mathcal{L}^1$.

Next, $\mathcal{A}$ and Ω are "inverses," that is, $\alpha(x, y) = \hat{\alpha}(x, y)\hat{\alpha}(y, x)$. Thus

$$\begin{aligned} |\alpha_{\mathcal{V}}(x, y) - \alpha_{\mathcal{V}_n}(x, y)| &= |\hat{\alpha}_{\mathcal{V}}(x, y)\hat{\alpha}_{\mathcal{V}}(y, x) - \hat{\alpha}_{\mathcal{V}_n}(x, y)\hat{\alpha}_{\mathcal{V}_n}(y, x)| \\ &\leq |\hat{\alpha}_{\mathcal{V}}(x, y) - \hat{\alpha}_{\mathcal{V}_n}(x, y)| + |\hat{\alpha}_{\mathcal{V}}(y, x) - \hat{\alpha}_{\mathcal{V}_n}(y, x)| \end{aligned}$$

since $|\alpha| \leq 1$. Hence, $\mathcal{V} \mapsto \hat{\alpha}$ is continuous implies that $\mathcal{V} \mapsto \alpha$ is continuous. QED

D. Completion of Step 2 of the Proof of Proposition 1

For $x_2 > x_1$, let us separate the difference of $T\mathcal{V}$ at those points into the portion $Q_1(x_1, x_2)$ due to changes in f and $\mathcal{V}$ and the portion $Q_2(x_1, x_2)$ due to changes in the opportunity set Ω. Thus $[T\mathcal{V}](x_2) - [T\mathcal{V}](x_1)$ equals

$$\begin{aligned} &\frac{\int_{\Omega_{\mathcal{V}}(x_2)} \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x_2)} u_{\mathcal{V}}(y)dy} - \frac{\int_{\Omega_{\mathcal{V}}(x_1)} \max\langle f(x_1, y), \mathcal{V}(x_1)\rangle u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x_1)} u_{\mathcal{V}}(y)dy} \\ &= \frac{\int_{\Omega_{\mathcal{V}}(x_1)} [\max\langle f(x_2, y), \mathcal{V}(x_2)\rangle - \max\langle f(x_1, y), \mathcal{V}(x_1)\rangle] u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x_1)} u_{\mathcal{V}}(y)dy} \\ &\quad + \left[\frac{\int_{\Omega_{\mathcal{V}}(x_2)} \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x_2)} u_{\mathcal{V}}(y)dy} - \frac{\int_{\Omega_{\mathcal{V}}(x_1)} \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle u_{\mathcal{V}}(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x_1)} u_{\mathcal{V}}(y)dy}\right] \\ &= Q_1(x_1, x_2) + Q_2(x_1, x_2). \end{aligned}$$

Since $\Omega_{\mathcal{V}}(x_1) \subseteq \Omega_{\mathcal{V}}(x_2) \subseteq [0, 1]$ for $x_1 < x_2$, we may define $\Delta(x_1, x_2) \equiv \Omega_{\mathcal{V}}(x_2) \setminus \Omega_{\mathcal{V}}(x_1)$. To shorten equations, let $\int_i g(y)$ denote $\int_{\Omega_{\mathcal{V}}(x_i)} g(y)u_{\mathcal{V}}(y)dy$ and $\int_\Delta g(y)$ denote $\int_{\Delta(x_1,x_2)} g(y)u_{\mathcal{V}}(y)dy$. Now $Q_2(x_1, x_2)$ equals

$$\begin{aligned} &\frac{\int_1 \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle + \int_\Delta \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle}{\psi + \int_1 1 + \int_\Delta 1} - \frac{\int_1 \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle}{\psi + \int_1 1} = \\ &\frac{(\psi + \int_1 1)\int_\Delta \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle - \int_1 \max\langle f(x_2, y), \mathcal{V}(x_2)\rangle \int_\Delta 1}{(\psi + \int_1 1 + \int_\Delta 1)(\psi + \int_1 1)} \end{aligned}$$

after a cancellation. Since the sets $\Omega_{\mathcal{V}}(x)$ are nested increasing if $\mathcal{V} \in \mathcal{F}$, the intervals $\Delta(x_j, x_{j+1})$ are disjoint if $(x_j, x_{j+1}]$ are: For if $x_1 < x_2 \leq x_3 < x_4$, then

[18] Steve Schochet sketched the argument of paragraphs 1 and 2, as well as the argument of Sec. *D*.

[19] This is a nonstandard version of this result: theorem 12.7 in Protter and Morrey (1977).

$\Delta(x_1, x_2) \subseteq \Omega_{\mathcal{V}}(x_2)$ whereas $\Delta(x_3, x_4) \subseteq \Omega_{\mathcal{V}}(x_3)^c \subseteq \Omega_{\mathcal{V}}(x_2)^c$, where S^c is the complement of S in [0, 1]. So consider a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. Since u is pointwise bounded by $\bar{l}$, $\mathcal{V} \leq \bar{f}$ for all $\mathcal{V} \in \mathcal{F}$. Also, there exists $\bar{f}_1 < \infty$ such that $|f_1(x, y)| \leq \bar{f}_1$ by the assumption of continuous differentiability. Finally, $|Q_2(x_1, x_2)| \leq K\lambda(\Delta(x_1, x_2))$, where λ is the Lebesgue measure, and $K = (\psi + 2)\bar{l}\bar{f}/\psi^2$. Since the sets $\Delta(x_j, x_{j+1}) \subset [0, 1]$ are disjoint,

$$\sum_{j=1}^{n} |Q_2(x_{j-1}, x_j)| \leq K\sum_{j=1}^{n} \lambda(\Delta(x_{j-1}, x_j)) = K\lambda\left(\bigcup_{j=1}^{n} \Delta(x_{j-1}, x_j)\right) \leq K.$$

We can bound $Q_1(x_1, x_2)$ using $|\max\langle a_2, b_2\rangle - \max\langle a_1, b_1\rangle| \leq |a_2 - a_1| + |b_2 - b_1|$:

$$\begin{aligned} |Q_1(x_1, x_2)| &\leq \frac{\int_1 |f(x_2, y) - f(x_1, y)| + |\mathcal{V}(x_2) - \mathcal{V}(x_1)|}{\psi + \int_1 1} \\ &\leq \frac{\int_1 \bar{f}_1 |x_2 - x_1| + |\mathcal{V}(x_2) - \mathcal{V}(x_1)|}{\psi + \int_1 1} \\ &= \bar{f}_1 |x_2 - x_1| + \theta|\mathcal{V}(x_2) - \mathcal{V}(x_1)|, \end{aligned}$$

where

$$\theta = \sup_{x\in[0,1], w\in\mathcal{F}, u} \frac{\int_{\Omega_{\mathcal{V}}(x)} u(y)dy}{\psi + \int_{\Omega_{\mathcal{V}}(x)} u(y)dy} \leq \frac{\int_0^1 l(y)dy}{\psi + \int_0^1 l(y)dy} = \frac{1}{1+\psi} < 1.$$

Let $\mathcal{P}$ be the space of partitions of [0, 1], namely all sets of subintervals characterized by finite increasing sequences in [0, 1]. Let $\langle\phi\rangle$ denote the total variation of the function ϕ. For any $\{x_k\} \in \mathcal{P}$,

$$\sum_{j=1}^{n} |Q_1(x_{j-1}, x_j)| \leq \sum \ [\bar{f}_1 |x_j - x_{j-1}| + |\mathcal{V}(x_j) - \mathcal{V}(x_{j-1})|]\theta \leq \theta\bar{f}_1 + \theta\langle\mathcal{V}\rangle.$$

Finally, $\langle T\mathcal{V}\rangle$ equals

$$\sup_{x\in\mathcal{P}} \sum_{j=1}^{n} |Q_1(x_j, x_{j+1}) + Q_2(x_j, x_{j+1})| \leq \sup_{x\in\mathcal{P}} \sum_{j=1}^{n} [|Q_1(x_j, x_{j+1})| + |Q_2(x_j, x_{j+1})|].$$

The above estimates show that $\langle T\mathcal{V}\rangle \leq (K + \theta\bar{f}_1) + \theta\langle\mathcal{V}\rangle$. Since $\theta < 1$, if $\langle\mathcal{V}\rangle \leq B$, then $\langle T\mathcal{V}\rangle \leq B$ for any $B \geq (K + \theta\bar{f}_1)/(1 - \theta)$.

E. Proof of Step 3 in Proposition 1

To prove continuity in the weak topology, we need $|\int_I T(\mathcal{V}_1)(x) - T(\mathcal{V}_2)(x)dx|$ small whenever $|\int_I \mathcal{V}_1(x) - \mathcal{V}_2(x)dx|$ is small, for any $I \subseteq [0, 1]$. Toward applying the triangle inequality, write $T\mathcal{V}_1(x) - T\mathcal{V}_2(x)$ as

$$\frac{\int_{\Omega_1(x)-\Omega_2(x)} \max\langle f(x,\ y),\ \mathcal{V}_1(x)\rangle u_{\mathcal{V}_1}(y)}{\psi + \int_{\Omega_1(x)} u_{\mathcal{V}_1}(y)} + \frac{[\mathcal{V}_1(x) - \mathcal{V}_2(x)]\int_{\Omega_2(x)} u_{\mathcal{V}_1}(y)}{\psi + \int_{\Omega_1(x)} u_{\mathcal{V}_1}(y)}$$

$$+ \frac{\int_{\Omega_2(x)} \max\langle f(x,\ y),\ \mathcal{V}_2(x)\rangle [u_{\mathcal{V}_1}(y) - u_{\mathcal{V}_2}(y)]}{\psi + \int_{\Omega_1(x)} u_{\mathcal{V}_1}(y)}$$

$$+ \left[\int_{\Omega_2(x)-\Omega_1(x)} u_{\mathcal{V}_2}(y) + \int_{\Omega_1(x)} u_{\mathcal{V}_2}(y) - u_{\mathcal{V}_1}(y)\right] \frac{\int_{\Omega_2(x)} \max\langle f(x,\ y),\ \mathcal{V}_2(x)\rangle u_{\mathcal{V}_2}(y)}{\psi + \int_{\Omega_2(x)} u_{\mathcal{V}_2}(y)},$$

where $\int_{I-J} \equiv \int_I - \int_J$. Since $\psi > 0$, f (and thus $\mathcal{V}_i$) is bounded, and $0 \leq u \leq l$, all factors above are uniformly bounded in absolute value, say by $C > 0$. When these are replaced with C, the absolute value of the integrated first, second, and fourth terms over I is small if $\mathcal{V}_1$ and $\mathcal{V}_2$ are weakly close, whereas the pointwise absolute value of the third and fifth terms is small given α_1 and α_2; so u_1 and u_2 are $\mathcal{L}^1$ close, by lemma 8. QED

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