A Model of Exchange where Beauty is in the Eye of the Beholder *

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this version: January 31, 2002
original version: January, 1994

Abstract

I explore the search-constrained exchange paradigm where traders differ *stochastically* in their consumption-valuations of any given good, and all goods are 'durable'. Individuals expect to engage in repeat trade before leaving the market. Thus, any good is valued not only for its own flow consumption value but also for its *option value* for future retrade.

The set of mutually agreeable trades is neatly analogous to the set of even-odds gambles preferred by a risk-loving individual. They are *risk-increasing* and possibly *valuation-decreasing*: Trades may result in a lower sum of own-valuations provided the 'spread' increases. Such risky trade stems from the twin properties of a falling demand curve and strict convexity of the value function — where convexity arises because individuals tend to hold onto their more prized possessions longer.

Furthermore, (i) the effect worsens as the search frictions *diminish,* and (ii) is less than a social planner would prefer. This environment produces a new *ex post* inefficiency of trade beyond the ex ante inefficiency due to Mortensen (1982). Finally, I show that the risk-increasing effect is muted but still present with positively correlated valuations.

*This is a long overdue version that incorporates an existence proof methodology only recently developed; some parts of the existence proof are missing, and will soon be added. This paper was previously entitled "A Theory of Risk-Increasing Trades", and reflects rather detailed suggestions from Peter Diamond and especially Richard Zeckhauser. I have also benefited from the comments of Daron Acemoglu, Abhijit Banerjee, Preston McAfee, and Robert Rosenthal at the April, 1995 Harvard-MIT Theory workshop, Ken Burdett at the NBER conference at the Cleveland Federal Reserve, and an NSF referee. Finally, I am grateful to Tom Louie (MIT undergrad) for numerous MATLAB simulations, and the NSF for financially supporting this research.

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1. INTRODUCTION

The simple paradigm of exchange arguably lies at the heart of all economics. The assumption that the opportunity for mutually advantageous exchange results in the transfer of goods to those who value them most is the basic insight underlying the welfare theorems. The neoclassical Walrasian world of frictionless trade under perfect information is the ideal framework for illustrating this postulate. Indeed, the resulting trade must (almost tautologically) result in an efficient allocation.

Adding market frictions to such an idyllic world somewhat ties Adam Smith's 'invisible hand'. For instance, goods migrate more slowly to their highest evaluators. Granted. This paper provides insights into the exact way that search frictions skew those trades that *do occur* — and explains why some seemingly advantageous trades are not undertaken, and other seemingly disadvantageous trades are.

Suppose for a moment, as goes the saying, that 'beauty is in the eye of the beholder', with goods initially not possessed by those most beholden to them. In the presence of search frictions, any given good may well be traded many times before arriving at its ultimate connoisseur. For this reason, an exchange is valued not only for how it improves a trader's *own-valuation* (or flow consumption-value), but also for its *option value* for future retrade. With such mixed motives for trade, it is neither necessary nor sufficient that exchange enhance the sum of own-valuations. Given the potentially complicated dynamic nature of the problem, it may come as a surprise that anything general can be said of this failure. In this paper, I show that exchanges are *risky*: They will sometimes result in an ex post *decrease* in the sum of own-valuations, *provided* the final 'spread' in the own-valuations becomes sufficiently riskier — a valuation-decreasing but risk-increasing trade; conversely, trades which increase this sum may well be turned down, if the exchange is sufficiently 'risk-decreasing'. These facts stem from the *combination* of a falling demand curve for units, and the strict convexity of the value function. The latter property in turn holds because individuals tend to hold onto more prized possessions longer.

search frictions flesh out and explores the robustness of this effect. I develop a simple search-theoretic model of exchange where goods and traders are heterogeneous, with traders differing stochastically in their valuations of any given good. Individuals must decide whether to search at cost, and if so, which proposed trades to agree upon. In equilibrium, individuals expect to engage in repeat trades before leaving the market. I consider this model in itself to be a contribution of the paper, as it embodies a great deal of the richness of trade heretofore not captured in the literature.[1] Indeed, the heterogeneity in valuation provides an incentive for individuals to 'trade-up' so long as goods are durable, providing a 'flow' rather than a 'flash' of utility. I thus simply assume that they are infinitely durable. I also suppose that there exists some tradable homogeneous good that can serve as a common currency of utility, and that serves to mediate transactions (Kiyotaki and Wright (1989)).

In this framework, to see why trades are risk-increasing one must focus on the *dynamic* rather than the *static* incentives to trade. Two individuals considering a swap of durables must carefully evaluate how long they will be in possession of either good. Obviously, a more rapid resale is anticipated the lower one values one's current good. This speaks to the convexity of the value function: A given rise in valuation is more helpful than the same magnitude decline is harmful. There have been other papers that have considered the case of heterogeneous preferences over goods; however, this risk-increasing effect has been missed as it only appears once one posits a falling demand curve for goods, and by implication forces *asset swaps,* which is the principal innovation of this paper.

My main result is that when individuals have unit demands, valuations for any good are purely idiosyncratic, and trade surplus is evenly split (the Nash bargaining solution), any search friction encourages trades that are *risky.* With the resulting 1-for-1 trades, I can compare the set of mutually agreeable trades to the set of even odds gambles preferred by a risk-loving individual. This is of some interest, for an analogue of the Arrow-Pratt risk aversion coefficient sheds light into comparative statics exercises. Despite the fact that the search frictions induce the risky trades, I prove that the effect actually worsens as search frictions diminish. Moreoever, I show that far from being a perverse market phenomenon, trade is even less risky than is socially optimal. For when surplus is split, individuals do not benefit from the full option value of resale. Mortensen explored this insight in (1982), and deduced his famous *ex ante* inefficiency of search (and related games), that individuals search too little. Here, we find a new *ex post* inefficiency for search models: After having met, two individuals may well not trade when they should.

[1] Of some note is that *repeat trade* emerges rather naturally in this search market: Goods may be acquired with the express intention of eventually unloading them. This effect has been identified in other settings, such as Wheaton (1990). But unlike that search-theoretic model of housing sales, which relied on stochastic shocks to demands for housing services, preferences here are stationary.

The main result is perfectly well formulated and most salient with unit demands and purely idiosyncratic valuations for goods (the so-called *pure variety* case). This paper also reformulates the main result when these assumptions do not hold, to underscore that this effect is not simply a property of hypothetical markets.

I show that the effect is muted with positively correlated valuations. Clearly, all trade is forestalled in a world of *pure quality,* without heterogeneous valuations over each good. But risky trade arises once we move away from this extreme, and I explore how it is damped by affiliation, or with correlation given complementarity or substitutability of quality and variety components. So long as there are heterogeneous valuations over a class of durable assets, the insights of this paper apply.

Extending the result beyond unit demands to demand schedules downward-sloping in quantity is an exciting but very hard generalization that I must leave open. For with multiple unit trades, the question also arises of *which* units change hands, and this decision margin immensely complicates the analysis. Likewise, in addition to markets marred by search frictions, the effect appears whenever one may be saddled with durable acquisitions for a while before facing a retrade opportunity.

The existence proof for this model merits some comment. The model that best and most simply captures the theory involves a continuous space of players and possible valuations. But such simplicity up front makes for difficulties later on. Walrasian existence proofs have recently tackled such hurdles; but in a search setting, trade is slow to occur, and depends on the mass of searching traders. Thus, one must also show that the stock of available trades is a continuous and well-defined function of the decisions agents make. In fact, I adapt a proof program recently developed in Shimer and Smith (1996) (hereafter, SS) for a matching setting.

The paper is organized to separately spotlight the conceptually linked issues of repeat trade, duration of possession, static inefficiency of trade, and risk-increasing trade. I first intuitively summarize the model and main result with a simple story. I then carefully develop and analyze the primary model with independent valuations and unit demands. For this setting, I derive distributional implications of the nature of trade, provide comparative statics results, and make pointed observations on the efficiency properties of the market. I then analyze the case of correlated valuations. In the conclusion, I describe an open problem for falling demand schedules, and discuss implications of my insight in a more traditional matching market, where the beauty-in-the-eye-of-the-beholder assumption is quite natural.

2. A PARABLE

I shall follow in the tradition of earlier search papers by providing a salient metaphor that encapsulates in a nutshell the basic story of this paper. The setting, as per usual, is a tropical island where now a famous resident French artist endows all inhabitants with one or more of his paintings the day they come of age and appreciate art. Art is only of value to the extent that it can be hung and enjoyed in one's hut. After all, a hut is one's home. Each hut has one ideal place where art may be displayed, and several progressively sorrier locations that multiplicatively diminish the aesthetic value of any hanging piece.

Some islanders treasure their endowed canvases; others find theirs rather wanting, and in all cases, such preferences are set for life. Fortunately, on this isle, *chacun à son gout.* Aware of the diversity of tastes, each islander has the option of trading some or all of his artwork with others. Transactions are permitted daily but only at high noon. To take advantage of this, he must stop climbing coconut trees, foregoing some production, and mill about the market place, showing Kodak photographs of his home art gallery to others. The trading period affords but one opportunity for a potential transaction.

Islanders are renowned for their lack of pretense, unable to hide their gut feelings for any art shown them. When two islanders meet, if both prefer their current art collection, they leave disappointed. But when a trade seems mutually beneficial, invariably one of the parties digs into his sack of coconuts (the island currency) to ensure that both equally profit from the exchange. Sometimes both traders walk out of the exchange with improved collections — but always if only one does, the other party returns to his hut laden with extra coconuts. Each then spends his evening in his hut, perhaps anticipating tomorrow's trek to the marketplace.

It has been observed that two islanders who each currently evaluate their art at 50 sometimes prefer a trade to 60 and 20: The new 20 will soon be compensated for his short-term loss, while the new 60 has secured a comfortable longer-term position. The swap is figuratively risk-increasing because it results in a greater spread $(20, 60)$ than before at $(50, 50)$, but it is not immediately 'consumption value enhancing', for the net static flow benefits of ownership has decreased from 100 to 80. By the same token, a swap to 50 apiece is often nixed in favor of remaining at 60 and 20.

I now provide a model that captures the curious nature of trade on this island.

3. THE PURE VARIETY MODEL

A. Individuals and Goods. Let there be a continuum of individuals $\mathcal{A}$, and indivisible goods Ω. Everyone has a lifetime unit demand, and each individual $a \in \mathcal{A}$ initially possesses one unit of some randomly assigned good. Suppressing time, denote by $\omega_a \in \Omega$ whatever good agent a happens to possess at any moment.

Time is continuous, with a constant total mass of agents in the market searching for partners. There is a steady inflow $\delta > 0$ of entrants, each owning one unit of some good. While searching, everyone pays a flow search cost $c > 0$; this provides individuals with an incentive to eventually *stop* searching, and sustains a steady-state if an inflow of entrants balances the outflow of individuals ceasing further search. This assumption is not essential to my ultimate purpose here, but merely provides a richer search environment; I could just as well posit that steady state is maintained by a Poisson death rate of agents.

B. Preferences. Assume that individual a attaches *valuation* $v_a(\omega) \in (0,1)$ to durable good $\omega \in \Omega$. This is its flow payoff, or its personal *consumption/reserve value,* per unit time. As goods do not depreciate, preferences are stationary.

All individuals have *unit demands.* The conclusion discusses the extension beyond this beachhead. Here, the flow payoff from owning exactly one good is simply its valuation, with subsequent units worthless. If everyone discounts future payoffs (including costs) using the interest rate $r > 0$, the *average payoff* from owning x until time t and y thereafter is $(1 - e^{-rt})x + e^{-rt}y$.

C. Statistical Properties. Given the potential complexity of a world with a continuum of agents having differing valuations over a continuum of possible goods, the statistical properties of the model are critical, and must be carefully laid out.

Let π be the *joint valuation density*: i.e. $\pi(v)$ is the density of all entering agents and goods $(a, \omega) \in \mathcal{A} \times \Omega$ with $v_a(\omega) = v$. WLOG, assume there is a constant unit entry flow of traders, so that $\int_0^1 \pi(v)dv = 1$. Anchoring this is the (Borel measurable) *entrants' own-valuation density* f. Thus, $\int_X f(v)dv$ is the fraction of all entrants a whose *endowed* goods satisfy $v_a(\omega_a) \in X$. The existence proof alone requires that f be boundedly finite and positive: $0 < \underline{f} < f(v) < \bar{f} < \infty$ for all v.

A-1 ***(Random Initial Goods Assignment)***. Individuals on average value their endowed good no more or less highly than other goods, or $\pi = f$.

The a-marginal π_a is the *goods-valuation density*; $\pi_a(v)$ is thus the density of goods $\omega \in \Omega$ with $v_a(\omega) = v$. For simplicity, everyone has 'average' preferences:

A-2 ***(Stochastically Equivalent Individuals)***. Agents share the same goods valuation distribution, or $\pi_a \equiv \pi$ for (almost) all $a \in \mathcal{A}$.

Similarly, the ω-marginal π_ω is the *agent-valuation density* for good ω, with $\pi_\omega(v)$ simply the density of all agents $a \in \mathcal{A}$ with $v_a(\omega) \leq v$. For most of the paper, I assume that all goods too are average, or commensurate in the eyes of traders.

A-3 ***(Stochastically equivalent goods / pure variety)***. The valuation of the goods across agents is identical in distribution, or $\pi_\omega \equiv \pi$ for (almost) all $\omega \in \Omega$.

This assumption is relaxed later on, as it rules out interesting quality effects.

D. The Matching Process. I have just described the demograpics of the model apart from the matching that occurs. I now address this more complex issue, by naturally assuming that those one meets are randomly drawn from the population at large. But what *exactly* does this mean? First, individuals do not just happen to meet others who especially value their good.

A-4 ***(Representative Sampling)***. Valuations for goods possessed by matched partners have density f.

Second, I need that agents and goods to a match be independent.[2] So 1's own valuation influences neither his partner 2's valuation of 1's good, nor 1's valuation of 2's good. These two assumptions are listed separately, as the first is later relaxed.

A-5 ***(Cross-Agent Independence)***. Any two matched individuals have independent valuations over any one of their goods.

A-6 ***(Cross-Good Independence)***. Each individual's valuation of his partner's good is independent of his own valuation.

Finally, the equilibrium dynamics will induce some *steady-state own-valuation density* g for traders remaining in the market. This will generally differ from f.

E. Actions and Values. Everyone has two margins of decision: Should I keep searching? If so, which trades should I agree to? Since individuals are stochastically equivalent, one's own current valuation is a sufficient statistic for any decision-making; therefore, I simply denote by $w(x)$ the *average* present discounted (Bellman)

[2]This may well abuse the law of large numbers — a typical sin in continuum agent models.

value of an individual with (a good having) own-valuation x. Think of $w(x)$ as the *dynamic* worth of having a good whose *static* flow worth is x. Thus, one quits searching with any valuation in the *exit set* $\mathcal{X} = \{y|\, w(y) \leq y\}$. Those entrants with valuations in $\mathcal{X}$ will never search, but will simply be 'wash-through', and immediately enjoy their endowed good. The trading decision is not so simple.

F. Mutually Agreeable Trades. To focus on the search-theoretic aspects of this problem, I shall assume away any adverse selection difficulties: When two potential traders ($i = 1, 2$) who find it individually rational to search meet, i values his own good x_i and the other's good y_i — *both common knowledge.* Not only does this preclude incentive problems,[3] but it also rules out *experience goods,* where one only gradually learns one's own valuation (briefly addressed in the conclusion).

By the unit demand assumption, trades are all 1-for-1 swaps. While I omit a tedious proof, intuitively, anyone acquiring two units forgoes any consumption value on the lesser, while not greatly improving his option value; the one foresaking his only good, even one that provides him with no consumption value, loses the maximum possible option value of retrade.

I assume that all *strictly mutually agreeable* trades are approved — namely, those tuples $\langle x_1, x_2; y_1, y_2 \rangle$ with positive *surplus* $s(x_1, x_2, y_1, y_2) \equiv (w(y_1) + w(y_2) - w(x_1) - w(x_2) > 0$, while negative surplus trades are declined. This falls under the rubric of a *transferable utility* model, with surplus evenly split (the Nash bargaining solution). I thus need that some tradable homogeneous good serve as a common currency of utility, and all exchanges will generally involve both goods and money, with the latter consumed at once. Intuitively, if a trade results in one party falling to a lower flow utility level then he is compensated by a one-shot utility 'spike'.

I now introduce some incredibly useful notation. Define respectively the *trade-away set* $\mathcal{T}(x)$ and and its inverse, the *trade-to set* $\mathcal{T}^{-1}(x)$, naturally as

$$\mathcal{T}(x) = \{(x'; y, y')|x' \notin \mathcal{X}, w(y) + w(y') \geq w(x) + w(x')\} \quad (1a)$$

$$\mathcal{T}^{-1}(x) = \{(x'; y, y')|y, y' \notin \mathcal{X}, w(x) + w(x') \geq w(y) + w(y')\} \quad (1b)$$

Given assumption A-1 through A-5, the mass density of mutually agreeable

[3]The characterization of the efficiency of trade with general two-sided incomplete information and continuum type spaces is unsolved. See Stole and Ellingsen (1993) for references.

randomly proposed matches for x is simply the following triple integral over $\mathcal{T}(x)$:

$$\mathcal{P}(x) = \int_{(x';y,y')\in\mathcal{T}(x)} g(x')f(y)f(y')dx'dydy'$$

Likewise, given A-1 through A-5, the mass density of mutually agreeable randomly proposed matches that would result in x is simply the following triple integral over $\mathcal{T}^{-1}(x)$:

$$Q(x) = \int_{(x';y,y')\in\mathcal{T}^{-1}(x)} f(x')g(y)g(y')dx'dydy'$$

G. Search Technology. Those wishing to trade must search for others, and may instantaneously exit the market at any time, even 'immediately following' a trade. Not only is the matching process random, but so also is the time at which any trader meets someone. To make the analysis as simple as possible, I assume a *quadratic search technology,* for which Shimer and Smith (1996) have just developed an existence theory for 'employment' matching models. This means that potential trades $\langle x_1, x_2; y_1, y_2\rangle$ are realized at a rate linearly proportional to the joint mass density $g(x_1)g(x_2)f(y_1)f(y_2)$. Given the constant of proportionality ρ (the rendevous rate), the mass density flow of trades to and away from the valuation x is therefore $\rho f(x)\mathcal{Q}(x)$ and $\rho g(x)\mathcal{P}(x)$, respectively. Together with the inflow $\delta f(x)$ to valuation x, the steady state condition demands that for all $x \notin \mathcal{X}$:

$$\delta f(x) + \rho f(x)\mathcal{Q}(x) = \rho g(x)\mathcal{P}(x) \tag{2}$$

H. Dynamic Values versus Static Valuations. Then the *expected surplus flow* of a random trade involving x equals ρ times[4]

$$\sigma(x) = \int_{(x';y,y')\in\mathcal{T}(x)} [w(y) + w(y') - w(x) - w(x')]\, g(x')f(y)f(y') \tag{3}$$

Next, since one can always abstain from search, forever enjoy whatever good one is in possession of, the average unmatched value $w(x)$ is the greater of x (absent further search) and the average *continuation value* $\hat{w}(x)$; this is the expected average present value of net consumption until a match is proposed, plus the expected discounted

[4]From now on, I shall omit the $dx'\,dy\,dy'$ notation at the end of triple integrals.

value and match surplus. Given the quadratic search technology, I have

$$\begin{aligned}\hat{w}(x) &= \int_0^\infty \rho e^{-\rho t}((1-e^{-rt})(x-c)+e^{-rt}[\hat{w}(x)+\sigma(x)/2])dt \\ &= \frac{r(x-c)}{r+\rho}+\frac{\rho}{r+\rho}[w(x)+\sigma(x)/2] \\ &= x-c+\phi\sigma(x) \qquad (4)\end{aligned}$$

where $\phi = \rho/2r$ measures the search frictions, with *greater* ϕ *corresponding to smaller frictions.* So $\phi\sigma(x)$ is a *flow option value* of resale, i.e. the half-share of the expected eventual match surplus share $\sigma(x)/r$ times the match arrival rate ρ.

Expression (4) encapsulates the following fact. The average present value of a good is a sum of the personal consumption value, less the flow search cost, plus the option value of future resale. *Nondurables* do not offer this bundle of a flow consumption and option value, and so their value would be constant if one wishes to trade again. With *durables,* w is an everywhere strictly increasing function. This provides a general incentive for everyone to 'trade-up' even if they plan to search again. Still, in light of the unit demand assumption, some may well 'trade-down'.

4. SEARCH EQUILIBRIUM AND VALUE FUNCTIONS

A. Equilibrium. Any notion of equilibrium requires that (i) everyone maximizes her present discounted payoffs, taking all other strategies as given, and (ii) a match is consummated iff both parties accept it. To rule out unreasonable equilibria, all matches with strictly positive mutual gains must be accepted; for definiteness, I also assume that zero surplus trades are accepted.

A *steady-state search equilibrium* (SE) is thus a 4-tuple $(w, \mathcal{X}, \mathcal{T}, g)$ fullfilling:

(a) VALUE ACCOUNTING: the unmatched *values* satisfy $w(x) = \max\langle x, \hat{w}(x)\rangle$, where $\hat{w}$ is given by (4);

(b) SEARCH OPTIMALITY: the *exit set* $\mathcal{X}$ satisfies $w(x) \leq x$ iff $x \in \mathcal{X}$;

(c) TRADE OPTIMALITY: the *trade-to* and *trade-away sets* $\mathcal{T}$ and $\mathcal{T}^{-1}$ satisfy (1);

(d) POINTWISE STEADY-STATE: g obeys the *balanced flow* condition (2)

Proposition 1 (Existence). *Given* A-1 *and* A-6*, a SE exists.*

To avoid getting sidetracked here, the proof is appendicized.

B. Properties of the Value and Flow Surplus Functions. Before moving on, I must nail down some basic properties of the value function. Despite the nontrivial option values of retrade, all dynamic values are well-behaved functions.

Lemma 1 (Monotonicity). *Given* A-1 *to* A-6*, in any SE, the value* $w \geq 0$ *is strictly increasing, and flow surplus* σ *strictly decreasing. Also, the trade-away sets* $\mathcal{T}$ *(resp. trade-to sets* $\mathcal{T}^{-1}$*) are strictly decreasing (resp. increasing) and nested. Thus,* $\mathcal{P}$ *and* $\mathcal{Q}$ *are decreasing (resp. increasing) functions.*

Proof: First, $w(x) = x$ for all $x \in \mathcal{X}$, which is obviously increasing. Suppose, by way of contradiction, that $w(x_2) \leq w(x_1)$ for some $x_2 > x_1$, both outside $\mathcal{X}$. Then (1a) implies $\mathcal{T}(x_1) \subseteq \mathcal{T}(x_2)$, and thus (3) yields $\sigma(x_2) \geq \sigma(x_1)$. Then by (4), $w(x_2) = x_2 - c + \phi\sigma(x_2) > x_1 - c + \phi\sigma(x_1) = w(x_1)$, a contradiction.

Next, (1) then implies $\mathcal{T}(x_2) \subset \mathcal{T}(x_1)$ and $\mathcal{T}^{-1}(x_1) \subset \mathcal{T}^{-1}(x_2)$ for all $x_2 > x_1$. The monotonicity of $\mathcal{P}$ (resp. $\mathcal{Q}$) follows from that of $\mathcal{T}$ and w (resp. $\mathcal{T}^{-1}$ and w). Finally, by (3), σ is strictly decreasing for w strictly increasing and $\mathcal{T}$ decreasing. □

Intuitively, since higher valuations always provide higher consumption value than lower ones, the better endowed can simply imitate the trading decision of lower agents and do better. If they optimize, they will do still better.

This intuition for monotonicity also works for continuity: Anyone can do almost as well as a slightly better endowed agent, simply by imitating her trading decision; thus, the value function cannot jump. In fact, as an appendix proves, a stronger form of continuity is possible, which is immediately useful in Lemma 3.

Lemma 2 (Continuity). *Given* A-1 *to* A-6*, in a SE, the value* w *and expected surplus* σ *are each continuous and Lipschitz;* w *has a Lipschitz constant of* 1.

I often refer to the derivative of the unmatched value function. This is justified:

Lemma 3 (Differentiability). *Given* A-1 *to* A-6*, in a SE, the value and flow surplus functions* w *and* σ*, as well as* $\mathcal{P}$ *and* $\mathcal{Q}$*, are a.e. differentiable. For a.e.* $x \notin \mathcal{X}$*,* $w'(x) = 1/(1 + \phi\mathcal{P}(x))$ *and* $\sigma'(x) = -\mathcal{P}(x)/(1 + \phi\mathcal{P}(x)) < 0$*.*

Proof: Monotone functions (Lemma 1) like σ, w, $\mathcal{P}$, and $\mathcal{Q}$ are a.e. differentiable.

If $x \notin \mathcal{X}$, and either σ or w is differentiable at x, then so is the other, since (4) implies $\hat{w}'(x) = \phi\sigma'(x) + 1$. Next, *provided* the trade-away set is differentiable (suitably) in x, computing $\sigma'(x)$ is a straightforward application of the Fundamental

Theorem of Calculus: Surplus vanishes all along the boundary of the trade-away set; therefore, we can safely ignore the effect on $\sigma'(x)$ of changes in the matching set, and simply differentiate (3) under the integral sign, and get $\sigma'(x) = -w'(x)\mathcal{P}(x)$. It suffices to solve for $w'(x) = 1 + \phi\sigma'(x) = 1 - \phi w'(x)\mathcal{P}(x) = 1/(1 + \phi\mathcal{P}(x))$. The appendix resolves the above qualifying proviso. □

Lemma 4 (Convexity). *Given* A-1 *to* A-6*, in a SE, the average value and flow surplus are strictly convex functions at all* $x \notin \mathcal{X}$*, and* $w''(x) = -\phi\mathcal{P}'(x)/(1 + \phi\mathcal{P}(x)) > 0$ *and* $\sigma''(x) = -\mathcal{P}'(x)/(1 + \phi\mathcal{P}(x)) > 0$ *a.e. exist.*

Proof: Now, $w(x) = x$ for $x \in \mathcal{X}$, which is trivially a weakly convex function. Otherwise, if $x \notin \mathcal{X}$, then $w'(x) = [1+\phi\mathcal{P}(x)]^{-1} < 1$ a.e. x. Since the RHS is strictly increasing by Lemma 1, w and thus σ are each convex. Next, $\mathcal{P}'(x) < 0$ a.e. exists, and so $w''(x)$ exists a.e., and is positive. Finally, $w'(x) = 1 + \phi\sigma'(x)$ a.e. implies that σ too is convex, or $\sigma''(x) = 2w''(x)/\phi > 0$. The results follows. □

5. THE MAIN RESULTS

5.1 When to Search

There are two decisions an individual cares about: (*i*) Should I search? (*ii*) Which trades should I agree to? I now tackle the first decision, and characterize the optimal stopping rule. As expected, with monotonic values, preferences too are monotonic, while search costs eventually force individuals out of the market. All results in this section maintain assumptions A-1 to A-6, and describe an SE.

Lemma 5 (Reservation Valuation Property). *There exists a unique threshold* $\theta = w(\theta) \in (0, 1)$ *such that one quits searching iff one's valuation* $x \in \mathcal{X} = [\theta, 1)$*.*

Proof: Since $\sigma'(x) < 0$, (4) implies $\hat{w}'(x) < 1$. Hence, if $\hat{w}(x) < x$ then $\hat{w}(y) < y$ for all $y > x$. So the exit set $\mathcal{X}$ has the reservation valuation property, and $\theta = w(\theta)$. Finally, $\theta > 0$ since $w(x) > 0$ for $x > 0$ by Lemma 1, while $\theta < 1$ as $w(x) < 1$ for all x, w being a strictly discounted average present value of payoffs all at most 1. □

What forces the reservation valuation rule is the fact that the option value of future resale $\phi\sigma(x)$ is diminishing in one's valuation by (3). This is intuitive: The higher one values one's own good, the smaller the surplus one expects from any future trade simply because it is harder to find an agreeable swap. Eventually, this option value falls below the flow search cost c, and the individual stops searching.

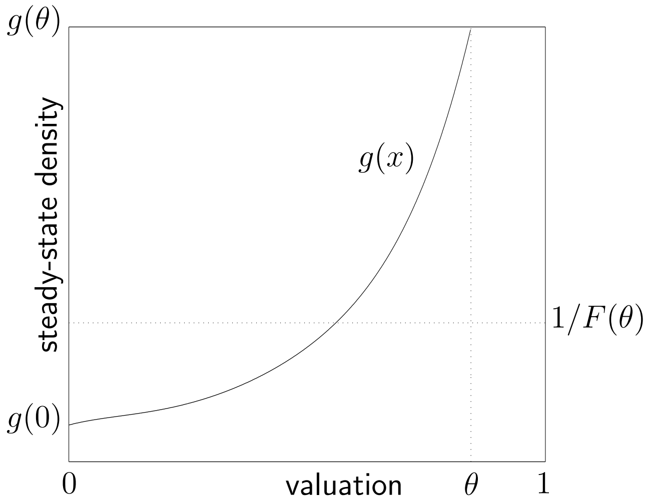

Figure 1: **Equilibrium Density of Valuations.** This figure depicts the graph of the steady-state density g given the uniform entering density f on $(0, 1)$ (thus reducing to $1/F(\theta)$ on $(0, \theta)$), and parameters $c = 0.02$, $r = 0.5$, and $\rho = 50$.

There is a natural cross-sectional implication of the monotonicity result alone.

Proposition 2 (Distributional Monotonicity). *The ratio of the steady-state to the entering density $g(x)/f(x)$ is strictly increasing and bounded above on $[0, \theta)$.*

Proof: It follows immediately from (2) that $g(x)/f(x) = [\delta + \rho \mathcal{Q}(x)]/\rho \mathcal{P}(x)$, and Lemma 1 implies that $\mathcal{Q}$ is increasing, and $\mathcal{P}$ decreasing. Finally, $\mathcal{P}(x)$ is bounded away from zero on $[0, \theta)$, because $[0, \theta) \times [\theta, 1)^2 \subseteq \mathcal{T}(x)$ for all $x \in [0, \theta)$. □

Figure 1 illustrates one such steady-state distribution.

Consider a typical trader's market experience. He will repeatedly engage in trades, first trading to get a particular unit, then trading it away, etc., until a final exchange lifts his valuation over the threshold θ, at which point he stops trading. Convexity of the value function helps to understand these trades.

That $w''(x) > 0$ admits a very simple pure economic intuition that also sheds light into exactly what is going on here, and how it might extend (see the conclusion). Individuals with higher valuations expect to hold onto higher valuation assets longer, as they are less likely to soon find a mutually agreeable trade than are those with lower valuations. Thus, welfare does not fall for a given valuation decrease as much as it rises for the same magnitude increase. Formalizing this intuition,

Proposition 3 (Duration). *An individual expects to hold onto his current good longer the greater is his valuation x. This length of time $\tau(x)$ is boundedly finite provided $x < \theta$, and thereafter it is infinite.*

Proof: The duration is a Poisson event with arrival time $\rho\mathcal{P}(x)$. By the standard inverse relationship between the mean and arrival time for such a stochastic process, I have $\tau(x)^{-1} = \rho\mathcal{P}(x)$. But then $\mathcal{P}'(x) < 0$ implies $\tau'(x) > 0$, while $\mathcal{P}(x)$ boundedly positive on $[0, \theta)$ (see proof of Proposition 2) yields $\lim_{x\uparrow\theta} \tau(x) < \infty$. Of course, one never trades if $x > \theta$, so that $\tau(x) = \infty$ for all $x \geq \theta$. □

5.2 Repeat Trade

An individual's current valuation serves as the state variable for an elaborate continuous time Markov process, where the exit set $\mathcal{X}$ constitutes the absorption states. Repeat trade is the descriptive term for the dynamics in the transitory states, and is most salient feature of market behavior. The next result speaks to a rather natural progression while in the market, with the higher valuation individuals expecting to trade to a higher valuation. Yet so long as one still searches, there is a boundedly positive chance that the next trade will lead one to 'trade down' to a lower valuation.

Proposition 4 (Up and Out). *The higher is one's valuation, the higher is one's expected next valuation. Yet anyone who is still searching expects to trade at least twice more with at least some boundedly positive probability.*

Proof: By (1a) and the fact that w is strictly increasing, the trade away-set is of the form $\mathcal{T}(x) = \{(x', y, y')|(y, y', -x) \geq u(y, y', x', x)\}$. If $b(x) = u_1(y, y', x', x)$ be the least y coordinate, then b is obviously increasing in x, since $\mathcal{T}$ is nested decreasing by Lemma 1. The expected next valuation $\int_{b(x)}^{1} yg(x')f(y)f(y')dx'dydy'/\mathcal{P}(x)$ for x thus increases in x because $b(x)$ does. The rest of the proof is appendicized. □

5.3 When to Trade

I now characterize the trades get consummated. For fixed (x_1, x_2), consider the *trade preference set* consisting of all mutually agreeable new valuations $\mathcal{MA}(x_1, x_2) = \{(y_1, y_2) \,|\, w(y_1) + w(y_2) \geq w(x_1) + w(x_2)\}$. This is analogous to the set of even odds lotteries (y_1, y_2) which a *risk-loving* individual will accept, as depicted in figure 2. Similarly, the *trade frontier* is the boundary $\{(y_1, y_2) \,|\, w(y_1) + w(y_2) = w(x_1) + w(x_2)\}$. For since w is convex, $\mathcal{MA}(x_1, x_2)$ is concave to the origin, or equivalently its complement (all rejected trades) is a convex set containing the origin.

Call a trade from (x_1, x_2) to (y_1, y_2) *risk-increasing* if $|y_1 - y_2| \geq |x_1 - x_2|$, and *valuation-decreasing* if $y_1 + y_2 \leq x_1 + x_2$; analogously define *risk-decreasing* and

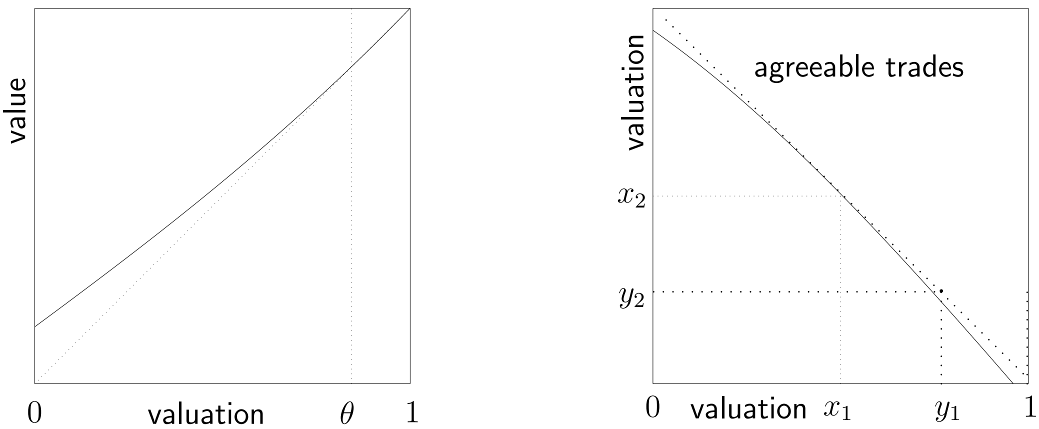

Figure 2: **The Value Function and Mutually Agreeable Trades.** For the parameters $\langle f$ uniform on $(0,1)$, $c = 0.02$, $\delta = 1$, $r = 1$, and $\rho = 50\rangle$, the two figures depict the graph of the Bellman value function, and the corresponding set of mutually agreeable trades $\mathcal{MA}(x_1, x_2)$. Note the resemblence of the left graph to the plot of the value of a standard financial option, where one can think of valuations $x > \theta$ as being 'in the money'. In the right graph, side-payments from the first to the second individual will make the trade from the *status quo ante* (x_1, x_2) to (y_1, y_2) strictly profitable for each. Both individuals plan to trade away their new goods eventually (since $y_1, y_2 < \theta$), but the second expects to retrade his new good sooner than the first.

valuation-increasing trades. Call the trade set $\mathcal{MA}(x_1, x_2)$ *risky* if it differs from $\{(y_1, y_2) \,|\, y_1 + y_2 \geq x_1 + x_2\}$, and contains all valuation- and risk-increasing trades, and only contains valuation-decreasing trades that are risk-increasing, and risk-decreasing trades that are valuation-increasing. The convexity of the continuation value w and the fact that all trades are swaps (given unit demands) together imply the main result: Search frictions induce risky trades.

Proposition 5. *The trade preference set* $\mathcal{MA}(x_1, x_2)$ *is risky.*

The message here is not an excessive propensity to trade when viewed from a purely static perspective. Rather, some seemingly disadvantageous trades (i.e. statically valuation-reducing) are undertaken, while other seemingly advantageous ones (statically valuation-enhancing) are not.

5.4 Comparative Steady States

I now define a partial order. The trade set M is *riskier* than M' if both are risky, $M' \neq M$, and if $(y_1, y_2) \in M \setminus M'$ and $(y_1', y_2') \in M'$ (resp. $(y_1', y_2') \in M' \setminus M$ and $(y_1, y_2) \in M$) with $y_1 + y_2 = y_1' + y_2'$, then (y_1, y_2) is more (resp. less) risk-increasing than (y_1', y_2'). Analogous to the theory of risk aversion, only here for

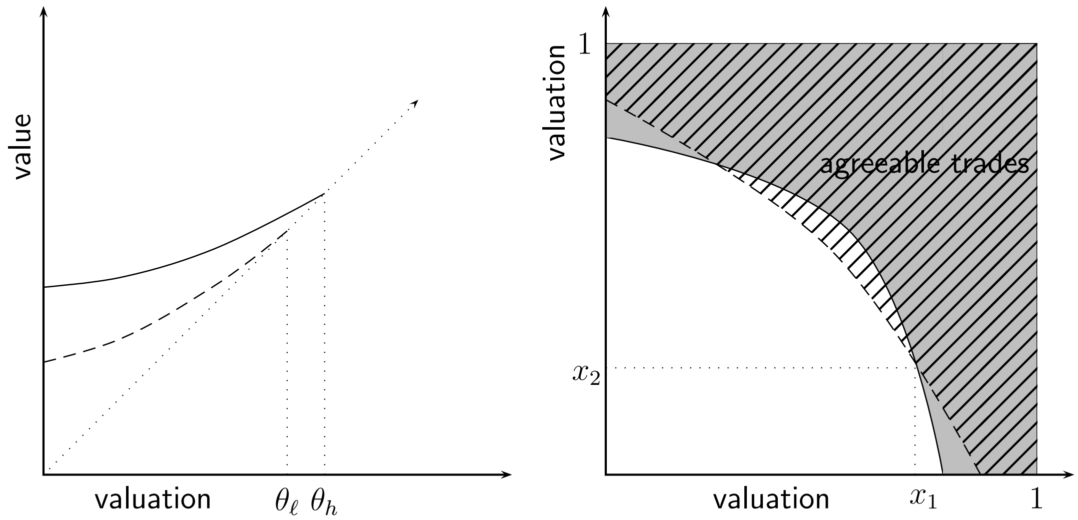

Figure 3: **The Effect of Search Frictions.** In the left panel, dashed lines denote the high search friction (low $\phi = \rho/r$) value function, while solid lines are the low search friction outcome. The less curved trade frontier through the point (x_1, x_2) corresponds to the higher search frictions. Even the left diagrams can illustrate how increased search frictions *lessen* the problem of risk-increasing trades.

risk-lovers, define $\xi(x) = w''(x)/w'(x)$. So the greater is $\xi(x)$, the more locally curved is the trade indifference frontier around the diagonal, and the riskier are the trades. Equivalently, riskier trade sets arise from more convex value functions w.

Consider the effect of worse search frictions — namely, a higher interest r rate or lower rendezvous rate ρ. Because risky trades springs from the search frictions, indirectly via the convexity of the value function, one might intuit that this ought to exacerbate this risk-increasing nature of trade: Strangely, this insight is wrong!

Proposition 6 (Changing Search Frictions). *For large enough search frictions, a local decrease in frictions, i.e. a greater rendezvous rate ρ or a lower interest rate r, renders the exit set $\mathcal{X}$ smaller and the trade set riskier.*

Proof: Increasing ϕ in principle entails recalculating the entire search equilibrium 4-tuple. Treat all variables implicitly as functions of ϕ, and proceed infinitesimally. Differentiating (3) and (4), we have

$$\begin{aligned}\frac{\partial}{\partial\phi}w(x) &= \sigma(x)/2 + \phi\frac{\partial}{\partial\phi}\sigma(x) \\ &= \sigma(x)/2 + \phi\int_{\mathcal{T}(x)} \tfrac{\partial}{\partial\phi}[w(y)+w(y')-w(x)-w(x')]f(y)f(y')g(x') \\ &\quad +\phi\int_{\mathcal{T}(x)}[w(y)+w(y')-w(x)-w(x')]f(y)f(y')(\partial g(x')/\partial\phi)\end{aligned}$$

where just as in the proof of Lemma 3, we can ignore the indirect feedback effects of a changing trade-away set on the surplus, since boundary trades entail zero surplus. For sufficient search frictions, or small enough ϕ, the first term uniformly dominates.[5] Thus, the value $w(x)$ rises for all x, and so $\theta = w(\theta)$ rises too (Lemma 5).

Next, consider the trading decision. By Lemma 3, $\xi(x) = w''(x)/w'(x) = -\phi\mathcal{P}'(x)/(1+\phi\mathcal{P}(x))$. This is an increasing function of ϕ since $\mathcal{P}'(x) < 0$, and the partial effect of $\mathcal{P}(x)$ changing with ϕ is negligible for small enough ϕ, as that term takes the factor ϕ. Thus, near the diagonal $x_1 = x_2$, the set $\mathcal{MA}(x_1, x_2)$ is bigger, or its frontier is locally more curved, the greater is ϕ.

For any off-diagonal valuation pair (x_1, x_2), the steepness of the trade frontier

$$|x_2'(x_1)| = w'(x_1)/w'(x_2) = \frac{1+\phi\mathcal{P}(x_2)}{1+\phi\mathcal{P}(x_1)}$$

is increasing in ϕ, whenever $\mathcal{P}(x_2) > \mathcal{P}(x_1)$ or equivalently $x_1 > x_2$, and is falling in ϕ for $x_2 > x_1$ (again, for small ϕ). Consider the trade frontier at two different values of ϕ that passes through the point (x_1, x_2). By symmetry, the frontier also passes through (x_2, x_1), and thus must pivot through those two points, as in figure 3. So the frontier with ϕ higher lies on the inside between these two points. □

The earlier false intuition ignored the fact that as search frictions lessen, individuals are more willing to trade down expecting to retrade sooner (given the very high option values $\phi\sigma(x)$). By way of contrast, with high search frictions, the option value of resale shrinks, and the overarching question in any trade becomes, "Is this trade valuation-enhancing?" For instance, if but one trade opportunity arises, only trades that enhance this static efficiency measure obtain.

Remark. The proposition only asserts the comparative static for low ϕ, where a proof is accessible; the general case for now is too intractable. But both this intuition and simulations suggest that it obtains for all search frictions.

5.5 Normative Implications

To say that two individuals 1 and 2 might well trade even though $y_1 + y_2 < x_1 + x_2$ does *not* imply that the trading process is ex post inefficient. Propostion 5 asserts that trades might be *statically* inefficient, in the sense that they need not enhance the

[5] Of course, if ρ and not r increases, that also affects the steady-state equation (2). This additional change now does not depend matter since we are ignoring the latter effect.

sum of valuations. But are such privately optimal risky trades necessarily *socially dynamically* inefficient too? Or, are trades more or less risk increasing than is socially optimal? To make efficiency assertions, I must compare the trading decisions with the private and social value functions.

Recall Mortensen's (1982) investigation of the purely *ex ante inefficiency,* such as everyone searches too little, or in this context their exit thresholds are too low. Here, the prime focus is on an *ex post inefficiency* via the trading decisions. This suggestion might appear rather counterintuitive, as transferable utility is often considered a guarantor of ex post efficiency of trade. But when the private and social values do not coincide as in a search environment, this is not so. Such an effect was not present in Mortensen's model, as it arises only in a bilateral exchange model with stochastic heterogeneity (like pure variety). Two traders *having met* might well trade when they should not, or not trade when they should, from the (constrained) social planner's perspective. For the option value of retrade for any good is worth less to each trader than to the planner because, as with Mortensen, each only enjoys but half of any match surplus he creates. With a *social values* different from private values, the ex post trading decision is skewed and generally inefficient.

Lemma 6 (Planner's Problem). *The planner's problem is solved by the same* 4*-tuple as is a search equilibrium, except that* w *and* σ *are replaced by* $\mathcal{V}$ *and* σ^**, where* $\mathcal{V}(x) = x - c + \phi\sigma^*(x)$ *and where* σ^* *is defined as in* (3) *for the values* $\mathcal{V}$*.*

Proof: I introduce a control variable for each decision margin. First, let $\beta(x) = 1$ or 0 as an individual with valuation x should or should not search. Thus, the planner's steady-state payoff from a given valuation x searching is $(x - c)$ if $\beta(x) = 1$ and a one-shot payoff of x/r if x exists at that moment. Given the steady-state density $g(x)$ of searchers, and a flow of $\delta f(x) + \rho f(x)\mathcal{Q}(x)$ of traders into valuation x, the current-valued Hamiltonian is

$$\begin{aligned} \mathcal{H} \;=\; & \int_0^1 \beta(x)\left(r(x-c)g(x) + \mathcal{V}(x)[\delta f(x) + \rho f(x)\mathcal{Q}(x) - \rho g(x)\mathcal{P}(x)]\right)dx \\ & + \int_0^1 x(1-\beta(x))(\delta f(x) + \rho f(x)\mathcal{Q}(x))dx \end{aligned} \tag{5}$$

where the constraint on the steady-state equation $\mathcal{V}(x)$ is interpreted as the cost of that constraint — namely, the social value of having someone with valuation x.

Next, let the control $\alpha(x, x', y, y') = 1$ or 0 as the trade from (x, x') to (y, y') is or is not efficient. Substituting for $\mathcal{Q}(x) = \int_0^1\int_0^1\int_0^1 \beta(y)\beta(y')\alpha(y, y', x, x')f(x')g(y)g(y')$

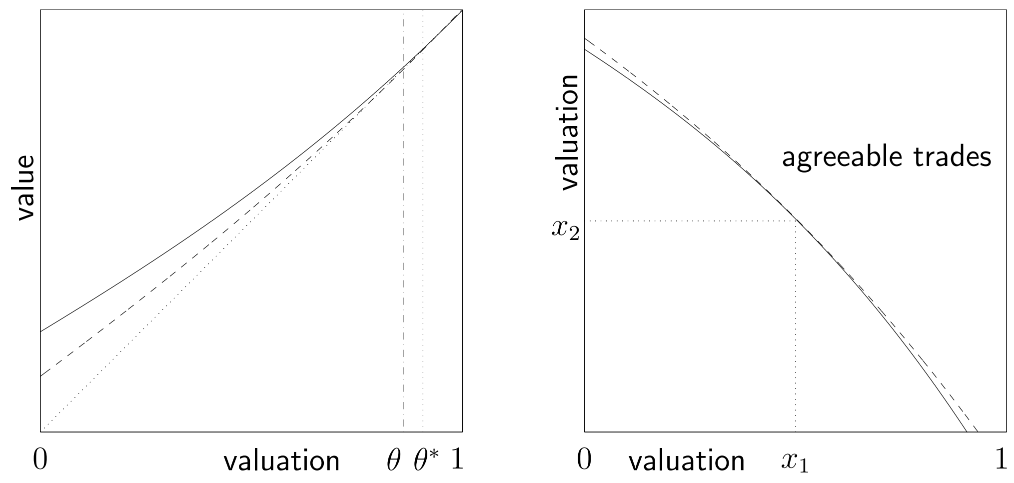

Figure 4: **Optimal versus Equilibrium Comparison.** The solid curves correspond to the social planner's solution, and the dashed curves to decentralized solution. Observe how despite the linear search technology, the optimal exit threshold θ^* exceeds its equilibrium level θ, and that the Bellman value function is higher in the social planner's solution. Near the diagonal, more trades occur in the social planner's world. Parameters are as before, but with $\rho = 25$.

and $\mathcal{P}(x) = \int_0^1 \int_0^1 \int_0^1 \beta(x')\alpha(x, x', y, y')g(x')f(y)f(y')$ in (5), yields:[6] first, the state equation $\mathcal{H}_{\mathcal{V}(x)} = 0$ as expected reproduces the original steady-state constraint (2); the costate equation $\mathcal{H}_{g(x)} = r\mathcal{V}(x)$ yields the formula for the continuation value analogous to (4), but without the factor 1/2; the Kuhn-Tucker boundary optimality condition $\alpha(y, y', x, x') = 1$ or 0 as $\mathcal{V}(x) + \mathcal{V}(x') - \mathcal{V}(y) - \mathcal{V}(y') \gtrless 0$ is the trade optimality condition; and likewise, search optimality follows from the optimality condition for $\beta = 1$ or 0. The algebraic details will very soon be appendicized. □

Proposition 7 (Search and Trade Inefficiency). *In search equilibrium, for high enough search frictions, individuals choose a lower than socially optimal exit threshold level θ (they search too little), and their trade set is not risky enough.*

Proof: A world where both parties to a trade retain the entire match surplus is rather neatly the same as one where individuals (mistakenly) believe that the interest rate is not r but $r/2$, by (4). So consider an infinitesimal movement towards such a regime with lower search frictions. The result now follows from Proposition 6. □

A striking illustration of this inefficiency is seen in the limit as search frictions vanish. For a high enough rendezvous rate, the planner approves a trade from $(1-\varepsilon, 1-\varepsilon)$ to $(1, 0)$; however, traders will not. For since $x \leq w(x) \leq 1$ always,

[6]For brevity, I describe the necessary first order conditions using the notational shortcuts like $\mathcal{H}_{\mathcal{V}(x)}$ and $\mathcal{H}_{g(x)}$.

$\sigma(x) \leq \int_0^1 2(1-x')g(x')dx'$ (integrating out (y, y')). Thus, an individual's option value of retrade is $\sigma(x)/2 < 1$, and thus by (4) has a continuation value $\hat{w}(x) < 1-c$. In particular, $w(0) + w(1) \leq 1 - c + 1 < 2(1-\varepsilon) < 2w(1-\varepsilon)$ for small enough c.

Given such a clearly inefficient outcome, it is natural to inquire of means that will decentralize the planner's constrained efficient solution; however, the fantastically large number of ex post trading decision margins that are potentially inefficient means that any efficient scheme must be formally the same as an ex post subsidy to each trader doubling his match surplus (calculated with respect to his private unsubsidized unmatched value). Most obviously, a simple choice of the right search technology as in Hosios (1990) will not work since operates at the ex ante level. There is a wide array of tax schemes that might improve upon the search equilibrium, but investigating any one is a high cost pursuit with dubious benefits.

6. CORRELATED VALUATIONS

I now assume that valuations for any given good are positively correlated, in some as-yet unspecified sense. Indeed, it rings truer that some goods are on average deemed 'high value' and others 'low value'. Yet even here, trades tend to be risky provided we shy away from the pure quality world (when trade ceases).

A. Common and Idiosyncratic Components. Suppose that goods have both common and individual-specific idiosyncratic components z and x to their valuations $v(x, z)$. As a normalization, v is assumed homogeneous of degree one, with $v_x > 0$ and $v_z > 0$. The 'high quality' goods have high values of z. Still assume stochastically identical traders A-2, but abandon pure variety A-3, and replace independence A-5 with a form of conditional independence: the entering joint distribution of (x, z) is independent, with probability densities f and κ on $(0, 1)$ for x and z, respectively.

The value $w(x, z)$ now takes as a state variable (x, z). Holding the net valuation fixed, goods with higher common valuations z have higher certain 'resale values', but a lower option value. This changes everything. First of all, one's exit decision now also depends on the common component of one's valuation, and search stops as soon as one's idiosyncratic component lies in the exit set $\mathcal{X}(z)$.

A potential trade is a 6-tuple $\langle x_1, y_1, z_1; x_2, y_2, z_2 \rangle$, where z_i denotes the common component for the good initially possessed by agent i. I assume that both the common and idiosyncratic components are common knowledge, thus avoiding a confounding learning problem. The trade-away set is then $\mathcal{T}(x,z) = \{(x', y, y', z') : w(x,z) + w(x', z') \leq w(y,z) + w(y',z')\}$, and has measure $\mathcal{P}(x,z)$. The surplus is $\sigma(x,z) = \int_{\mathcal{T}(x,z)} [w(y,z) + w(y',z') - w(x,z) - w(x',z)]\kappa(z')f(y)f(y')g(x')$.

Proposition 8. *The exit set has the reservation valuation property, $\mathcal{X}(z) = [\theta(z), 1)$. The trade set is less (more) risky in the idiosyncratic component when the two components x and z are complements (substitutes): $v_{xz} \gtrless 0$.*

Proof: Compute the derivative $\sigma_x(x,z) = -w_x(x,z)\mathcal{P}(x,z)$. Then as in Lemma 3, $w_x(x,z) = v_x(x,z) + \phi\sigma_x(x,z) = v_x(x,z)/(1+\phi\mathcal{P}(x,z))$. Because $w_x(x,z) < v_x(x,z)$, we have $\mathcal{X}(z) = [\theta(z), 1)$ for some $\theta(z)$. Finally, as with the proof of Proposition 6, it suffices to check whether the risk-loving coefficient $w''(x)/w'(x)$ rises or falls.

$$\frac{w_{xx}(x,z)}{w_x(x,z)} = \frac{v_{xx}(x,z)}{v_x(x,z)} - \frac{\phi\mathcal{P}_x(x,z)}{1+\phi\mathcal{P}(x,z)}$$

Euler's Theorem then implies $v_{xx} \lessgtr 0$ as $v_{xz} \gtrless 0$, and so the result follows. □

There is a simple intuition for the knife-edge. If there are no cross effects, or $v_{xz} \equiv 0$, then $v(x,z) = bv(x) + (1-b)v(z)$, for some increasing function v. The common component is then just like currency stapled to the back of the good and does not affect trade riskiness in x. But with substitutes $v_{xz} < 0$, the static valuation encourages risky trades as measured by the idiosyncratic component since $v_{xx} > 0$.

This variety of correlation can so skew trade that own-valuations seemingly bear little relationship to the trades that occur. As an example,[7] assume that valuations lie in $[0, 100]$. Let A have a good that whose consumption valuation is 40 to him, 80 to B, and 30 to most others; let B own a good whose valuation to him is 90, to A is 30, and to most others is 80. Then quite plausibly this trade will be mutually agreeable even though the consumption values of *both* individuals will fall, the total going from $40 + 90$ to $80 + 30$. But now B has a good that is worth much more to him than it was to A, while A has one whose retrade value is quite high.

B. Affiliation. The complements case above, where the effect is muted, in fact corresponds to the net valuations being positively correlated. To shed more light on positively correlated valuations, I next employ the stronger concept from auction theory of *affiliation.* Let $h(x,y)$ be the joint valuation density for a given good in a match. So $h(x,y) = f(x)g(y)$ if there is pure variety. But with affiliation, we have

[7] I thank Richard Zeckhauser for this example.

$h(x,y) \equiv f(x)g(y|x) \equiv f(x|y)g(y)$, where the family of conditional densities $\{f(\cdot|\cdot)\}$ or $\{g(\cdot|\cdot)\}$ obeys the monotone likelihood ratio principle (MLRP): $f(x'|y)/f(x|y)$ is increasing in y for fixed $x' > x$. By a property of the MLRP, this increases any term of the form $\int_A \psi f(y|x)h(x',y')$, where A is an upper contour set, and ψ is positive and weakly increasing in y.[8] Since $\mathcal{P}(x) = \int_{(x';y,y')\in\mathcal{T}(x)} f(y|x)h(x',y')dx'dydy'$, affiliation tempers the fall in $\mathcal{P}(x)$ as x increases. Hence, the duration function $\tau(x)$ is not so steeply increasing, which intuitively reduces risk-increasing trades, and suggests:

Proposition 9. *Trades are not as risky when valuations are slightly affiliated.*

Proof: Let the surplus σ be defined as in (3), except with the new affiliated density. By the chain rule, its derivative is $\sigma'(x) = -w'(x)\mathcal{P}(x) + \mathcal{D}(x)$, where $\mathcal{D}(x) \equiv \int_{\mathcal{T}(x)}[w(y) + w(y') - w(x) - w(x')](\partial/\partial x)f(y|x)h(x',y')$. As in Lemma 3,

$$w'(x) = \frac{1+\mathcal{D}(x)}{1+\phi\mathcal{P}(x)} \qquad \Longrightarrow \qquad \frac{w''(x)}{w'(x)} = \frac{\mathcal{D}'(x)}{1+\mathcal{D}(x)} - \frac{\phi\mathcal{P}'(x)}{1+\phi\mathcal{P}(x)}$$

Now, $\mathcal{P}'(x) = \int_{\mathcal{T}(x)}(\partial/\partial x)f(y|x)h(x',y') + \int_{\partial\mathcal{T}(x)} f(y|x)h(x',y')$.[9] Thus, $-\mathcal{P}'(x) > 0$ is lower, since the first term is new and positive, is not swamped by the ambiguous sign second term for small enough affiliation. So the second term of $w''(x)/w'(x)$ is lower with slight affiliation.

Likewise, for small enough affiliation, the ambiguous sign second term of

$$\mathcal{D}'(x) = \int_{\mathcal{T}(x)}\left(-w'(x)\frac{\partial}{\partial x}f(y|x) + [w(y)+w(y')-w(x)-w(x')]\frac{\partial^2}{\partial x^2}f(y|x)\right)h(x',y')$$

is swamped by the negative first term, and so $\mathcal{D}'(x)/(1+\mathcal{D}(x)) < 0$. □

7. CONCLUSION AND EXTENSIONS

A. Summary. Consider *any* world with asset swaps, where for whatever reason assets acquired today might well later be traded away. The basic insight of this paper applies when, for whatever reason, more highly valued assets are expected to be retained longer than those less prized. As a result, one's Bellman value function will be convex in static flow valuation, and trades will be risk-increasing. Inasmuch as

[8] Namely, $z \in A \Rightarrow z' \in A$ for all $z' \geq z$.

[9] Here, I introduce the notation $\int_{\partial A(x)} \psi(x,z)dz = (\partial/\partial x')\int_{A(x')}\psi(x,z)dz|_{x'=x}$.

consumption- or use-values are sometimes measurable, this paper offers predictions that are in principle falsifiable for some markets. I hope that this insight will shed some simple light on sometimes paradoxical transactions that occur in the economy.

B. Falling Demand Curves. I have studied the unit demand case. But what if individuals derive consumption value from more than one durable good, and enter the search market already in possession of one or more goods? In this case, if the flow consumption valuation of a bundle is simply the sum of all valuations, then its average present value is additively separable across goods possessed, $\sum_k w(x_k)$. Search frictions will force an increasing duration function over each separate valuation, and thus $w''(x) > 0$; however, in any trade the units will necessarily go to the highest evaluators. Search-induced convexity alone will not force risky trades.

For risk-increasing trades, there must be 'diminishing returns'. For instance, if n goods are separately valued at $x_1 \geq x_2 \geq \cdots \geq x_n$, then the flow consumption payoff from that bundle is $v = \boldsymbol{\alpha} \cdot \boldsymbol{x} \equiv \sum_{k=1}^{n} \alpha_k x_k$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$. The paper has considered $\alpha_1 = 1$ and $\alpha_k = 0$ for all $k \geq 2$.

Falling demand curves intuitively induces swaps even when a new owner has a lower valuation for a good. The special case of the unit demands — while arguably true in the housing and maybe car markets — is often not valid. With the natural many good analogues of assumption A-1 through A-5, we may let $w(\boldsymbol{x})$ denote the average present value of a goods bundle with vector-valuation $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. It is optimal to stop searching whenever $\boldsymbol{x} \in \mathcal{X}$, the new exit set in $\mathbb{R}^n$. The question now is not simply, 'Does the individual search?' and 'Do two individuals trade?' but also 'What do they trade?'. This additional decision margin (with $C(2n, n)$ possible swaps) renders the dynamic problem much richer and more complex.

To briefly explain the difficulty, consider a two-good context. Given a *potential exchange* between individuals 1 and 2, let vector valuations over one's own goods be $\langle (x_1^1, x_2^1), (x_1^2, x_2^2) \rangle$, and over the other's be $\langle (y_1^1, y_2^1), (y_1^2, y_2^2) \rangle$. The *realized exchange* is the 4-tuple $\langle \boldsymbol{z}^1, \boldsymbol{z}^2 \rangle = \langle (z_1^1, z_2^1), (z_1^2, z_2^2) \rangle$ that maximizes $w(\boldsymbol{z}^1) + w(\boldsymbol{z}^2)$, where either x_j^i or y_j^i appears as some z_ℓ^k, for all i, j. The trade-away set $\mathcal{T}(\boldsymbol{x})$ consists of all $(\boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}') \in \mathbb{R}^6$ with $\boldsymbol{x}' \notin \mathcal{X}$ that induce a trade away from $\boldsymbol{x}$. The surplus $\sigma(\boldsymbol{x})$ is the integral of $[w(\boldsymbol{z}^1) + w(\boldsymbol{z}^2) - w(\boldsymbol{x}) - w(\boldsymbol{x}')]$ over the domain $\mathcal{T}(\boldsymbol{x})$. This is no longer so easily differentiated as before, for either x_1 or x_2 may reappear as one of the four possible arguments of $w(\boldsymbol{z}^1) + w(\boldsymbol{z}^2)$, and each of these domains depends on $\boldsymbol{x}$. This invalidates the critical convexity argument in Lemma 4.

I conjecture that the *sine qua non* of the risk-increasing trade are search frictions, falling demand curves, and stochastically heterogeneous preferences, but I must leave this as an open problem for a much more technically-oriented paper by someone else.

C. Implications for Matching Paradigms. That beauty is in the eye of the beholder is a natural assumption in the search-theoretic matching world.[10] This paper affords some simple insights into the nature of matches that break up or form. Assume that search while matched is feasible, and that break-ups occur, with damages due. If one can search while matched, then the model is much like the exchange paradigm here. To fix ideas, given matches (a, b) and (c, d), the new match (a, c) forms when a meets c if $w(a, c)+w(b, c) > w(a, b)+w(c, d)$. Since higher quality matches are expected to last longer, the value function w will be convex. But as all four parties share in the surplus from the new match, the potential matching inefficiency discussed in the paper is amplified. For instance, two matches with consumption values $(5, 5)$ may dissolve in favor of new matches with outputs $(7, 2)$.

A. APPENDIX: SEARCH EQUILIBRIUM EXISTENCE

A.1 Proof Overview

By Lemma 2, we consider value functions w as elements of the space $C[0, 1]$ of continuous functions on $[0, 1]$ with the sup norm: $\|w\| = \sup_{x\in[0,1]} |w(x)|$. Instead of trade-away sets $\mathcal{T}$, we work with the associated trade indicator functions α, as introduced in the proof of Lemma 6. These are endowed with the weak topology: $\alpha \in \mathcal{L}^1([0, 1]^4)$, having norm $\|\alpha\|_{\mathcal{L}^1} = \int_{[0,1]^4} |\alpha(x, x', y, y')|$. Next, densities g are given a weak topology, so that $\|g\|_{\mathcal{L}^1} = \int_0^1 |g(x)|dx$.

The proofs of the next two results are deferred for now, but will be modifications of similar purely technical arguments in SS.

Lemma A.1. *Posit* A-1 *to* A-6*. Any Borel measurable map $w \mapsto \alpha_w$ from value functions to trade indicator functions is continuous.*

[10] In fact, Peter Diamond tells me that in a similar spirit but an unrelated employment context, Diamond and Maskin (1984) also must consider a four-tuple of convex value functions, and conclude that too few double breaches occur. In other words, there is too little trade.

Lemma A.2. *The map $\alpha \mapsto g_\alpha$ from trade indicator functions to the density implied by the steady-state equation* (2) *is both well-defined and continuous.*

Proof of Proposition 1:
• STEP 1: THE BEST RESPONSE VALUE: I proceed by means of a best response map T from values to values, which is the analogue of the map in price space for Walrasian equilibria. It would be very natural to choose for the best-response map either $T(w(x)) = \max\langle x - c + \phi\sigma(x), x\rangle$, or the implied value of $w(x)$ from expanding out $\sigma(x)$. But neither mapping is closed on a small enough family $\mathcal{G}$. Consider instead the map $T : C[0,1] \mapsto C[0,1]$ given by $Tw(x) = \max\langle x, \hat{T}w(x)\rangle$, where

$$\hat{T}w(x) \equiv \frac{x - c + \phi \int_{[0,1]^3} \max\langle w(y) + w(y') - w(x'), w(x)\rangle g_w(dx')f(y)f(y')}{1 + \phi \bar{g}_w \bar{f}^2}$$

where g_w is the steady-state density implied by the value function w, as in Lemmas A.1 and A.2, and $\bar{g}_w \equiv \int_0^1 g(x)dx$ is the steady-state mass of agents, and $\bar{f} \equiv \int_0^1 f(x)dx$ is the entering total flow of agents. By the characterization in section 4, a fixed point of the mapping $Tw = w$ is a search equilibrium.

As described in Stokey and Lucas (1989), to establish the existence of a fixed point in the mapping T, we need a nonempty, closed, bounded convex subset $\mathcal{G} \subseteq C[0,1]$ such that (*i*) $Tw \in \mathcal{G}$ when $w \in \mathcal{G}$; (*ii*) $T(\mathcal{G})$ is an equicontinuous family; (*iii*) T is continuous as an operator.
• STEP 2: THE FAMILY $\mathcal{G}$: Let $\mathcal{G}$ be the space of Lipschitz functions w on $[0,1]$ satisfying $x \leq w(x) \leq 1$ and $0 \leq w(x_2) - w(x_1) \leq (x_2 - x_1)$ for $x_2 > x_1$. This subset of $C[0,1]$ is clearly nonempty, closed, bounded, and convex. We can also immediately verify that if $w(x) \in [x,1]$ always, then $Tw(x) \in [x,1]$. Also, since w is nondecreasing in x, so is $\max\langle w(y) + w(y') - w(x'), w(x)\rangle$, and thus its integral over (x', y, y'). Thus $\hat{T}w$ is nondecreasing, and hence so is Tw. Applying pointwise the inequality $\max\langle A, B\rangle - \max\langle C, D\rangle \leq \max\langle A - C, B - D\rangle$, for $x_2 > x_1$ we have:

$$\hat{T}w(x_2) - \hat{T}w(x_1) \leq \frac{\phi \int_{[0,1]^3} \max\langle 0, w(x_2) - w(x_1)\rangle g_w(x')f(y)f(y')}{1 + \phi \bar{g}_w \bar{f}^2}$$

Then $\hat{T}w$ has Lipschitz constant 1. Once more appealing to $\max\langle A, B\rangle - \max\langle C, D\rangle \leq \max\langle A - C, B - D\rangle$, Tw has Lipschitz constant 1 as well. So $Tw \in \mathcal{G}$ if $w \in \mathcal{G}$.
• STEP 3: EQUICONTINUITY OF $T(\mathcal{G})$: We prove that for all $\varepsilon > 0$, there is an $\eta > 0$

so that for all $x_2 \in [0,1]$ and $w \in \mathcal{G}$, if $|x_1 - x_2| < \eta$, then $|Tw(x_1) - Tw(x_2)| < \varepsilon$. But this follows from $|\max\langle A, B\rangle - \max\langle A, C\rangle| \leq |B - C|$, as it implies

$$\left|\hat{T}w(x_1) - \hat{T}w(x_2)\right| \leq \frac{\rho \int_{[0,1]^3} |w(x_1) - w(x_2)|\, g_w(x')f(y)f(y')}{1 + \phi\bar{g}\bar{f}^2} < |w(x_1) - w(x_2)|$$

• STEP 4: CONTINUITY OF T: This algebraic exercise is deferred for now. □

B. APPENDIX: OMITTED PROOFS

B.1 Proof of Lemma 2:

By (3) and (4), for any agent x and any trading set $T \subseteq [0,1]^3$,

$$w(z) \geq z - c + (\phi/2)\int_T (w(y) + w(y') - w(z) - w(x'))\, g(x')f(y)f(y')dx'dydy' \qquad (6)$$

By subtracting (6) from (4), with $T = \mathcal{T}(x)$, we find that

$$w(x) - w(z) \leq x - z + (\phi/2)\int_{\mathcal{T}(x)} (w(z) - w(x))\, g(x')f(y)f(y')dx'dydy'$$

and so $w(x) - w(z) \leq (x - z)/(1 + \phi\mathcal{P}(x)/2) \leq x - z$. By inverting (3), we see that σ is a linear function, plus a Lipschitz one, and thus is Lipschitz. □

B.2 Deferred Portion of Proof of Lemma 3:

Just as in SS, the implication of the Fundamental Theorem of Calculus only requires that the trade-to sets $\mathcal{T}$ be a.e. continuous,[11] and the value function Lipschitz w. For their argument is quite general, and only requires that the pointwise surplus s be continuous in (x, x', y, y'). This is true by Lemma 2.

B.3 Completion of Proof of Lemma 4:

As $N(x) = 0$ for all $x \in \mathcal{X}$, the expected number $N(x)$ of further trades obeys the recursion

$$N(x) = 1 + \int_{\mathcal{T}(x)\setminus\mathcal{X}} N(y)g(x')f(y)f(y')dx'dydy'/\mathcal{P}(x)$$

[11] This is in the topology induced by the *Hausdorff* metric for sets.

But $A = \{(x', y, y')|w(y) + w(y') \geq \theta + w(x'), x', y < \theta\} \subseteq \mathcal{T}(x) \setminus \mathcal{X}$. Because $\{(x', y, y')|x' \leq \bar{x}, y + y' \geq \theta + w(\bar{x}), y < \theta\} \subseteq A$ for any $0 < \bar{x} < \theta$, the set A measure $\geq m > 0$. Then $N(x) \geq 1 + m$ for all $x \in (0, \theta)$ since $N(y) \geq 1$ at all $y < \theta$.

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