Sociology 376 Exam 1 Spring 2009 Prof Montgomery

Answer all questions. 220 points possible.

[HINT: Somewhere on this exam, it may be useful to know that

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then inv(A)} = A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.]$$

1) [70 points] Consider a one-sex model of intergenerational social mobility. Each individual has either high income (H) or low income (L), and the individual's income depends probabilistically on his parent's income and potentially his grandparent's income. (Note that, because this is a one-sex model, each individual has only one grandparent.) To be more precise, if your father was an H, then, regardless of your grandfather's income, you become an H with probability p (and thus become an L with probability 1-p). If your father was an L and your grandfather was an H, you become an H with probability q (and thus become an L with probability 1-q). If your father was an L and your grandfather was an L, you become an H with probability r (and thus become an L with probability 1-r).

a) This intergenerational social mobility process can be specified as a Markov chain process. Give the possible states of the Markov chain, and then give the transition matrix. [HINT: Is it possible to "lump" any states together?]

b) Suppose that the probabilities p, q, and r are all strictly greater than 0 and strictly less than 1 (i.e., $0 < p, q, r < 1$). Use high-school algebra (i.e. solve a system of simultaneous equations) to find the limiting (long-run) probability distribution over states. [HINT: Your answers will be functions of the parameters p, q, and r. These equations do not simplify very well, so don't waste a lot of time attempting to simplify.]

c) Did the limiting distribution in part (b) depend on the initial probability distribution over states? What property of the transition matrix guarantees this result? Draw the zero-pattern transition diagram (under the assumption that p, q, and r are positive). Briefly explain how, simply from this diagram (without any matrix computation), you can tell that the transition matrix satisfies the relevant property.

d) Now suppose that r = 0 while p and q remain strictly greater than 0 and strictly less than 1. In the long-run, what proportion of the population are H's? Use the zero-pattern of the transition matrix to determine the communication classes of this Markov chain, and then draw the reduced transition diagram on the set of communication classes. Briefly discuss how the long-run outcome is revealed by the reduced graph.

2) [65 pts] Every morning, two dormitory students pass each other in the hallway. Each student chooses whether or not to say "hello" to the other student based upon their interaction the previous day. If student 2 said hello yesterday, then student 1 says hello today with probability 4/5. Similarly, if student 1 said hello yesterday, then student 2 says hello today with probability 4/5. [Assume that these probabilities are independent. Thus, if both said hello yesterday, the probability that both say hello today is $(4/5)^2$.] If student 2 did not say hello yesterday, then student 1 will not say hello today (i.e., says hello with probability 0). Similarly, if student 1 did not say hello yesterday, then student 2 will not say hello today.

a) Conceptualizing this process as a Markov chain, list the possible states of the chain, and then give the transition matrix. Which state(s) are absorbing? [HINT: If you ignore the identities of the students, the chain has three possible states. If you get stuck on the transition matrix, it might be helpful to draw probability trees.]

b) Assuming that both students initially say hello to each other, use the transition matrix to compute the expected number of days that the chain will spend in each non-absorbing state before absorption. What is the total expected number of days before absorption?

c) Suppose now that student 1 is somewhat friendlier than student 2. In particular, suppose that, if student 2 said hello yesterday, then student 1 says hello today with probability 9/10. In contrast, if student 1 said hello yesterday, then student 2 says hello today with probability 6/10. (As before, these probabilities are independent, and each student would not say hello today if the other did not say hello yesterday.)

Given these new assumptions, list the possible states of the chain, and give the transition matrix. [HINT: Can you now ignore the identities of students?] Then draw the zero-pattern transition diagram. Identify the communication classes of the Markov chain, then draw the reduced transition diagram. Based on this reduced diagram, what can you say about the long-run outcome of this process?

3) [25 pts] Consider the two-player game below.

	L	M	R
T	2,2	4,3	6,4
B	5,3	8,1	2,2

a) Find the Nash equilibrium (or Nash equilibria) of this game. Is M ever a best response for the column player?

b) What are some potential problems with the standard game-theoretic analysis, and how are these problems addressed by Young's evolutionary approach? In particular, how might Young's approach address the issue of multiple equilibria? Would this approach make a clear prediction for the game above?

4) [30 points] Consider a population with the following Leslie matrix (using 20-year age categories) and initial population (in millions):

$$L = \begin{bmatrix} .2 & 1 & 0 & 0 \\ .8 & 0 & 0 & 0 \\ 0 & .9 & 0 & 0 \\ 0 & 0 & .6 & 0 \end{bmatrix} \qquad n_0 = \begin{bmatrix} 25 \\ 25 \\ 25 \\ 25 \end{bmatrix}$$

a) Compute the probability of survival (from birth) to each age category. Then compute life expectancy (from birth).

b) Project this population forward for the next two periods, giving the number of people in each age category and the total population size in each period.

c) In the long run, will population size be growing or shrinking or constant? Explain how you can determine this directly from the Leslie matrix (without population projection). Be sure to give the relevant computation.

5) [30 points] Consider a social process with dynamics given by

$$x_t = x_0 (RP)^t$$

where x_t is a frequency distribution at time t, x_0 is the initial distribution at time 0,

$$R = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} .90 & .05 & .03 & .02 \\ .04 & .92 & .03 & .01 \\ .01 & .05 & .91 & .03 \\ 0 & .03 & .06 & .91 \end{bmatrix}$$

a) Will this process reach a "stable-growth" equilibrium for any initial condition x_0? How do you know?

b) Use the Matlab handout (next page) to determine the

i) long-run growth rate λ

ii) limiting (long-run) probability distribution x

[HINT: Not all of the computations are relevant.]

c) Restate the dynamics of the process to show how these dynamics depend on eigenvectors and eigenvalues of the relevant matrix. [HINT: You don't need to solve numerically for anything. I'm merely looking for an equation linking x_t to the initial condition.]

Matlab computations for problem 5

```
>> R

R =
   1.1000        0        0        0
        0   1.3000        0        0
        0        0   1.2000        0
        0        0        0   1.0000

>> P

P =
   0.9000   0.0500   0.0300   0.0200
   0.0400   0.9200   0.0300   0.0100
   0.0100   0.0500   0.9100   0.0300
        0   0.0300   0.0600   0.9100

>> [a,b] = eig(P')

a =
   0.3288   0.7024   0.7796  -0.4819
   0.6818   0.2425  -0.3669   0.6244
   0.5607  -0.5000  -0.5000  -0.5000
   0.3357  -0.4448   0.0873   0.3575

b =
   1.0000        0        0        0
        0   0.9067        0        0
        0        0   0.8748        0
        0        0        0   0.8585

>> sum(a)

ans =
   1.9070  -0.0000  -0.0000   0.0000

>> [c,d] = eig((R*P)')

c =
  -0.2113   0.0674   0.9367   0.1303
  -0.9130  -0.0484  -0.1773   0.4245
  -0.3377  -0.2983  -0.2739  -0.8850
  -0.0885   0.9509   0.1267  -0.1398

d =
   1.2338        0        0        0
        0   0.8996        0        0
        0        0   0.9767        0
        0        0        0   1.0779

>> sum(c)

ans =
  -1.5505   0.6716   0.6122  -0.4700
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1a) [15 pts] This process could be specified as a Markov chain with four states {HH, LH, HL, LL} where the first H or L gives grandfather's income and the second H or L gives father's income. However, because rows HH and LH of the transition matrix will be the same (because grandfather's income doesn't affect transition probabilities if father was an H), you really need only three states {xH, HL, LL}. For the 3-state model (with xH as state 1, HL as state 2, LL as state 3), the transition matrix is given by

$$\mathrm{P} = \begin{bmatrix} p & 1-p & 0 \\ q & 0 & 1-q \\ r & 0 & 1-r \end{bmatrix}.$$

b) [20 pts] The long-run equilibrium is determined by the condition x = xP, which yields the 3-equation system

(1) x(1) = x(1) p + x(2) q + x(3) r
(2) x(2) = x(1)(1-p)
(3) x(3) = x(2)(1-q) + x(3)(1-r)

Using any two of these equations along with the requirement that x is a probability vector

(4) x(1) + x(2) + x(3) = 1,

we obtain the limiting (long-run) distribution over states. For instance, using equations (2) and (3), we obtain

x(3) = x(2)(1-q)/r = x(1)(1-p)(1-q)/r

Substitution into equation (4) yields

x(1) + x(1)(1-p) + x(1)(1-p)(1-q)/r = 1

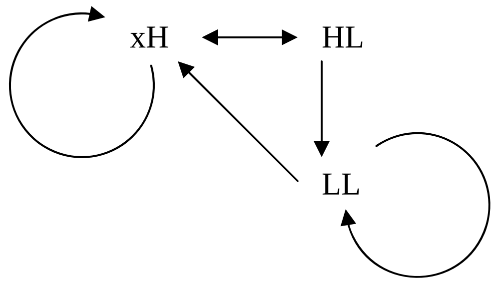
and hence the limiting distribution is

x(1) = 1/[1 + 1-p + (1-p)(1-q)/r] = r/[r(2-p) + (1-p)(1-q)]

x(2) = x(1)(1-p) = r(1-p)/[r(2-p) + (1-p)(1-q)]

x(3) = x(1)(1-p)(1-q)/r = (1-p)(1-q)/[r(2-p) + (1-p)(1-q)]

c) [15 pts] The limiting probability distribution does not depend on the initial state because the transition matrix P is primitive. The zero-pattern transition diagram is



From this diagram, we see that every state can reach (directly or indirectly) every other state. Thus, the matrix is irreducible. Further, because the diagram has loops (cycles of length 1), the matrix is primitive.

d) [20 pts] Given r = 0, equation (3) implies that x(2) = 0. Hence, equation (2) implies that x(1) = 0. And because x is a probability vector, we obtain x(3) = 1. Thus, in the long-run everyone is an L. Intuitively, given r = 0, the transition matrix is no longer regular, and state LL is now absorbing. More formally, to determine communication classes of the Markov chain, let Z denote the zero-pattern of the transition matrix. Given r = 0, the zero-pattern is given by

$$Z = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{which implies the reachability matrix } R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence the can-reach-and-be-reached-by matrix $R \,\&\, R' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The equivalence classes of the can-reach-and-be-reached-by relation are the communication classes of the Markov chain. Thus, there are two communication classes: {xH, HL} and {LL}. The reduced graph is

$$\{xH, HL\} \longrightarrow \{LL\}$$

Intuitively, once a family line enters state LL, it can never leave this state. In the long run, every family line will be in state LL.

2a) [18 pts] The three states are: neither said hello (0), one said hello (1), or two said hello (2). Given that each student says hello with probability p = .8 (and that the probabilities are independent), the transition matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 1-p & p & 0 \\ (1-p)^2 & 2p(1-p) & p^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ .2 & .8 & 0 \\ .04 & .32 & .64 \end{bmatrix}$$

State 0 (neither said hello) is absorbing.

b) [17 pts] The submatrix from non-absorbing to non-absorbing states is given by

$$Q = \begin{bmatrix} .8 & 0 \\ .32 & .64 \end{bmatrix} \text{ so that } I\text{-}Q = \begin{bmatrix} .2 & 0 \\ -.32 & .36 \end{bmatrix}$$

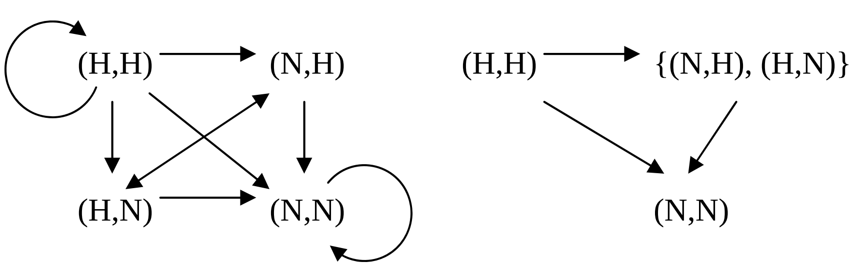
Using the hint at the beginning of the exam, the fundamental matrix is thus

$$N = (I\text{-}Q)^{-1} = \begin{bmatrix} 5 & 0 \\ 4.44 & 2.77 \end{bmatrix}$$

Thus, starting from state 2, we would expect to spend 4.44 days in state 1 and 2.77 days in state 2. Overall, we would expect to spend 7.22 days before absorption into state 0.

c) [30 pts] Given that each student could either not say hello (N) or say hello (H), there are now 4 states: (N,N), (N,H), (H,N), (H,H). Given that student 1 says hello with probability p = .9 and student 2 says hello with probability q = .6, the transition matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 1-q & q & 0 & 0 \\ (1-p)(1-q) & (1-p)q & p(1-q) & pq \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .1 & 0 & .9 & 0 \\ .4 & .6 & .0 & 0 \\ .04 & .06 & .36 & .54 \end{bmatrix}$$



Thus, the long-run outcome is the same as before: the chain is eventually absorbed in state (N,N).

3a) [10 pts] There are 2 (pure-strategy) Nash equilibria: (T,R) and (B,L). The action M is never a best response for the column player.

b) [15 pts] Standard game-theoretic analysis does not specify the process by which players reach Nash equilibrium. Relatedly, when there are multiple Nash equilibria, it does not indicate which equilibrium is more likely to occur.

To explain how players reach equilibrium, Young specifies a Markov chain process. Given no randomization ($\varepsilon = 0$), the chain is absorbing, and the absorbing states correspond to the Nash equilibria. (Note, however, that the states of the chain are configurations of the social memory, not merely pairs of actions.)

Implicitly, Young addresses multiple equilibria by focusing on "accidents of history" (i.e., mistakes in the social memory). More explicitly, he addresses multiple equilibria using the concept of stochastic stability. If randomization does sometimes occur ($\varepsilon > 0$), the chain is regular. Thus, as t becomes large, the probability distribution x_t will converge to the limiting distribution x (which depends implicitly on ε). State i is stochastically stable when x(i) remains positive as ε becomes very small.

For coordination games in which one of the Nash equilibria is preferred by both players, there is a unique stochastically stable state. The game in part (a) satisfies this condition because players would prefer to coordinate on (T,R) rather than (B,L). Thus, the evolutionary approach would predict (T,R). (In its current form, the game in part (a) may not look like a coordination game. But because M is never a best response for the row player, it essentially "reduces" to the 2×2 coordination game with actions {T,B} for the row player and {L,R} for the column player.)

4a) [10 pts] survival probability vector = $\begin{bmatrix} 1 \\ .8 \\ .72 \\ .432 \end{bmatrix}$

and thus life expectancy (from birth) is 1 + .8 + .72 + .432 = 2.952 (× 20 years)

b) [10 pts] $n_1 = \begin{bmatrix} 30 \\ 20 \\ 22.5 \\ 15 \end{bmatrix}$, $n_2 = \begin{bmatrix} 26 \\ 24 \\ 18 \\ 13.5 \end{bmatrix}$, $pop_1 = \text{sum}(n_1) = 87.5$, $pop_2 = \text{sum}(n_2) = 81.5$

c) [10 pts] Population size will be constant because NRR = (.2)(1) + (1)(.8) = 1.

5a) [6 pts] Yes, it will reach a stable-growth equilibrium because P is primitive.

b) [18 pts] $\lambda = 1.2338$, v = [.1363 .5888 .2178 .0571] (= normalized eigenvector)

c) [6 pts] $x_t = VL^tc$ where V is matrix of eigenvectors of $(RP)'$, L is diagonal matrix of eigenvalues of $(RP)'$, and initial condition $x_0 = Vc$ where c is a vector of constants.

Soc 376 Spring 2009 Final Exam Prof Montgomery

Answer all questions. 220 points possible.

1) [50 pts] Suppose that you are managing a fishery. When undisturbed by humans, the fish population grows according to the logistic growth model,

$$\Delta P = r P (1 - P/K)$$

where P is the size of the fish population, ΔP is the change in population size per period, r is the intrinsic growth rate, and K is the carrying capacity of the environment. Of course, to make a profit, the fishery must harvest some of the fish. Thus, the dynamics of the fish population is given by

$$\Delta P = r P (1 - P/K) - H$$

where H denotes the number of fish harvested each period.

a) Restate the dynamics in the form $P_{t+1} = f(P_t)$ where f is the generator function.

b) Using a cobweb diagram, discuss how the equilibria (and the stability of these equilibria) change as H is increased from 0 to higher values. [NOTE: You merely need to give a qualitative discussion and graphical analysis. Analytical solutions are possible (using the formula for roots of a quadratic equation) but ugly. Make sure that your graph is well-labeled and qualitatively correct.]

c) Again using a diagram, illustrate the case where H is set at the highest sustainable level. What concern might you (in your role as fishery manager) have with setting H at this level? Briefly discuss.

d) In the absence of harvesting, it is possible that the fish population has no stable fixed point. In this case, can imposing harvesting lead to stability? Briefly discuss, again using a cobweb diagram.

e) Suppose the model had been specified in continuous time, so that

$$dP/dt = r P (1 - P/K) - H$$

Assuming parameter values (r, K, H) so that this equation has two fixed points P*, is it possible for both fixed points to be unstable? Briefly discuss the graphical analysis of the continuous-time model, using a phase diagram.

2) [60 points] Each individual in a large community can choose either to recycle or not recycle. Suppose that individual i's utility for recycling is given by

$$U_R(i) = v(i) + 6x^2$$

where x denotes the proportion of the community that is expected to recycle. For all individuals, the utility of not recycling is given by

$$U_{NR} = 2.$$

The value v(i) differs across individuals. Specifically, assume that v is distributed uniformly between –3 and 3. The probability density function for v is thus

$$f(v) = \begin{cases} 1/6 & \text{if } -3 \le v \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Given that proportion x of the community is expected to recycle, find the critical value v* at which someone would be indifferent between recycling and not recycling. [HINT: v* is a function of x. Individuals with $v > v^*(x)$ will strictly prefer to recycle, while those with $v < v^*(x)$ will strictly prefer not to recycle.]

b) Given the probability density function f(v), what is the associated cumulative distribution function F(v)? [HINT: Make sure your answer is correct for $v < -3$ or $v > 3$ as well as $v \in [-3,3]$.]

c) Suppose that people have adaptive expectations, so that actual participation in period t ($= x_t$) becomes expected participation in period t+1. Write the equation (generator function) that determines the dynamics of participation. Use your answers to parts (a) and (b) to simplify your answer as much as possible. [HINT: Make sure your answer is correct for all x_t between 0 and 1.]

d) Plot the generator function from part (c). On your diagram, indicate the fixed point(s) and the stability (or instability) of each fixed point. [HINT: Your graph doesn't need to be perfect, but should be well-labeled and qualitatively correct.]

e) Solve numerically for each fixed point. [HINT: Recall that the zeros of the function $ax^2 + bx + c$ are given by $x = [-b \pm \text{sqrt}(b^2 - 4ac)]/2a$.]

f) Formally assess stability at each fixed point. [HINT: You've already assessed stability informally in part (d) based on your plot of the generator function. But here I'm looking for the precise (numerical) test. You can use calculus or the non-calculus approach.]

3) [110 pts] Consider a society in which each individual is assigned to one of three racial classes: white (W), mulatto (M), or black (B). Further suppose that the population distribution in generation t is given by

$$x_t = [W_t \quad 1-W_t-B_t \quad B_t]$$

where W_t = proportion of the population that is white in generation t
$1-W_t-B_t$ = proportion of the population that is mulatto in generation t
B_t = proportion of the population that is black in generation t

Following our lectures on two-sex models, suppose that population dynamics are determined by the equation

$$x_{t+1} = x_t N_t R P$$

where N_t is the matching matrix in generation t, R is the reproduction matrix, and P is the intergenerational transition matrix. To simplify, we'll assume random matching so that

$$N_t = \begin{bmatrix} W_t & 1-W_t-B_t & B_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_t & 1-W_t-B_t & B_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_t & 1-W_t-B_t & B_t \end{bmatrix}$$

where the rows of this matrix correspond to the 3 types of individuals (W, M, B) and the columns of this matrix correspond to the 9 types of couples, listed as

WW, WM, WB, MW, MM, MB, BW, BM, BB

where the first element is the female's class and the second element is the male's class. To further simplify, we will ignore differential reproduction, so that R is the 9×9 identity matrix. Finally, we'll assume that intergenerational transitions occur according to

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

where the rows correspond to the 9 types of couples (listed in the order given above) and the columns correspond to the 3 types of individuals (W, M, B).

3a) Give the N_tP matrix for generation t. Then give the x_tN_tP vector.

b) Using your answer to part (a), write the two-equation system in the form

$$W_{t+1} = f(W_t, B_t)$$
$$B_{t+1} = g(W_t, B_t)$$

Briefly explain why don't we need a third equation for mulattos.

c) Solve for the nullcline(s) for W and the nullcline(s) for B. Then plot these nullclines on a two-dimensional phase diagram. [NOTE: Make sure your graph is properly labeled. You'll use this diagram again for parts (d) and (e) and (i), so please make it large and legible.]

d) For this model, the (relevant part of the) phase diagram is a "simplex." Indicate this simplex on your phase diagram. Is the simplex the unit square? Why or why not?

e) State the inequalities which determine whether W and B are rising or falling. Then use the inequalities to draw the appropriate vectors (arrows) on your phase diagram to indicate dynamics in each region.

f) Solve for all of the equilibria of the model. [HINT: You should be able to give numerical coordinates for every equilibrium.]

g) Provide a formal stability analysis for each equilibrium in part (f). [HINT: You should already have a good guess about stability based on your phase diagram, but here I'm looking for the precise (numerical) test. You can use calculus or the non-calculus approach. It may be useful to know that

the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues $\lambda_1 = (1/2)(a + d + \text{sqrt}(a^2 + 4bc - 2ad + d^2))$
$\lambda_2 = (1/2)(a + d - \text{sqrt}(a^2 + 4bc - 2ad + d^2))$]

h) Suppose that the initial population distribution is given by $x_0 = [0.1\ \ 0.8\ \ 0.1]$. Compute the distributions for the next 4 periods ($x_1, \ldots, x_4$).

i) Using the answer to part (h), plot the corresponding trajectory on your phase diagram. To which equilibrium is this trajectory converging? Does this trajectory provide a good indication of the stability of this fixed point? Briefly discuss.

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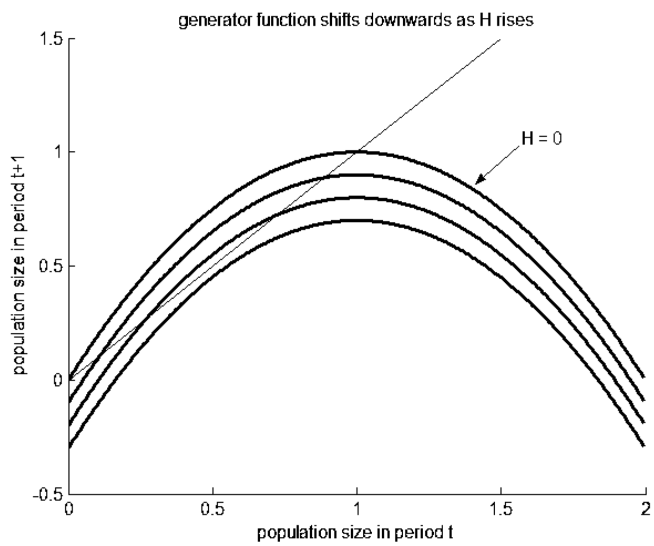
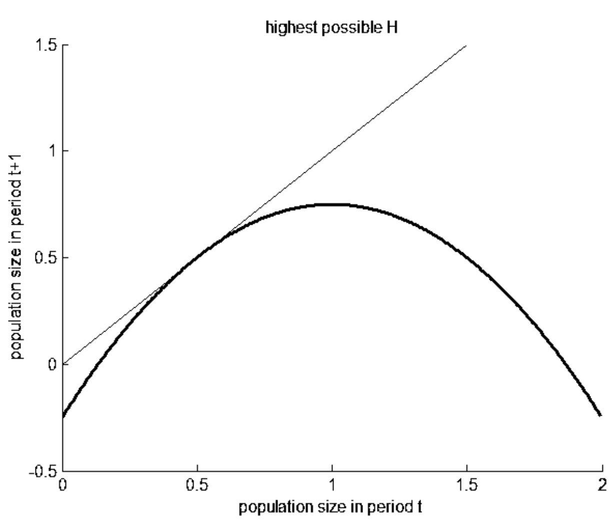
1a) [5 pts] $P_{t+1} = f(P) = P_t + r\,P_t\,(1 - P_t/K) - H$

$= (1 + r)\,P_t - (r/K)\,P_t^2 - H$

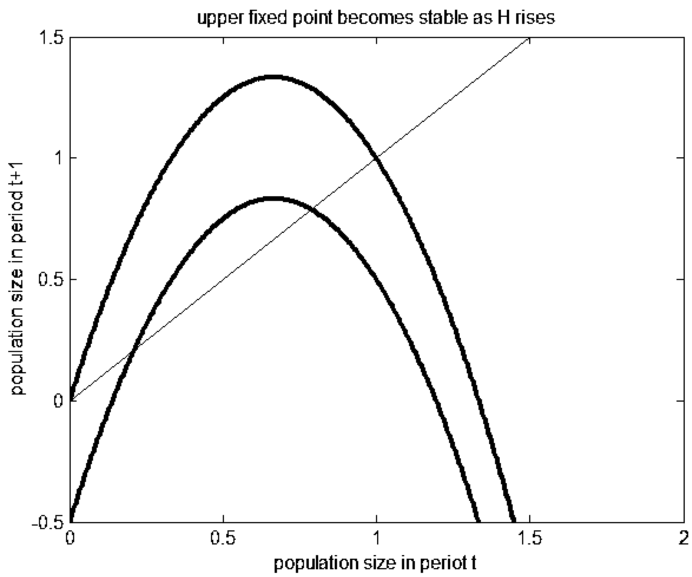
b) [15 pts] Note that the generator function is quadratic (initially rising, then falling). Given H = 0, there is a fixed point at P* = 0 (which is unstable if r > 0) and another fixed point at positive P* (which is stable if $|f'(P^*)| < 1$). Graphically, an increase in H causes the generator function to shift downwards. This causes the lower (unstable) equilibrium to rise and the upper (stable) equilibrium to fall. If the fishery manager sets H too high, she induces a catastrophe – there is no equilibrium with positive P*.



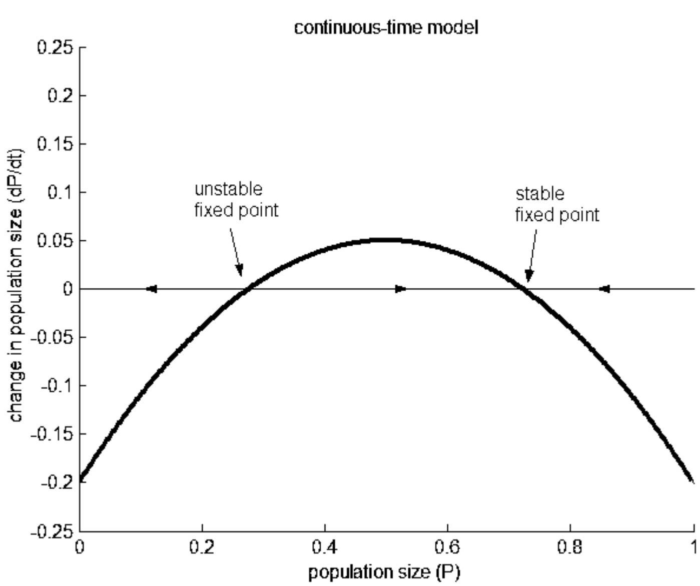
c) [10 pts] At this level of H, the generator function is shifted downwards until it is just tangent to the 45-degree line. The positive fixed point is "marginally" or "borderline" stable. Any small negative shock would cause the fish population to fall to zero.



d) [10 pts] When r is very high, the upper fixed point P* may be unstable (because $|f'(P^*)| > 1$). As the curve shifts downward, P* falls, and $f'(P^*)$ becomes smaller in absolute value. Thus, harvesting can lead to stability.



e) [10 pts] In the continuous-time version of the model, there is no possibility of "overshooting" an equilibrium, and thus the upper fixed point is always stable. Plotting dP/dt, we see that P is falling at very low or very high values, but P is rising between the two fixed points.



For a one-dimensional model, the phase diagram is simply the horizontal axis of this diagram, along with the indicated fixed points and the arrows indicating dynamics.

2a) [4 pts] An individual is indifferent between recycling and not recycling if

$$U_R = U_{NR} \quad \rightarrow \quad v + 6x^2 = 2 \quad \rightarrow \quad v^*(x) = 2 - 6x^2$$

As indicated, individuals with $v > v^*(x)$ will recycle; those with $v < v^*(x)$ will not.

b) [8 pts]

$$F(v) = \begin{cases} 0 & \text{for } v < -3 \\ (1/6)v - 1/2 & \text{for } v \in [-3, 3] \\ 1 & \text{for } v > 3 \end{cases}$$

c) [15 pts] Given adaptive expectations, everyone with $v > v^*(x_t)$ will recycle in period t+1. Thus,

$$x_{t+1} = 1 - F(v^*(x_t)) = 1 - F(2 - 6x_t^2)$$

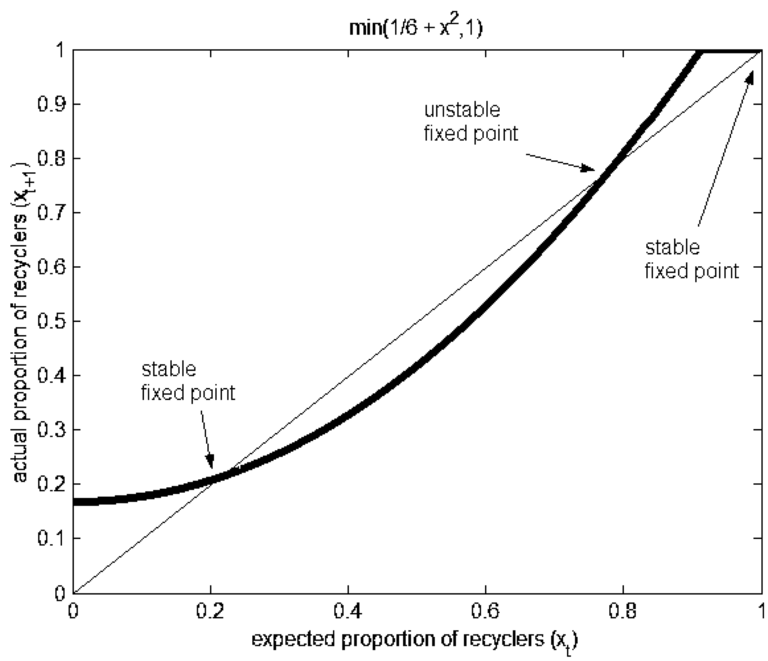
Substituting for F(v),

$$x_{t+1} = \begin{cases} 1 - 0 & \text{for } 2-6x_t^2 < -3 \\ 1 - [(1/6)(2-6x_t^2) - 1/2] & \text{for } 2-6x_t^2 \in [-3,3] \\ 1 - 1 & \text{for } 2-6x_t > -3 \end{cases}$$

which (restricting attention to x_t between 0 and 1) can be rewritten as

$$x_{t+1} = \begin{cases} 1 & \text{for } x_t > \mathrm{sqrt}(5/6) \\ 1/6 + x_t^2 & \text{for } x_t \in [0, \mathrm{sqrt}(5/6)] \end{cases}$$

d) [12 pts]



2e) [6 pts] The upper fixed point is obviously at $x^* = 1$. The lower and intermediate fixed points are solutions to the equation

$$x^* = 1/6 + (x^*)^2$$

Equivalently, these fixed points are the solutions to the equation

$$(x^*)^2 - x^* + 1/6 = 0$$

Using the quadratic equation,

$$x^* = [1 \pm \text{sqrt}(1 - 4(1)(1/6))]/2 = 1/2 \pm \text{sqrt}(1/3)/2 = .7887 \text{ or } .2113$$

f) [15 pts] Suppose the system is initially at the lower or intermediate fixed point x^*, so that

$$x^* = 1/6 + (x^*)^2$$

Further suppose there is a small shock in period t, so that

$$x_t = x^* + \varepsilon_t \qquad \text{and} \qquad x_{t+1} = x^* + \varepsilon_{t+1}$$

where the ε's reflect the (small) departure from equilibrium in each period. Substituting into the equation

$$x_{t+1} = 1/6 + x_t^2$$

yields $x^* + \varepsilon_{t+1} = 1/6 + (x^* + \varepsilon_t)^2$

$$x^* + \varepsilon_{t+1} = 1/6 + (x^*)^2 + 2x^*\varepsilon_t + \varepsilon_t^2$$

Recall that x^* is a fixed point, so that $x^* = 1/6 + (x^*)^2$. Further, because ε_t is small, ε_t^2 is extremely small and can be ignored. Thus, the preceding equation simplifies to

$$\varepsilon_{t+1} = 2x^*\varepsilon_t \quad \rightarrow \quad \varepsilon_{t+1}/\varepsilon_t = 2x^*$$

Thus, the fixed point is stable if and only if $|2x^*| < 1$

In particular, the lower fixed point is stable because $|2 \times .2113| < 1$, while the intermediate fixed point is unstable because $|2 \times .7887| > 1$

[Equivalently, given the generator function $f(x) = 1/6 + x^2$ (for $x < \text{sqrt}(5/6)$), the slope of the generator function is given by $f'(x) = 2x$, and each fixed point is stable if and only if this slope has an absolute value less than 1.]

Because the generator function has slope of 0 at $x = 1$, the upper equilibrium is stable.

3a) [10 pts] $N_t P = \begin{bmatrix} 1-B_t & B_t & 0 \\ W_t & 1-W_t-B_t & B_t \\ 0 & W_t & 1-W_t \end{bmatrix}$

$$x_t N_t P = \left[W_t(2-2B_t-W_t) \quad 2W_tB_t + (1-W_t-B_t)^2 \quad B_t(2-2W_t-B_t) \right]$$

b) [9 pts] $W_{t+1} = f(W_t, B_t) = W_t(2-2B_t-W_t)$
$B_{t+1} = g(W_t, B_t) = B_t(2-2W_t-B_t)$

We don't need a third equation because $M_t = 1-W_t-B_t$ for all t

c) [12 pts] $\Delta W = 0 \rightarrow W(1-2B-W) = 0 \rightarrow W = 0$ or $W = 1-2B$
$\Delta B = 0 \rightarrow B(1-2W-B) = 0 \rightarrow B = 0$ or $B = 1-2W$

See nullclines on phase diagram below.

d) [6 pts] The simplex is the triangle defined by the inequalities $W \geq 0$, $B \geq 0$, and $W+B \leq 1$. Any distribution x = [W 1−W−B B] corresponds to a point in this triangle. The remainder of the unit square is not part of the simplex, because $W + B > 1$ implies $M = 1-W-B$ would need to be negative.

e) [12 pts] $\Delta W > 0 \rightarrow W < 1-2B$ $\qquad \Delta W < 0 \rightarrow W > 1-2B$
$\Delta B > 0 \rightarrow B < 1-2W$ $\qquad \Delta B < 0 \rightarrow B < 1-2W$

See arrows on phase diagram below.

f) [12 pts] From the phase diagram, we find 4 equilibria:

(W = 0, B = 0), (W = 1, B = 0), (W = 0, B = 1), (W = 1/3, B = 1/3)

g) [31 pts] Here we generalize the analysis from question (2f) for a two-dimensional model. Suppose that the system is initially in an equilibrium (W*, B*), and that there is a small shock in period t, so that

$W_t = W^* + w_t \qquad B_t = B^* + b_t$
$W_{t+1} = W^* + w_{t+1} \qquad B_{t+1} = B^* + b_{t+1}$

where the small w's and b's reflect (small) departures from equilibrium. Substitution into the first function from part (b) yields

$$W^* + w_{t+1} = (W^* + w_t)[2 - (W^*+w_t) - 2(B^*+b_t)]$$

$$= W^*[2 - (W^*+w_t) - 2(B^*+b_t)] + w_t[2 - (W^*+w_t) - 2(B^*+b_t)]$$

3g cont'd) Because $W^* = W^*[2 - W^* - 2B^*]$, this simplifies to

$$w_{t+1} = W^*[-w_t - 2b_t] + w_t[2 - (W^* + w_t) - 2(B^* + b_t)]$$

The interaction terms w_t^2 and $w_t b_t$ are very small, and can thus be ignored. This leaves

$$w_{t+1} = W^*[-w_t - 2b_t] + w_t[2 - W^* - 2B^*] = [2 - 2W^* - 2B^*]\, w_t + [-2W^*]\, b_t$$

We've thus obtained w_{t+1} as a linear function of w_t and b_t. Going through the same steps using the second equation from part (b), we obtain the analogous equation

$$b_{t+1} = [2 - 2W^* - 2B^*]\, b_t + [-2B^*]\, w_t$$

We've thus obtained a linear two-equation system, with w_{t+1} and b_{t+1} as functions of w_t and b_t. Writing this two-equation system in matrix form,

$$\begin{bmatrix} w_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} 2 - 2W^* - 2B^* & -2W^* \\ -2B^* & 2 - 2W^* - 2B^* \end{bmatrix} \begin{bmatrix} w_t \\ b_t \end{bmatrix}$$

[You could have also found the elements of the Jacobian matrix using calculus:

$\partial f(W,B)/\partial W = 2 - 2W - 2B \qquad \partial f(W,B)/\partial B = -2W$
$\partial g(W,B)/\partial W = -2B \qquad \partial g(W,B)/\partial B = 2 - 2W - 2B$]

To evaluate stability of an equilibrium (W^*,B^*), we evaluate the Jacobian matrix at this equilibrium, and then determine whether the dominant eigenvalue has an absolute value less than 1 (indicating stability) or greater than 1 (indicating instability).

$(W^*,B^*) = (0,0) \rightarrow J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = 2$ (unstable)

$(W^*,B^*) = (1,0) \rightarrow J = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = 0$ (stable)

$(W^*,B^*) = (0,1) \rightarrow J = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = 0$ (stable)

$(W^*,B^*) = (1/3,1/3) \rightarrow J = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix} \rightarrow \lambda_1 = 4/3, \lambda_2 = 0$ (unstable)

Thus, the all-white and all-black equilibria are stable; the all-mulatto and interior equilibria are unstable.

h) [8 pts] $x_1 = [.17\ .66\ .17]$
$x_2 = [.2533\ .4934\ .2533]$
$x_3 = [.3141\ .3717\ .3141]$
$x_4 = [.3322\ .3356\ .3322]$

i) [10 pts] See trajectory on phase diagram below. The trajectory is heading toward the unstable interior equilibrium. It does not provide a good indication of the stability of this equilibrium, since any other nearby trajectory would eventually "veer off" toward one of the stable equilibria. Rather, the trajectory lies on the "separatrix" which constitutes the boundary between two basins of attraction.

Phase diagram (for parts c, d, e, and i)