

## 12 Threshold Models

### 12.1 Introduction

### 12.2 Small population

Consider a small (finite) population composed of  $n$  individuals. Each individual  $i$  chooses to participate in some form of collective action (e.g., a protest rally) if and only if she expects sufficient participation by others. To be more precise, suppose that individual  $i$ 's *threshold level* is given by  $\theta(i) \in \{0, 1, \dots\}$ . Further letting  $x$  denote the expected number of participants, individual  $i$  chooses to participate if

$$x \geq \theta(i)$$

and chooses not to participate if

$$x < \theta(i).$$

Note that individuals with lower threshold levels are *more* willing to participate. Indeed, any individual  $i$  with threshold  $\theta(i) = 0$  will *always* participate regardless of the expected number of participants (i.e., for any  $x \in \{0, \dots, n\}$ ). Conversely, any individual  $i$  with threshold  $\theta(i) > n$  will *never* participate. The remaining individuals – those with intermediate threshold levels – may (or may not) participate depending on the precise number of participants expected.

For any expected participation level  $x$ , let  $f(x)$  denote the number of individuals with threshold levels equal to  $x$ . Adopting statistical terminology,  $f(x)$  is the *frequency distribution* of thresholds. We may further let  $F(x)$  denote the number of individual with threshold levels *less than or equal to*  $x$ . This *cumulative frequency distribution* is given by

$$F(x) = \sum_{x'=0}^x f(x').$$

This cumulative function plays an important role in threshold models. Because each individual participates if and only if her threshold is less than or equal to  $x$ ,  $F(x)$  indicates the number of individuals who would choose to participate if the expected level of participation was equal to  $x$ .

To develop a process model, suppose that individuals choose whether to participate in each period  $t \in \{0, 1, 2, \dots\}$ . Suppose further that individuals have *adaptive expectations*, so that expected participation in period  $t + 1$  is equal to actual participation in period  $t$ . Under this assumption, the generator function for the process

is simply the cumulative distribution function, and dynamics are determined by the equation

$$x_{t+1} = F(x_t).$$

Given the initial condition  $x_0$ , we thus obtain

$$\begin{aligned} x_1 &= F(x_0) \\ x_2 &= F(x_1) = F(F(x_0)) \\ x_3 &= F(x_2) = F(F(x_1)) = F(F(F(x_0))) \end{aligned}$$

and so on. It is possible to show that this sequence of participation levels must be monotonic (either weakly rising or else weakly falling over time), and that the process must eventually reach a fixed point  $x^*$  such that  $x^* = F(x^*)$ .<sup>1</sup>

To illustrate, consider the following example with  $n = 10$  individuals.

```
>> theta; % theta(i) is threshold level for individual i

h =
     3     5     4  Inf     6     5     2     8     4     5

>> f = histc(theta, 0:10) % frequency distribution

f =
     0     0     1     1     2     3     1     0     1     0     0

>> F = cumsum(f) % cumulative frequency distribution

F =
     0     0     1     2     4     7     8     8     9     9     9
```

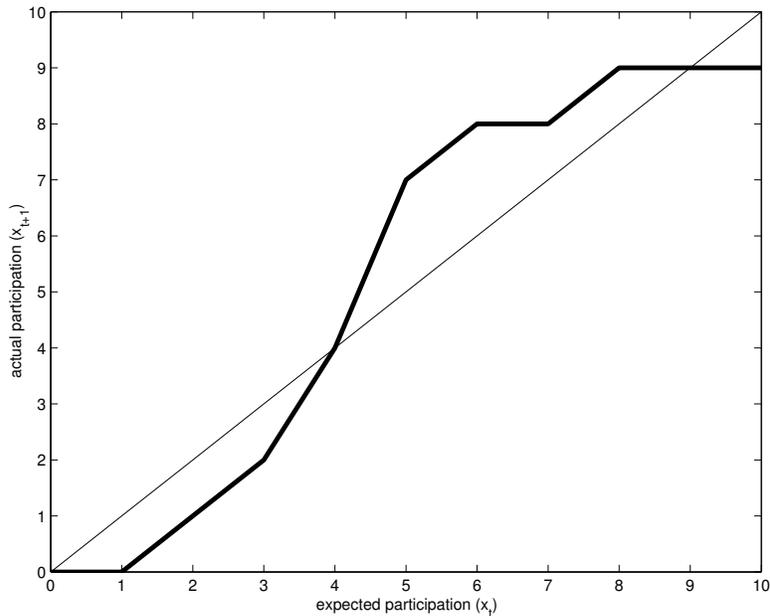
Note that the vectors  $\mathbf{f}$  and  $\mathbf{F}$  have 11 elements, corresponding to values of  $x \in \{0, \dots, 10\}$ . Further, because individual 4 has a threshold greater than 10 (indeed, has an infinite threshold), we find that  $F(10) < 10$ . Substantively, even if everyone was expected to participate, individual 4 would not choose to participate.

To analyze the dynamics of the system, we can plot the generator function  $F(x)$  against the 45-degree line.

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<sup>1</sup>These claims may become obvious after the example below, but a formal proof may still be instructive. To see that the sequence  $\{x_0, x_1, \dots\}$  must be monotonic, note that any cumulative function  $F(x)$  must be non-decreasing in  $x$ . That is,  $x' \geq x''$  implies  $F(x') \geq F(x'')$ . Thus,  $F(x_t) \geq x_t$  implies  $F(F(x_t)) \geq F(x_t)$ . Conversely,  $F(x_t) \leq x_t$  implies  $F(F(x_t)) \leq F(x_t)$ . To see that the process eventually reaches a fixed point, suppose  $F(x_t) \neq x_t$ . Because the population has a finite number of individuals, either  $F(x_t) \geq x_t + 1$  or else  $F(x_t) \leq x_t - 1$ . However,  $F(x)$  is bounded from below and above:  $F(0) \geq 0$  and  $F(n) \leq n$ . Thus, the process must reach a fixed point within  $n$  periods.

```
>> plot(0:10,F,0:10,0:10) % threshold diagram
```



From this diagram, we see that the process has 3 fixed points, given by  $x^* = 0$ ,  $x^* = 4$ , and  $x^* = 9$ . Inspection further reveals that the upper and lower fixed points are stable, while the intermediate fixed point is unstable. Note that, consistent with the claim made above, the process converges to one of the stable fixed points for any initial condition  $x_0 \in \{0, \dots, 10\}$ . You can also verify that any sequence is monotonic. For example, given the initial condition  $x_0 = 5$ , we obtain

$$\begin{aligned} x_1 &= F(5) = 7 \\ x_2 &= F(7) = F(F(5)) = 8 \\ x_3 &= F(8) = F(F(7)) = F(F(F(5))) = 9 \end{aligned}$$

and the process remains at the upper fixed point  $x^* = 9$  for every subsequent period. Alternatively, given the initial condition  $x_0 = 3$ , we obtain

$$\begin{aligned} x_1 &= F(3) = 2 \\ x_2 &= F(2) = F(F(3)) = 1 \\ x_3 &= F(1) = F(F(2)) = F(F(F(3))) = 0 \end{aligned}$$

and the process remains at the lower fixed point  $x^* = 0$  for every subsequent period.

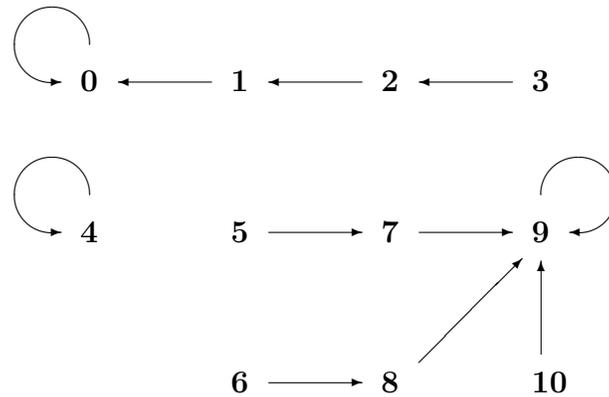
### 12.2.1 A Markov chain specification

To help bridge the divide between linear and non-linear models – the two halves of this of this book – it may be useful to see that our present example can be respecified

as a Markov chain process. In each period, the state of the process is given by the number of individuals currently articpating. Thus, the set of possible states is

$$\{0, \dots, 10\}.$$

Transitions are completely deterministic. More precisely, if the process occupies in state  $x$  in period  $t$ , then it surely (with probability 1) occupies state  $F(x)$  in period  $t + 1$ . We thus obtain the (zero-pattern) transition diagram below.



Alternatively, we can specify the transition matrix, and then obtain the long-run outcome through iterated multiplication.

```
>> P = zeros(11); for x = 1:11; P(x,F(x)+1) = 1; end; P % transition matrix
```

```
P =
    1    0    0    0    0    0    0    0    0    0    0
    1    0    0    0    0    0    0    0    0    0    0
    0    1    0    0    0    0    0    0    0    0    0
    0    0    1    0    0    0    0    0    0    0    0
    0    0    0    0    1    0    0    0    0    0    0
    0    0    0    0    0    0    0    1    0    0    0
    0    0    0    0    0    0    0    0    1    0    0
    0    0    0    0    0    0    0    0    0    1    0
    0    0    0    0    0    0    0    0    0    0    1
    0    0    0    0    0    0    0    0    0    0    1
```

```
>> P^3 % long-run outcome (from each initial state)
```

```
ans =
    1    0    0    0    0    0    0    0    0    0    0
    1    0    0    0    0    0    0    0    0    0    0
    1    0    0    0    0    0    0    0    0    0    0
    1    0    0    0    0    0    0    0    0    0    0
```

0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0

Using either the transition diagram or transition matrix, we thus find that the process reaches one of the absorbing states within 3 periods. More precisely, the process is absorbed in state **0** for any initial condition  $x_0 \in \{\mathbf{0}, \dots, \mathbf{3}\}$ , is absorbed in state **4** given the initial condition  $x_0 = \mathbf{4}$ , and is absorbed in state **9** for any initial condition  $x_0 \in \{\mathbf{5}, \dots, \mathbf{10}\}$ .

We thus find a correspondence between the fixed points of dynamical systems and the absorbing states of Markov chains. However, while the dynamical systems perspective permits further distinction between stable and unstable fixed points (determined by the slope of the generator function at the fixed point), there is no direct analog to this stability concept in Markov chain analysis.<sup>2</sup> Further, it is apparent that the Markov chain specification will become impractical as the number of individuals in the population grows larger.

### 12.3 Large population

We now consider threshold models with a large (infinite) population. Conceptually, the model remains quite similar to the small-population version developed in the preceding section. However, we now assume that thresholds depend on the *proportion* of the population participating in the collective action. That is, individual  $i$  chooses to participate if

$$x \geq \theta(i)$$

and chooses not to participate if

$$x < \theta(i)$$

where  $x \in [0, 1]$  is the expected *proportion* of the population participating in the collective action. Relatedly, we now specify  $f(x)$  as a *probability density function* (pdf). Like any pdf, this function must satisfy the condition

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

---

<sup>2</sup>We could potentially employ the concept of stochastic stability discussed in Chapters 5 and 6. However, there is no necessary correspondence between the stable fixed points of dynamical systems and the stochastically stable states of Markov chains. The dynamical systems stability concept is “local” (reflecting dynamics very close to the fixed point) while stochastic stability involves more “global” considerations (involving the “distance” from the absorbing state to intermediate states which can “reach” other absorbing states).

Graphically, the area under the  $f(x)$  curve must be equal to 1. The corresponding *cumulative distribution function* (cdf) is given by

$$F(x) = \int_{-\infty}^x f(x')dx'.$$

Graphically,  $F(x)$  gives the area under the  $f(x')$  curve for  $x' \leq x$ .

Because  $F(x)$  indicates the proportion of the population with thresholds less than or equal to  $x$ , an expected participation rate of  $x$  implies an actual participation rate of  $F(x)$ . Retaining our assumption of adaptive expectations, the dynamics of the model are again determined by the equation

$$x_{t+1} = F(x_t)$$

and  $F(x)$  is again the generator function.

Obviously, the proportion of the population participating in the collective action is bounded between 0 and 1. Thus, it might also seem natural to require that  $\theta(i) \in [0, 1]$  for all  $i$ . However, to capture the possibility that some individuals always participate, we allow some individuals to have negative thresholds. Hence,  $F(0) \geq 0$  indicates the proportion of the population who always participate. Similarly, to capture that possibility that some individuals never participate, we assume that some individuals have thresholds greater than 1. Hence,  $F(1) \leq 1$  indicates the proportion of the population who might be willing to participate, and  $1 - F(1) \geq 0$  indicates the proportion of the population who never participate.

To illustrate, suppose that thresholds are distributed uniformly between  $-2/5$  and  $6/5$ . That is,

$$f(x) = \begin{cases} 5/8 & \text{for } x \in [-2/5, 6/5] \\ 0 & \text{otherwise} \end{cases}.$$

Graphically, this pdf is a rectangle with height  $5/8$  and width  $8/5$  (and hence area equal to 1 as required). Thus, the proportion of the population with thresholds less than or equal to  $x$  is given by

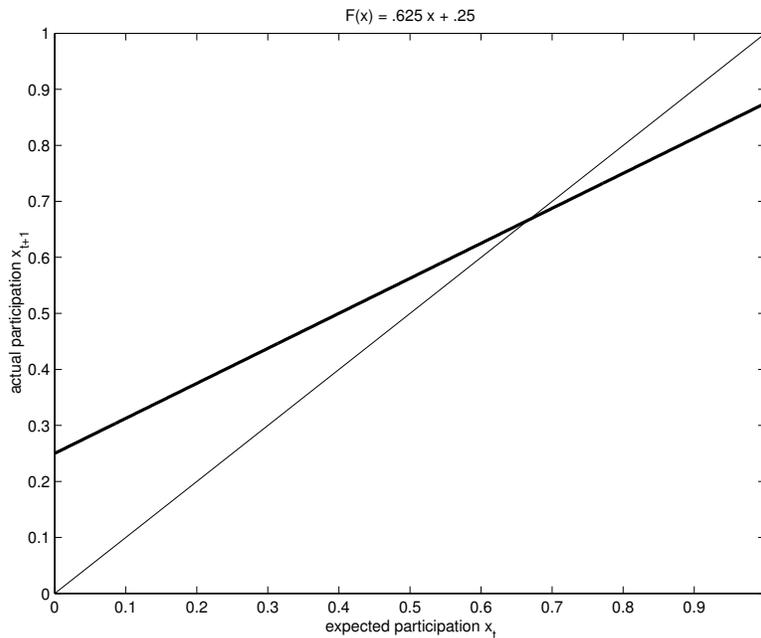
$$F(x) = (5/8)(x + 2/5) = (5/8)x + 1/4$$

for any  $x \in [0, 1]$ .<sup>3</sup> In particular, note that  $F(0) = 1/4$  and  $F(1) = 7/8$ . Thus, 25% of the population will always participate while 12.5% of the population will never participate. To analyze the dynamics of the model, we plot the generator function  $F(x)$  against the 45-degree line.

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<sup>3</sup>While this cdf can be found through integration of the pdf, it is also easy to derive graphically, since  $F(x)$  is the area of a rectangle with height  $5/8$  and width  $(x + 2/5)$ .

```
>> hold on; ezplot(' .625*x + .25', [0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold diagram
```



Solving algebraically for the unique fixed point shown on this diagram, we obtain

$$\begin{aligned} x^* &= F(x^*) \\ x^* &= (5/8)x^* + 1/4 \\ x^* &= 2/3 \end{aligned}$$

This fixed point is stable because

$$|F'(2/3)| = |5/8| < 1$$

Further, it is apparent from the diagram that the process will eventually converge to this fixed point for any initial condition  $x_0 \in [0, 1]$ . For instance, given  $x_0 = 0.1$ , we obtain the time path  $\{x_0, x_1, \dots\}$  given below.

```
>> x = .1; y = x; for t = 1:15; x = .5*x + .25; y = [y; x]; end; y
```

```
y =
    0.1000
    0.3000
    0.4000
    0.4500
    0.4750
    0.4875
    0.4938
```

0.4969  
 0.4984  
 0.4992  
 0.4996  
 0.4998  
 0.4999  
 0.5000  
 0.5000  
 0.5000

To offer another example, suppose now that thresholds are distributed uniformly between  $1/10$  and  $3/5$ . This implies

$$f(x) = \begin{cases} 2 & \text{for } x \in [1/10, 3/5] \\ 0 & \text{otherwise} \end{cases}$$

so that

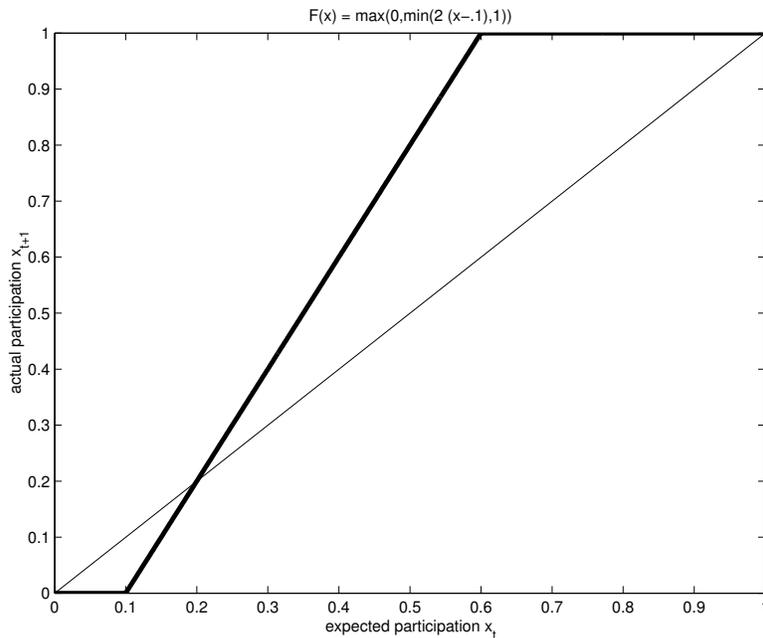
$$F(x) = \begin{cases} 0 & \text{for } x \in [0, 1/10] \\ 2(x - 1/10) & \text{for } x \in [1/10, 3/5] \\ 1 & \text{for } x \in [3/5, 1] \end{cases}$$

Rewriting this piecewise linear function as

$$F(x) = \max\{0, \min\{2(x - 1/10), 1\}\},$$

we can use Matlab to plot it against the 45-degree line.

```
>> hold on; ezplot('max(0,min(2*(x-.1),1))',[0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold diagram
```



For this example, there are no individuals who would always participate ( $F(0) = 0$ ) and no individuals who never participate ( $F(1) = 1$ ). Consequently, both  $x^* = 0$  and  $x^* = 1$  are fixed points. Further, both of these fixed points are stable because

$$|F'(0)| = |F'(1)| = |0| < 1.$$

From the diagram, we see there is another fixed point which lies on the upward-sloping segment of the threshold curve. This fixed point can be determined algebraically by setting

$$\begin{aligned} x^* &= F(x^*) \\ x^* &= 2(x^* - 1/10) \\ x^* &= 1/5 \end{aligned}$$

This fixed point is unstable because

$$|F'(1/5)| = |2| > 1.$$

Further, it is apparent from the diagram that the process will converge to the lower stable fixed point ( $x^* = 0$ ) for any initial condition  $x_0 \in [0, 1/5)$ , and will converge to the upper stable fixed point ( $x^* = 1$ ) for any initial condition  $x_0 \in (1/5, 1]$ . For instance, given the initial condition  $x_0 = 1/4$ , we obtain the time path  $\{x_0, x_1, \dots\}$  computed below.

```
>> x = .25; y = x; for t = 1:5; x = max(0, min(2*(x-.1), 1)); y = [y; x]; end; y
y =
    0.2500
    0.3000
    0.4000
    0.6000
    1.0000
    1.0000
```

Both of the preceding examples assumed that thresholds are uniformly distributed. While this simplifies derivation of the cdf, perhaps it is more realistic to assume a smoother distribution function. For a final example, we'll assume that  $F(x)$  is the cdf for the normal distribution. To make use of Matlab functions, this distribution can be written as

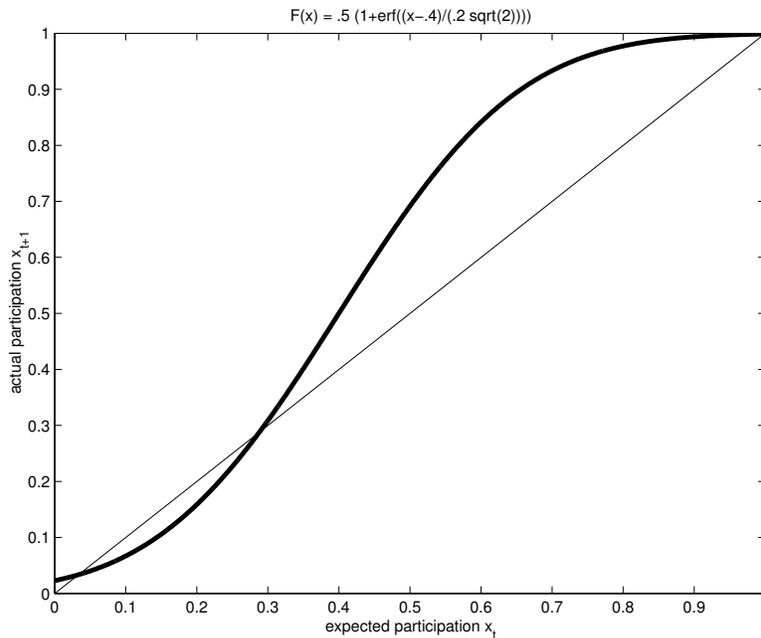
$$F(x) = (1/2) \left( 1 + \operatorname{erf} \left( \frac{x - \mu}{\sigma\sqrt{2}} \right) \right)$$

where  $\mu$  is the mean of the distribution,  $\sigma$  is the standard deviation, and

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$$

Setting  $\mu = 0.4$  and  $\sigma = 0.2$ , we can once again plot  $F(x)$  against the 45-degree line to analyze the dynamics of the model.

```
>> hold on; ezplot(' .5*(1+erf((x-.4)/(.2*sqrt(2))))', [0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold diagram
```



Qualitatively, this example is similar to the previous one, since there are stable upper and lower fixed points and an unstable intermediate fixed point. However, because the normal distribution has positive density  $f(x)$  for values of  $x$ , there will be some individuals who always participate and others who never participate. More precisely, computing  $F(0)$  and  $F(1)$ , we find

```
>> x = 0; .5*(1+erf((x-.4)/(.2*sqrt(2)))) % F(0)
```

```
ans =
    0.0228
```

```
>> x = 1; .5*(1+erf((x-.4)/(.2*sqrt(2)))) % F(1)
```

```
ans =
    0.9987
```

which indicates that 2.28% of the population will always participate while 0.13% of the population will never participate. Consequently, for the present example, the lower fixed point is greater than 0, while the upper fixed point is less than 1. To compute these stable fixed points, we can choose the initial conditions  $x_0 = 0$  and  $x_0 = 1$ , and iterate until  $x_t$  converges to the corresponding fixed point.

```
>> x = 0; y = x; for t = 1:8; x = .5*(1+erf((x-.4)/(.2*sqrt(2))))); y = [y; x]; end; y
```

```

y =
    0
    0.0228
    0.0296
    0.0320
    0.0329
    0.0332
    0.0333
    0.0334
    0.0334

```

```
>> x = 1; y = x; for t = 1:5; x = .5*(1+erf((x-.4)/(.2*sqrt(2)))); y = [y; x]; end; y
```

```

y =
    1.0000
    0.9987
    0.9986
    0.9986
    0.9986
    0.9986

```

Thus, the lower fixed point is at  $x^* = .0334$  while the upper fixed point is at  $x^* = .9986$ .

## 12.4 Removing individuals from the group

When thresholds are based on the group participation rate, it may sometimes be possible to alter the equilibrium participation rate by removing individuals from the group. For instance, leaders of some political or religious movements might attempt to increase participation rates by excluding those followers with the highest thresholds – those individuals who are least inclined to participate in group activities. The removal of these individuals has an obvious “direct” effect on the participation rate: individuals who were not participating anyway are no longer included in the denominator of this rate. But perhaps more crucially, removal of these individuals may also have an “indirect” effect: the higher participation rate causes some remaining individuals to switch from non-participation to participation.

In other settings, leaders might remove individuals from the group in order to *decrease* the equilibrium participation rate. For instance, consider a grade-school classroom in which each child chooses whether to “act out.” To help preserve order, the teacher might send those children with the lowest thresholds – those most inclined to misbehave – to the principal’s office. Having (at least temporary) altered the distribution of thresholds in the group, the teacher may thus decrease the equilibrium level of acting out, and perhaps alter the behavior of some of the remaining children. Presumably, similar logic underlied the (rather more insidious) Soviet policy of exiling dissidents to Siberia.

More formally, consider a group in which thresholds are originally distributed according to the density function  $f(x)$  with cumulative function  $F(x)$ . By removing from the group those individuals with thresholds above  $z$ , we are truncating (“cutting off”) the upper tail of the density function  $f(x)$ . The area under the density function is now less than 1. To renormalize the density function, we divide by  $F(z)$ , the proportion of the original group which remains. Thus, for the remaining (“lower”) group, the probability density function is

$$l(x) = \begin{cases} f(x)/F(z) & \text{for } x \leq z \\ 0 & \text{for } x > z \end{cases}$$

and the cumulative distribution function is

$$L(x) = \begin{cases} F(x)/F(z) & \text{for } x \leq z \\ 1 & \text{for } x > z \end{cases}$$

Given this new threshold curve, we can analyze the model as before (plotting the threshold curve against the 45-degree line), so fixed points are now determined by the condition  $x^* = L(x^*)$ .

Conversely, if we remove from the original group those individuals with thresholds *below*  $z$ , we truncate the lower tail of the density function. The portion of the group which remains is now given by  $1 - F(z)$ . Thus, for the remaining (“higher”) group, the probability density function is

$$h(x) = \begin{cases} 0 & \text{for } x < z \\ f(x)/(1 - F(z)) & \text{for } x \geq z \end{cases}$$

and the cumulative distribution function is

$$H(x) = \begin{cases} 0 & \text{for } x < z \\ (F(x) - F(z))/(1 - F(z)) & \text{for } x \geq z \end{cases}$$

and fixed points are given by the condition  $x^* = H(x^*)$ .

To illustrate, let’s return to our first example from the preceding section where

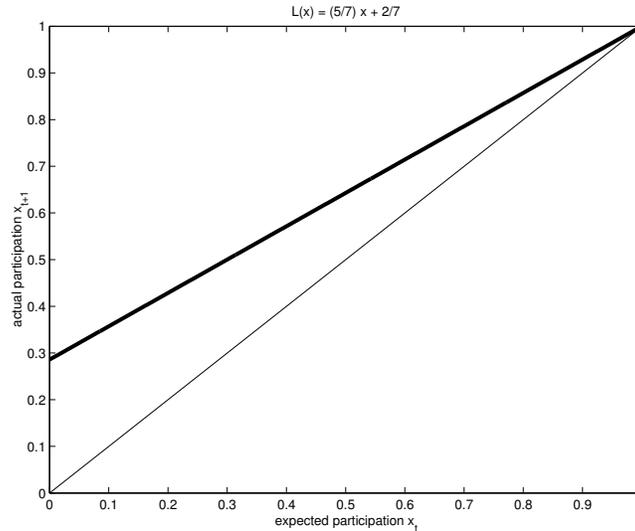
$$F(x) = (5/8)x + 1/4$$

which implied a unique equilibrium participation rate of  $x^* = 2/3$ . If we remove from this group those individuals with thresholds above  $z = 1$ , we obtain

$$L(x) = F(x)/F(1) = (5/7)x + 2/7$$

for all  $x \in [0, 1]$ . Plotting this generator function, it is apparent that the participation rate converges to the unique fixed point  $x^* = 1$  for any initial condition.

```
>> hold on; ezplot('(5/7)*x + 2/7',[0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold curve
```



Further, decomposing the total change in the equilibrium participation rate (from  $2/3$  to  $1$ ) into direct and indirect effects, it is interesting to note that most of the increase in participation is due to the latter effect. More precisely, the fraction of the remaining group with thresholds below the original participation rate is  $L(2/3) = 16/21$ . Thus, the participation rate rose from  $2/3$  to  $16/21$  as a consequence of the direct effect.<sup>4</sup> The remaining change in the participation rate (from  $16/21$  to  $1$ ) is thus the consequence of the indirect effect. Some individuals (those with thresholds between  $2/3$  and  $1$ ) who initially chose not to participate have now changed their behavior.

Alternatively, if we remove from the group those individuals with thresholds below  $z = 1/2$ , we obtain

$$H(x) = \begin{cases} 0 & \text{for } x < 1/2 \\ (10/7)x - 5/7 & \text{for } x \geq 1/2 \end{cases}$$

which may be restated as

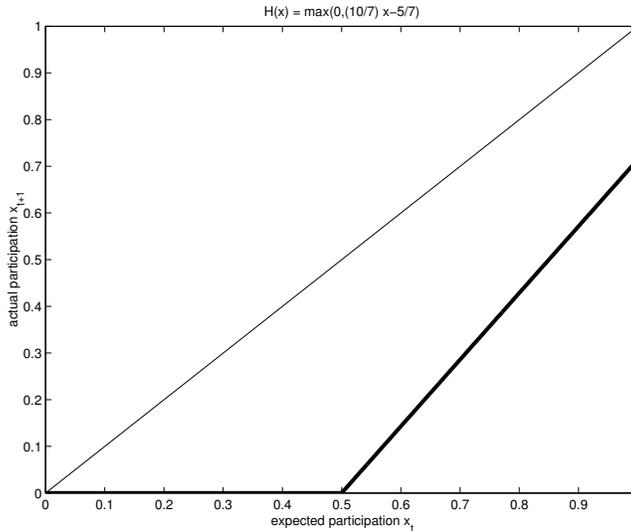
$$H(x) = \max\{0, (10/7)x - 5/7\}.$$

Plotting this generator function, we find that the participation rate will converge to  $0$  for any initial condition.

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<sup>4</sup>To obtain this result in a different way, note that the non-participants removed from the group constituted  $(1/8)$ th of the original group. Thus, the direct effect causes participation to rise from  $(2/3)/1$  to  $(2/3)/(7/8) = 16/21$ .

```
>> hold on; ezplot('max(0,(10/7)*x-5/7)',[0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold curve
```



For this example, the direct effect of removing participants accounts for a drop in equilibrium participation from  $2/3$  to  $H(2/3) = 5/21$ , while the remaining drop in participation (from  $5/21$  to  $0$ ) results from the indirect effect.

## 12.5 Declining threshold curves

To this point, we have assumed that an increase in expected participation will never *decrease* actual participation. But for some applications of threshold models – suppose that individuals are deciding whether to attend a community festival or to eat at a particular restaurant – overcrowding could potentially make participation less attractive.<sup>5</sup> To proceed formally, we now assume that each individual  $i$  is characterized by both a lower threshold  $\underline{\theta}(i)$  and an upper threshold  $\bar{\theta}(i)$ , and will choose to participate if and only if

$$\underline{\theta}(i) \leq x \leq \bar{\theta}(i)$$

where  $x$  is the expected participation rate. To interpret, if  $x < \underline{\theta}(i)$ , then individual  $i$  is unwilling to participate because she expects that participation will be too low. Conversely, if  $x > \bar{\theta}(i)$ , then individual  $i$  is unwilling to participate because she expects that participation will be too high. Essentially, our preceding analysis incorporated only the lower threshold, implicitly setting  $\bar{\theta}(i) = \infty$  for all  $i$ .

Given that each individual has two thresholds, we will now need to keep track of two distributions. As before, let  $f(x)$  and  $F(x)$  denote the pdf and cdf for the

---

<sup>5</sup>To quote Yogi Berra: “No one goes there anymore; it’s too crowded.” Of course, taken literally, this claim makes no sense. But it does indicate that some individual are driven away by overly high levels of participation.

lower threshold  $\underline{\theta}$ . Further, let  $g(x)$  and  $G(x)$  denote the pdf and cdf for the upper threshold  $\bar{\theta}$ . Because we assume  $\underline{\theta}(i) \leq \bar{\theta}(i)$  for all  $i$ , this implies

$$F(x) \geq G(x) \text{ for all } x.$$

Adopting statistical terminology, we say that the  $F$  distribution is *stochastically dominated* by the  $G$  distribution (or, equivalently, that  $G$  *stochastically dominates*  $F$ ). Continuing to assume adaptive expectations, the dynamics of the model are determined by

$$x_{t+1} = F(x_t) - G(x_t).$$

Intuitively, given the expected participation rate  $x_t$ ,  $F(x_t)$  indicates the proportion of the population above their lower threshold (who feel that expected participation is high enough to warrant their own participation). However, to determine actual participation, we must subtract  $G(x_t)$ , the proportion of the population above their upper threshold (who have “dropped out” because of overcrowding).

To illustrate, suppose that the cdf for lower thresholds is given by

$$F(x) = \begin{cases} 1/5 + 2x & \text{for } x \in [0, 2/5] \\ 1 & \text{for } x \in [2/5, 1] \end{cases}$$

which can be restated as

$$F(x) = \min\{1/5 + 2x, 1\}.$$

Further suppose that the cdf for upper thresholds is given by

$$G(x) = \begin{cases} 0 & \text{for } x \in [0, 1/5] \\ -1/10 + (1/2)x & \text{for } x \in [1/5, 1] \end{cases}$$

which can be restated as

$$G(x) = \max\{0, -1/10 + (1/2)x\}.$$

Thus, we obtain

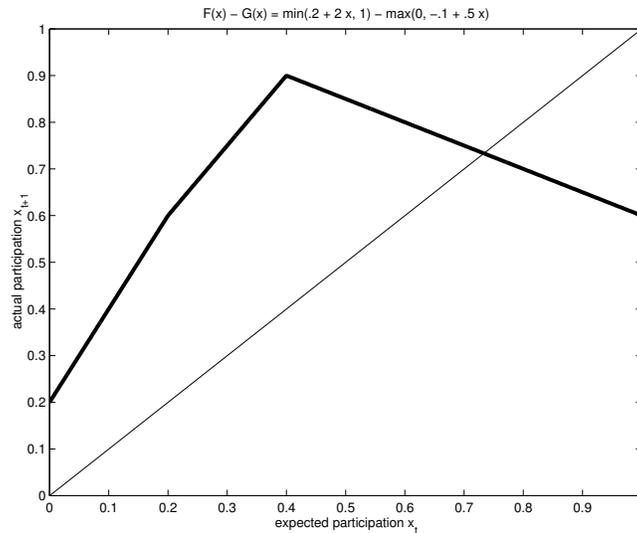
$$F(x) - G(x) = \begin{cases} 1/10 + 2x & \text{for } x \in [0, 1/5] \\ 3/10 + 3/2x & \text{for } x \in [1/5, 2/5] \\ 11/10 - (1/2)x & \text{for } x \in [2/5, 1] \end{cases}$$

or we may simply write

$$F(x) - G(x) = \min\{1/5 + 2x, 1\} - \max\{0, -1/10 + (1/2)x\}.$$

Plotting this generator function, we find a unique fixed point.

```
>> hold on; ezplot('min(.2+2*x, 1) - max(0, -.1+.5*x)', [0,1,0,1]); plot(0:1,0:1); hold off
>> % threshold curve
```



Noting that this fixed point lies in the interval  $[2/5, 1]$ , some algebra yields

$$\begin{aligned} x^* &= F(x^*) - G(x^*) \\ x^* &= 11/10 - (1/2)x^* \\ x^* &= 22/30 \end{aligned}$$

This fixed point is stable because

$$|F'(22/30) - G'(22/30)| = |-1/2| < 1.$$

However, as indicated by the following time paths (which assume the initial conditions  $x_0 = 0.2$  and  $x_0 = 0.4$ , the cobweb spirals inwards to this fixed point.

```
>> x = .2; y = x; for t = 1:10; x = min(.2 + 2*x, 1) - max(0, -.1 + .5*x); y = [y; x]; end; y
```

```
y =
    0.2000
    0.6000
    0.8000
    0.7000
    0.7500
    0.7250
    0.7375
    0.7312
    0.7344
    0.7328
    0.7336
```

```
>> x = .4; y = x; for t = 1:10; x = min(.2 + 2*x, 1) - max(0, -.1 + .5*x); y = [y; x]; end; y
```

```
y =
```

```
0.4000  
0.9000  
0.6500  
0.7750  
0.7125  
0.7438  
0.7281  
0.7359  
0.7320  
0.7340  
0.7330
```

## 12.6 Further reading

Schelling (*Micromotives and Macrobehavior*, 1978, Chap 3)

Granovetter (*Am J Soc* 1978)