last revised: 20 February 2009

10 Population Momentum

10.1 Further analysis of the stable growth equilibrium

We saw in the preceding chapter that age-structured populations will generally reach the stable growth equilibrium, which is characterized by the condition

$$\lambda \mathbf{x} = L\mathbf{x}$$

where λ is the dominant eigenvalue of the Leslie matrix and $\mathbf{x}$ is the associated eigenvector. To gain some further insight into this equilibrium, it will be useful to write out this simultaneous equation system. For simplicity, we'll continue to assume 4 age classes (though results again extend easily to the general case with n age classes). We can thus write these equations as

$$\begin{aligned}
\lambda \mathbf{x}(1) &= f(1)\mathbf{x}(1) + f(2)\mathbf{x}(2) + f(3)\mathbf{x}(3) + f(4)\mathbf{x}(4) \\
\lambda \mathbf{x}(2) &= s(1)\mathbf{x}(1) \\
\lambda \mathbf{x}(3) &= s(2)\mathbf{x}(2) \\
\lambda \mathbf{x}(4) &= s(3)\mathbf{x}(3)
\end{aligned}$$

Using the last three equations, we obtain

$$\begin{aligned}
\mathbf{x}(2) &= s(1)\mathbf{x}(1)\lambda^{-1} \\
\mathbf{x}(3) &= s(2)\mathbf{x}(2)\lambda^{-1} = s(1)s(2)\mathbf{x}(1)\lambda^{-2} \\
\mathbf{x}(4) &= s(3)\mathbf{x}(3)\lambda^{-1} = s(1)s(2)s(3)\mathbf{x}(1)\lambda^{-3}
\end{aligned}$$

and thus

$$\mathbf{x} = \mathbf{x}(1) \begin{bmatrix} 1 \\ s(1)\lambda^{-1} \\ s(1)s(2)\lambda^{-2} \\ s(1)s(2)s(3)\lambda^{-3} \end{bmatrix}.$$

Intuitively, if population size is stable (so that $\lambda = 1$), the limiting distribution simply reflects the survival probabilities reported in the first row of the life table. In contrast, if the population is growing (so that $\lambda > 1$), the limiting distribution becomes skewed towards younger age classes.

To begin to see the relationship between the long-run growth rate (λ) and the net reproduction rate (NRR), we can substitute the results from the last three equations back into the first equation. Simplifying, we obtain

$$1 = f(1)\lambda^{-1} + s(1)f(2)\lambda^{-2} + s(1)s(2)f(3)\lambda^{-3} + s(1)s(2)s(3)f(4)\lambda^{-4}.$$

If we assume that $\lambda = 1$, this becomes

$$1 \;=\; f(1) + s(1)f(2) + s(1)s(2)f(3) + s(1)s(2)s(3)f(4)$$

and it is apparent that NRR = 1. Unfortunately, for other values of λ, there is no simple equation relating λ to NRR. However, if λ is close to 1, then we can write

$$\lambda \;=\; 1 + \epsilon$$

where ϵ is close to 0, and adopt the approximation

$$\lambda^{-t} \;\approx\; 1 - t\epsilon.$$

Substitution into our previous equation (along with some simplification) yields

$$\epsilon \;=\; \lambda - 1 \;\approx\; \frac{\text{NRR} - 1}{f(1) + 2s(1)f(2) + 3s(1)s(2)f(3) + 4s(1)s(2)s(3)f(4)}.$$

While this approximation is not very accurate when λ is not close to 1, it does reveal that NRR > 1 if and only if $\lambda > 1$, and that NRR < 1 if and only if $\lambda < 1$. The denominator of the final term also has a demographic interpretation, as

$$\frac{f(1) + 2s(1)f(2) + 3s(1)s(2)f(3) + 4s(1)s(2)s(3)f(4)}{\text{NRR}}$$

represents mean age at fertility (i.e., the average age of mothers weighted by births), which is related to the demographic concept of "generation length."

10.2 The momentum of population growth

Although world population growth has been a concern for many years, policymakers in some countries did not always view the problem as urgent, especially when regions of their countries were not yet thickly inhabited. But their focus on current population size is short-sighted because it neglects the "momentum" of population growth. Consider a country in a stable growth equilibrium with a high NRR. Even if policymakers were able to instantly reduce fertility levels to replacement level (so that NRR = 1), population size would continue to increase for generations. As we saw in the previous section, the current age distribution would be skewed towards younger age classes. Even if fertility levels fall, the survival of those already born will increase (perhaps dramatically) the number of individuals in older age classes. Eventually, the country will reach a new zero-growth equilibrium with a somewhat smaller birth cohort but a larger (perhaps much larger) population size overall. To employ the "momentum" metaphor: even if the country "slams on the brakes" by instantly cutting NNR to 1, population growth will not "stop on a dime."

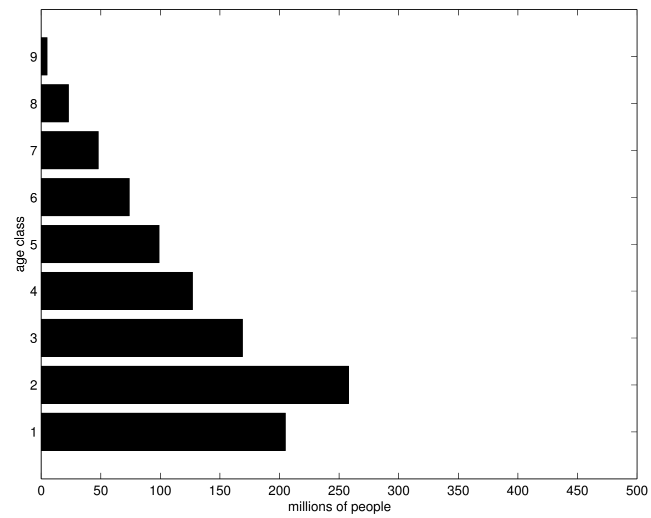
To illustrate, we'll consider population growth in China. The first step is to make some population projections using the initial distribution and Leslie matrix from the early 1980's (reported in Bradley and Meeks, 1986, p 166). For this data, age classes (and hence periods) are 10-year intervals. Thus, to project the population forward for 100 years, we consider the next 10 periods.

```
>> L   % Leslie matrix for China, 1981

L =
         0    0.4500    0.6900    0.1300         0         0         0         0         0
    0.9700         0         0         0         0         0         0         0         0
         0    0.9930         0         0         0         0         0         0         0
         0         0    0.9870         0         0         0         0         0         0
         0         0         0    0.9810         0         0         0         0         0
         0         0         0         0    0.9620         0         0         0         0
         0         0         0         0         0    0.9070         0         0         0
         0         0         0         0         0         0    0.7610         0         0
         0         0         0         0         0         0         0    0.5100         0

>> S = [zeros(1,9); L(2:9,:)]; N = inv(eye(9)-S); NRR = L(1,:)*N(:,1)

NRR =
    1.2247

>> x0'   % population (in millions) by 10-year age classes for China, 1982

ans =
   205   258   169   127    99    74    48    23     5

>> popfreq = []; for t = 0:10; popfreq = [popfreq; (L^t * x0)']; end; popfreq

popfreq =

  205.0000  258.0000  169.0000  127.0000   99.0000   74.0000   48.0000   23.0000    5.0000
  249.2200  198.8500  256.1940  166.8030  124.5870   95.2380   67.1180   36.5280   11.7300
  287.9407  241.7434  197.4581  252.8635  163.6337  119.8527   86.3809   51.0768   18.6293
  277.9028  279.3025  240.0512  194.8911  248.0591  157.4157  108.7064   65.7358   26.0492
  316.6573  269.5658  277.3474  236.9305  191.1882  238.6328  142.7760   82.7256   33.5253
  343.4753  307.1576  267.6788  273.7419  232.4289  183.9230  216.4400  108.6525   42.1900
  358.5057  333.1710  305.0075  264.1990  268.5408  223.5966  166.8182  164.7108   55.4128
  394.7280  347.7506  330.8388  301.0424  259.1792  258.3362  202.8021  126.9486   84.0025
  423.9020  382.8861  345.3163  326.5379  295.3226  249.3304  234.3110  154.3324   64.7438
  453.0169  411.1850  380.2059  340.8272  320.3337  284.1003  226.1427  178.3107   78.7095
  491.6829  439.4264  408.3067  375.2633  334.3515  308.1610  257.6790  172.0946   90.9384

>> popsize = sum(popfreq,2)

popsize =
  1.0e+003 *

    1.0080
    1.2063
    1.4196
```

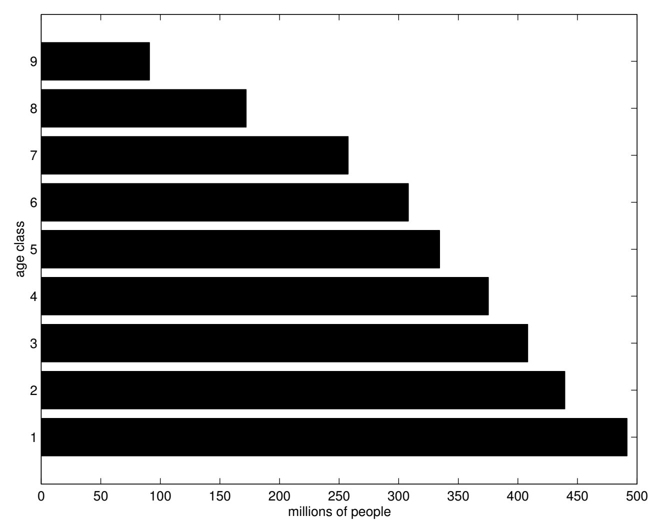
```
    1.5981
    1.7893
    1.9757
    2.1400
    2.3056
    2.4767
    2.6728
    2.8779
```

Thus, if the Leslie matrix remains unchanged, these projections indicate that the population size will grow from 1 billion to 2.87 billion. To help visualize this change, we adopt another demography convention, plotting the initial (period 0) and final (period 10) age distributions as horizontal bar charts.

```
>> barh(popfreq(1,:))    % age distribution in 1982
```



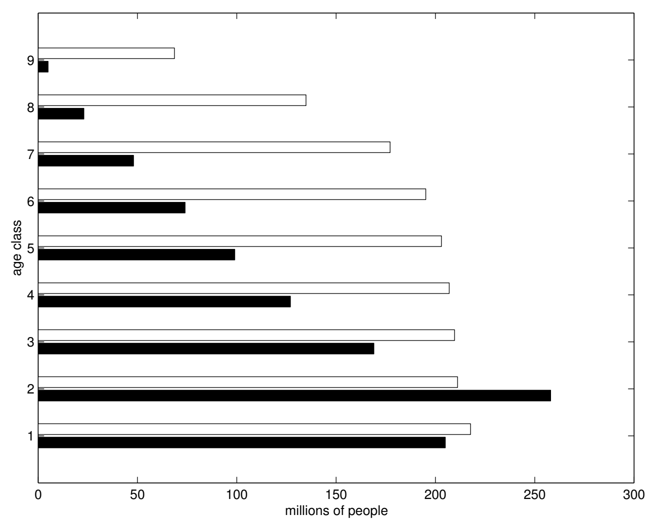

```
>> barh(popfreq(11,:))    % projected age distribution in 2082
```



Next, we'll consider what would happen if the NRR fell instantly to 1. Of course, this change could happen in various ways, through decreases in any combination of the fertility levels in the first row of the Leslie matrix. For the sake of this numerical example, we'll suppose that each fertility level falls by the same percentage.

```
>> L(1,:) = L(1,:)/NRR

L =
         0    0.3674    0.5634    0.1061         0         0         0         0         0
    0.9700         0         0         0         0         0         0         0         0
         0    0.9930         0         0         0         0         0         0         0
         0         0    0.9870         0         0         0         0         0         0
         0         0         0    0.9810         0         0         0         0         0
         0         0         0         0    0.9620         0         0         0         0
         0         0         0         0         0    0.9070         0         0         0
         0         0         0         0         0         0    0.7610         0         0
         0         0         0         0         0         0         0    0.5100         0

>> popfreq = []; for t = 0:20; popfreq = [popfreq; (L^t * x0)']; end; popfreq

popfreq =
  205.0000  258.0000  169.0000  127.0000   99.0000   74.0000   48.0000   23.0000    5.0000
  203.4940  198.8500  256.1940  166.8030  124.5870   95.2380   67.1180   36.5280   11.7300
  235.1104  197.3892  197.4581  252.8635  163.6337  119.8527   86.3809   51.0768   18.6293
  210.6169  228.0571  196.0075  194.8911  248.0591  157.4157  108.7064   65.7358   26.0492
  214.9145  204.2984  226.4607  193.4594  191.1882  238.6328  142.7760   82.7256   33.5253
  223.1901  208.4670  202.8683  223.5167  189.7836  183.9230  216.4400  108.6525   42.1900
  214.6203  216.4944  207.0078  200.2310  219.2699  182.5719  166.8182  164.7108   55.4128
  217.4303  208.1817  214.9789  204.3167  196.4266  210.9376  165.5927  126.9486   84.0025
  219.3006  210.9074  206.7245  212.1842  200.4346  188.9624  191.3204  126.0160   64.7438
  216.4867  212.7216  209.4311  204.0370  208.1527  192.8181  171.3889  145.5949   64.2682
  217.8133  209.9921  211.2325  206.7085  200.1603  200.2429  174.8860  130.4270   74.2534
  218.1089  211.2789  208.5221  208.4865  202.7810  192.5542  181.6203  133.0883   66.5178
  217.2435  211.5657  209.8000  205.8113  204.5253  195.0753  174.6467  138.2131   67.8750
  217.7848  210.7262  210.0847  207.0726  201.9009  196.7533  176.9333  132.9061   70.4887
  217.7706  211.2513  209.2511  207.3536  203.1382  194.2287  178.4552  134.6462   67.7821
  217.5237  211.2375  209.7725  206.5308  203.4139  195.4190  176.1654  135.8044   68.6696
  217.7251  210.9980  209.7589  207.0455  202.6067  195.6842  177.2450  134.0619   69.2603
  217.6841  211.1934  209.5211  207.0320  203.1116  194.9077  177.4855  134.8834   68.3716
  217.6204  211.1536  209.7150  206.7973  203.0984  195.3934  176.7813  135.0665   68.7906
  217.6902  211.0918  209.6755  206.9887  202.8681  195.3807  177.2218  134.5305   68.8839
  217.6655  211.1595  209.6142  206.9497  203.0559  195.1591  177.2102  134.8658   68.6106

>> popsize = sum(popfreq,2)

popsize =
  1.0e+003 *

    1.0080
    1.1605
```

```
    1.3224
    1.4355
    1.5280
    1.5990
    1.6271
    1.6288
    1.6206
    1.6249
    1.6257
    1.6230
    1.6248
    1.6247
    1.6239
    1.6245
    1.6244
    1.6242
    1.6244
    1.6243
    1.6243

>> barh([popfreq(1,:)' popfreq(21,:)'],'group')

>> % initial (1982) and projected (long-run) age distributions
```



Thus, the momentum of population growth causes population size to rise by more than 60% even after fertility has been reduced to replacement level. Moreover, while the age distribution was initially skewed toward younger age classes, the long-run age distribution is now “fatter” because it reflects survival probabilties.

10.3 A formal result

Our projections in the preceding section started from the actual age distribution in China in 1982. Clearly, this population had not yet reached a stable growth equilibrium.[1] However, if we assume a hypothetical population which has reached this equilibrium, it is possible to state a precise result concerning population momentum (due to Keyfitz 1971). Namely, if the NRR falls suddenly to 1 in period 0, population momentum will cause population size to increase by factor

$$\frac{\mathrm{P}_\infty}{\mathrm{P}_0} = \mathrm{b}_0 \ \mathrm{e} \ \frac{\mathrm{B}_\infty}{\mathrm{B}_0}$$

where P_t denotes population size in period t, B_t denote the size of the birth cohort in period t, $\mathrm{b}_t = \mathrm{B}_t/\mathrm{P}_t$ denote the birth rate in period t, and e denotes life expectancy at birth.

To see why this result holds, let's move from this (standard demography) notation to our own notation from section 6.1. Given 4 age classes, and normalizing population size so that $\mathrm{B}_0 = \mathbf{x}_0(1) = 1$, we can write

$$\begin{aligned} \mathrm{P}_0 &= 1 + s(1)\lambda^{-1} + s(1)s(2)\lambda^{-2} + s(1)s(2)s(3)\lambda^{-3} \\ \mathrm{e} &= 1 + s(1) + s(1)s(2) + s(1)s(2)s(3) \end{aligned}$$

and hence

$$\mathrm{b}_0 \ \mathrm{e} = \frac{1 + s(1) + s(1)s(2) + s(1)s(2)s(3)}{1 + s(1)\lambda^{-1} + s(1)s(2)\lambda^{-2} + s(1)s(2)s(3)\lambda^{-3}}$$

where λ is the initial growth factor. Intuitively, this expression reflects the "fattening" of the age distribution already illustrated by the bar graph above. If the size of the birth cohort remained unchanged as the population reached the new zero-growth equilibrium (i.e., if $\mathrm{B}_\infty/\mathrm{B}_0 = 1$), then the ratio $\mathrm{P}_\infty/\mathrm{P}_0$ would be determined entirely by this effect. However, somewhat more subtlely, the size of the birth cohort falls as the population reaches its new equilibrium (so that $\mathrm{B}_\infty/\mathrm{B}_0 < 1$). Given the initial skew of the age distribution, the age classes with positive fertility are smaller (in period 0) than they will be eventually (in period ∞). Thus, the decrease in NNR initially causes a fall in the size of the birth cohort. But then, as girls who were already born in period 0 survive into child-bearing age classes, the birth cohort begins to rise again. Eventually, after some further oscillations, the size of the birth cohort stabilizes, converging to $\mathrm{B}_\infty < \mathrm{B}_0$.

To illustrate, we'll again use the Chinese data from the preceding section. But now our first step is to determine the stable growth equilibrium associated with the original Leslie matrix. While we could obtain the limiting distribution and growth factor through population projection, we'll simply compute the eigenvalues and eigenvectors of this matrix.

```
>> [eigvec, eigval] = eig(L);    %  L is Leslie matrix for China, 1981
```

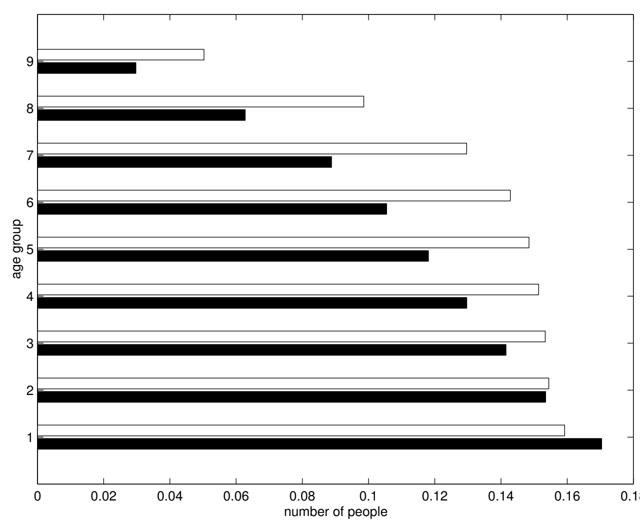
[1] Perhaps the most obvious indication is that there were fewer individuals in age class 1 than age class 2 even though NRR was greater than 1. This presumably reflects social upheavals in China during the preceding decades.

```
>> abs(diag(eigval))'   % absolute values of the eigenvalues

ans =
         0         0         0         0         0    1.0771    0.7347    0.7347    0.2126

>> % thus, the dominant eigenvalue (= 1.0771) and associated eigenvector are in 6th column

>> v1 = eigvec(:,6); x0 = v1/sum(v1); x0'   % limiting distribution

ans =
    0.1704    0.1535    0.1415    0.1297    0.1181    0.1055    0.0888    0.0628    0.0297
```

Given this new initial condition, our second step is the same as before. Using the empirical Leslie matrix, each of the fertility levels in the first row is divided by NRR. We then project population growth for the next 20 periods to obtain the eventual zero-growth equilibrium.

```
>> L(1,:) = L(1,:)/NRR;   % modified Leslie matrix with NRR = 1

>> popfreq = []; for t = 0:20; popfreq = [popfreq; (L^t * x0)']; end; popfreq

popfreq =
    0.1704    0.1535    0.1415    0.1297    0.1181    0.1055    0.0888    0.0628    0.0297
    0.1499    0.1653    0.1524    0.1397    0.1272    0.1136    0.0957    0.0676    0.0320
    0.1614    0.1454    0.1642    0.1504    0.1370    0.1224    0.1031    0.0728    0.0345
    0.1619    0.1566    0.1444    0.1620    0.1476    0.1318    0.1110    0.0784    0.0371
    0.1561    0.1570    0.1555    0.1425    0.1589    0.1420    0.1195    0.0845    0.0400
    0.1604    0.1514    0.1559    0.1535    0.1398    0.1529    0.1288    0.0910    0.0431
    0.1598    0.1556    0.1503    0.1539    0.1506    0.1345    0.1387    0.0980    0.0464
    0.1582    0.1550    0.1545    0.1484    0.1510    0.1448    0.1220    0.1055    0.0500
    0.1598    0.1535    0.1539    0.1525    0.1456    0.1452    0.1314    0.0928    0.0538
    0.1593    0.1550    0.1524    0.1519    0.1496    0.1400    0.1317    0.1000    0.0473
    0.1589    0.1545    0.1539    0.1504    0.1490    0.1439    0.1270    0.1002    0.0510
    0.1594    0.1542    0.1534    0.1519    0.1476    0.1433    0.1305    0.0967    0.0511
    0.1592    0.1546    0.1531    0.1514    0.1490    0.1419    0.1300    0.0993    0.0493
    0.1591    0.1544    0.1536    0.1511    0.1485    0.1433    0.1287    0.0989    0.0507
    0.1593    0.1544    0.1533    0.1516    0.1482    0.1429    0.1300    0.0980    0.0505
    0.1592    0.1545    0.1533    0.1513    0.1487    0.1426    0.1296    0.0989    0.0500
    0.1592    0.1544    0.1534    0.1513    0.1485    0.1430    0.1293    0.0986    0.0505
    0.1592    0.1544    0.1533    0.1514    0.1484    0.1428    0.1297    0.0984    0.0503
    0.1592    0.1545    0.1533    0.1513    0.1486    0.1428    0.1295    0.0987    0.0502
    0.1592    0.1544    0.1534    0.1513    0.1485    0.1429    0.1295    0.0986    0.0504
    0.1592    0.1544    0.1534    0.1514    0.1485    0.1428    0.1296    0.0985    0.0503

>> popsize = sum(popfreq,2)

popsize =
    1.0000
    1.0434
```

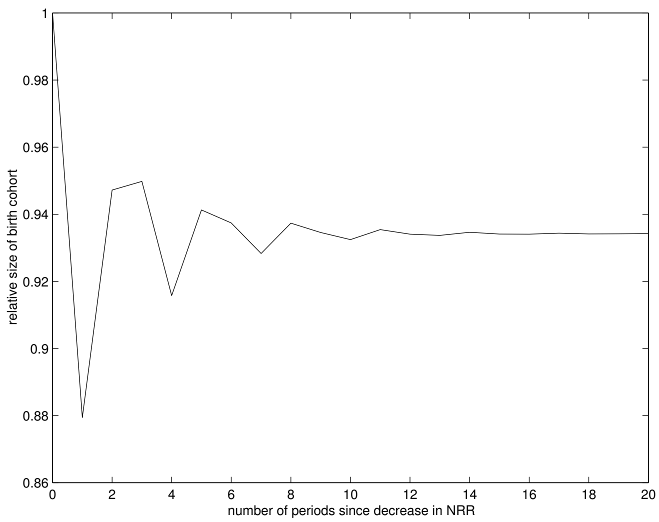
```
    1.0911
    1.1308
    1.1560
    1.1767
    1.1877
    1.1894
    1.1884
    1.1872
    1.1889
    1.1881
    1.1879
    1.1884
    1.1881
    1.1881
    1.1883
    1.1881
    1.1882
    1.1882
    1.1882

>> barh([popfreq(1,:)' popfreq(21,:)'], 'group')

>> % equilibrium age distributions before and after fall in NRR
```



Note that the size of the birth cohort falls slightly as the population moves from the initial stable-growth equilibrium to the new zero-growth equilibrium.

To illustrate the short-run oscillations in the size of this cohort, we can normalize and plot the first column of the popfreq matrix.

```
>> plot(0:20, popfreq(:,1)/popfreq(1,1))  % relative size of birth cohort
```



Thus, we see that the size of the birth cohort eventually stabilizes at about 93% of the size of the initial birth cohort.

Finally, to verify the analytical result from Keyfitz (1971), consider the following computations.

```
>> b0 = x0(1)/sum(x0)   % x0 is the initial age distribution

b0 =
    0.1704

>> e = sum(N(:,1))   % N is the fundamental matrix

e =
    7.4625

>> b0 * e

ans =
    1.2718

>> Bratio = popfreq(21,1)/popfreq(1,1)

Bratio =
    0.9343

>> Pratio = b0 * e * Bratio

Pratio =
    1.1882
```

If there was no change in the size of the birth cohort, the momentum of population growth would have generated a 27.18% increase in population size. However, because the birth cohort will eventually stabilize at 93.43% of its initial size, the momentum effect increases population size by only 18.82%. Note that this corresponds to the result obtained from the population projections above.

10.4 Further reading

This chapter is based on Keyfitz's (Demography 1971) original paper on the momentum of population growth. The approximation argument connecting the equilibrium growth factor to the NRR is found in Farina and Rinaldi (2000, p xx). The empirical data on China was taken from Bradley and Meeks (1986, Ch xx, p xx), who provide another discussion of population projection methods.