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4 Absorbing Markov Chains

So far, we have focused on *regular* Markov chains for which the transition matrix P is primitive. Because primitivity requires $P(i,i) < 1$ for every state i, regular chains never get "stuck" in a particular state. However, other Markov chains may have one or more *absorbing states*. By definition, state i is absorbing when $P(i,i) = 1$ (and hence $P(i,j) = 0$ for all $j \neq i$). In turn, the chain itself is called an *absorbing chain* when it satisfies two conditions. First, the chain has at least one absorbing state. Second, it is possible to transition from each non-absorbing state to some absorbing state (perhaps in multiple steps). Consequently, the chain is eventually "absorbed" into one of these states.[1]

This chapter focuses on absorbing Markov chains, developing some special analysis of this type of chain. We begin with a simple social-psychological application, and then consider a more elaborate model of network formation.

4.1 Conformity to group pressure

In a famous psychology experiment, Asch (1951) examined the degree to which people conform to group pressure. Ostensibly, the experiment concerned visual perception. A subject was led into a room and seated with some other participants. The experimenter presented them with a simple visual task – to determine which of three lines was the same length as a reference line. In sequence, each participant was asked to give his response, with the subject answering last. The task was designed so that the correct response was obvious. However, unbeknownst to the subject, all of the other participants were confederates of the experimenter, and had been instructed to give the same incorrect answer. Thus, when it was finally the subject's turn to answer, the subject had to choose whether to give the correct answer (ignoring pressure to conform to the group) or the incorrect answer (giving into this pressure).

After recording the responses, the experimenter presented another similar task, and participants again answered sequentially with the subject going last. In one version of this experiment (reported by Cohen 1958), this procedure was repeated 35 times in all. Thus, for each subject, the data might consist of a sequence of responses such as

aababbaabbbabbbaaabbabbbbbbbbbbbbbbbbbbbbbbb

[1] We've already seen one example of an absorbing state at the end of Chapter 2. In the mover-stayer model, any individual who is a stayer (occupying some absorbing state iS) never changes state. However, in that example, the chain itself was not absorbing because it was not possible to transition (even indirectly) from any of the non-absorbing (mover) states to some absorbing (stayer) state. The general observation is that a Markov chain can be neither regular nor absorbing. We will develop a more complete typology of states and chains in Chapter 7.

where a denotes the correct answer (against the group) and b denotes the incorrect answer (with the group). This experiment was repeated with other subjects, each of whom generated his own response sequence. While hypothetical, the sequence above illustrates a common pattern. Namely, most subjects initially waver between a's and b's, but eventually "lock into" a single response (with some subjects choosing a and others choosing b).

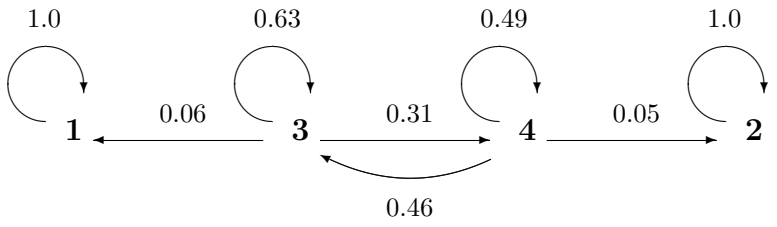
What type of process might generate such sequences? A regular Markov chain could potentially produce the initial portion (when subjects appear to be alternating stochastically between responses) but cannot account for the final portion (when the process seems to have been "absorbed"). To reconcile these observations, Cohen (1958) proposed a Markov chain model with 4 states:

1 permanent non-conformist
2 permanent conformist
3 temporary non-conformist
4 temporary conformist

Observationally, subjects in state 1 or 3 give response a, while subjects in state 2 or 4 give response b. But in Cohen's model, states 1 and 2 are absorbing, while transitions can occur from state 3 to 1 or 4, and from state 4 to 3 or 2. More precisely, using the experimental data, Cohen estimated the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.06 & 0 & 0.63 & 0.31 \\ 0 & 0.05 & 0.46 & 0.49 \end{bmatrix}$$

which can be represented as the transition diagram below.



Because it is possible to transition from each of the non-absorbing states (3 and 4) into an absorbing state (1 or 2), this Markov chain is absorbing.[2]

Regardless of the type of Markov chain (e.g., regular or absorbing), we can continue to apply the matrix analysis developed in Chapter 1.3. That is, for any Markov

[2]In this example, it is possible to move directly from each non-absorbing state to some absorbing state. But the definition of an absorbing chain merely requires indirect (multi-step) transitions.

chain, $P^t(i,j)$ can always be interpreted as the probability that the process that occupied state i in period 0 will occupy state j in period t. Thus, the following computations reveal the (probabilistic) future of the chain.

```
>> P = [1 0 0 0; 0 1 0 0; .06 0 .63 .31; 0 .05 .46 .49]

P =
    1.0000         0         0         0
         0    1.0000         0         0
    0.0600         0    0.6300    0.3100
         0    0.0500    0.4600    0.4900

>> P^2

ans =
    1.0000         0         0         0
         0    1.0000         0         0
    0.0978    0.0155    0.5395    0.3472
    0.0276    0.0745    0.5152    0.3827

>> P^3

ans =
    1.0000         0         0         0
         0    1.0000         0         0
    0.1302    0.0329    0.4996    0.3374
    0.0585    0.0936    0.5006    0.3472

>> P^10

ans =
    1.0000         0         0         0
         0    1.0000         0         0
    0.3073    0.1334    0.3324    0.2269
    0.2375    0.1958    0.3368    0.2299

>> P^100

ans =
    1.0000         0         0         0
         0    1.0000         0         0
    0.6618    0.3351    0.0019    0.0013
    0.5967    0.4001    0.0019    0.0013

>> P^1000

ans =
    1.0000         0         0         0
         0    1.0000         0         0
    0.6638    0.3362    0.0000    0.0000
    0.5987    0.4013    0.0000    0.0000
```

Following Cohen (1958), we might assume that subjects began the experiment in one of the non-absorbing states. In the "short run" – after only 2 or 3 responses – most subjects would still occupy one of the non-absorbing states. For instance, consider a subject who was initially a temporary non-conformist (state 3). For the third response, she has a 49.96% chance of being a temporary non-conformist and a 33.74% chance of being a temporary conformist (state 4).

But in the "long run" – after many responses – each subject will eventually become either a permanent non-conformist (state 1) or a permanent conformist (state 2). In particular, if she was initially a temporary non-conformist (state 3), the subject has a 66.38% chance of ending up as a permanent non-conformist and a 33.62% chance of ending up a permanent conformist. If she was initially a temporary conformist (state 4), she has a 59.87% chance of ending up a permanent non-conformist and a 40.13% chance of ending up as a permanent conformist. This illustrates an important general result: the limiting distribution for an absorbing chain typically depends on the initial state of the process. In contrast, the Theorem from Chapter 1 guarantees that the limiting distribution for a regular chain does not depend on the intial state.

4.2 Analysis of expected time until absorption

Following the preceding example, the transition matrix for any absorbing chain can be written in the "canonical" form

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

where R is the rectangular submatrix giving transition probabilities from non-absorbing to absorbing states, Q is the square submatrix giving these probabilities from non-absorbing to non-absorbing states, I is an identity matrix, and 0 is a rectangular matrix of zeros.[3] Iterated multiplication of the P matrix yields

$$P^2 = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} = \begin{bmatrix} I & 0 \\ R+QR & Q^2 \end{bmatrix},$$

$$P^3 = \begin{bmatrix} I & 0 \\ R+QR & Q^2 \end{bmatrix} \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} = \begin{bmatrix} I & 0 \\ R+QR+Q^2R & Q^3 \end{bmatrix},$$

and hence by induction we obtain

$$P^t = \begin{bmatrix} I & 0 \\ (I+Q+Q^2+\ldots+Q^{t-1})R & Q^t \end{bmatrix}.$$

[3]Note that R and 0 are not necessarily square. More precisely, if there are n_A absorbing states and n_N non-absorbing states, then R is $n_N \times n_A$, Q is $n_N \times n_N$, I is $n_A \times n_A$, and 0 is $n_A \times n_N$.

The preceding example illustrates the general result that $Q^t \to 0$ as $t \to \infty$.[4] Thus,

$$P^\infty \;=\; \begin{bmatrix} I & 0 \\ NR & 0 \end{bmatrix}$$

where the matrix

$$N \;=\; I + Q + Q^2 + \ldots \;=\; (I - Q)^{-1}$$

is called the *fundamental matrix* for the absorbing chain.[5]

The elements of the N matrix can be given an interesting interpretation. However, we should first clarify the indexing of states in this matrix. Given its derivation from the Q matrix, N is a square matrix with rows and columns corresponding to the non-absorbing states. Thus, row i corresponds to the ith non-absorbing state, not the ith state overall. For instance, in the preceding example, row 1 and 2 of the N matrix correspond to states 3 and 4 (i.e., the 1st and 2nd non-absorbing states).

With that clarification, here's the interpretation: $N(i, j)$ is the expected number of periods that the chain spends in the jth non-absorbing state given that the chain began in the ith non-absorbing state. Perhaps this interpretation is apparent from the specification of the N matrix as the infinite sum

$$N \;=\; I + Q + Q^2 + \ldots$$

which implies

$$N(i, j) \;=\; Q^0(i, j) + Q^1(i, j) + Q^2(i, j) + \ldots$$

where $Q^t(i, j)$ is the probability that the process which began in the ith non-absorbing state will occupy the jth non-absorbing state in period t. However, $Q^t(i, j)$ can also be understood as the expected *proportion* of period t spent in the jth state.[6] Summing over all time periods t, we thus obtain the total number of periods that the chain is expected to occupy the jth state.

To arrive at this interpretation in a different way, we can also specify these expectations recursively. To begin, we can write

$$N(i, i) \;=\; 1 + \sum_k N(i, k)Q(k, i)$$

where the summation is taken over all non-absorbing states k. Intuitively, if the chain begins in the ith non-absorbing state, then it must (obviously) occupy the ith state for the one initial period. Further, for each of the $N(i, k)$ periods that the chain is expected to occupy the kth non-absorbing state, the chain transitions back to the ith

[4] Because the sum of every row of Q is strictly less than 1, the largest eigenvalue of Q is less than 1, and hence $Q^t \to 0$ as $t \to \infty$.

[5] To simplify the infinite sum, note that $NQ = Q + Q^2 + \ldots$, and hence $N - NQ = I$.

[6] Alternatively, averaging over many chains, $Q^t(i, j)$ is the proportion of chains in the jth non-absorbing state in period t.

non-absorbing state with probability $Q(k,i)$. Summing over all non-absorbing states, we obtain the total number of periods (in addition to the first) that the process is expected to occupy the ith non-absorbing state. Similarly, we can write

$$N(i,j) \;=\; \sum_k N(i,k)Q(k,j) \quad \text{for all } j \neq i.$$

This equation is similar to the first, but omits the initial period since the chain did not begin in the jth non-absorbing state. Rewriting this system of equations in matrix notation, we obtain

$$N \;=\; I + QN$$

and hence

$$N \;=\; (I-Q)^{-1}.$$

Given this interpretation of the fundamental matrix, the sum of each row i reveals the expected number of periods spent in *any* non-absorbing state given that the chain initially occupied the ith absorbing state. Equivalently,

$$\sum_j N(i,j)$$

is the expected number of periods before absorption (into any absorbing state) given that the chain began in the ith non-absorbing state.

To illustrate, let's return to the example from the previous section.

```
>> R = P(3:4,1:2)   % the submatrix R

R =
    0.0600         0
         0    0.0500

>> Q = P(3:4,3:4)   % the submatrix Q

Q =
    0.6300    0.3100
    0.4600    0.4900

>> N = inv(eye(2)-Q)   % the fundamental matrix N

N =
   11.0629    6.7245
    9.9783    8.0260

>> sum(N,2)   % expected number of periods before absorption

ans =
   17.7874
   18.0043
```

```
>> N*R   % lower left submatrix of P^t as t becomes large

ans =
    0.6638    0.3362
    0.5987    0.4013
```

If the chain began in the 1st non-absorbing state (i.e., state 3), we can thus expect the chain to spend 11.06 periods in that state, 6.72 periods in the 2nd non-absorbing state (i.e., state 4), and hence 17.78 periods overall before absorption (into either state 1 or 2). Alternatively, if the chain began in the 2nd non-absorbing state (i.e., state 4), we can expect it to spend 9.98 periods in the 1st non-absorbing state, 8.03 periods in the 2nd non-absorbing state, and hence 18.00 periods before absorption. Of course, in the context of the example, each "period" corresponds to one response made by the subject.

4.3 Formation of dominance hierarchies

Sociologists have long been interested in the structure of social networks. In some types of networks, social ties between actors are not always reciprocated. For instance, within an organization, we might observe worker i giving advice to worker j but not receiving advice in return. Similarly, in a study of middle-school students, we might find that student i admires student j, but also find that j does not admire i. Depending on the context, such networks are sometimes labeled *dominance hierarchies*, and researchers might try to infer the "pecking order" within the group from the structure of the network.

It is sometimes difficult to determine the pecking order, as illustrated by the annual controversy in the US over the ranking of college football teams. Let iBj indicate that team i beat team j this year. There would be no controversy about the annual ranking if (i) every pair of teams played each other and (ii) there were never any upsets. More formally, the who-beats-whom relation B on the set of teams S would need to be *complete*, which requires

$$iBj \text{ or } jBi \quad \text{for all } i, j \in S,$$

and *transitive*, which requires

$$(iBj \text{ and } jBk) \text{ implies } iBk \quad \text{for all } i, j, k \in S.$$

Of course, in reality, many college football teams do not play each other, and we sometimes see upsets corresponding to "cycles" where iBj and jBk but kBi. Thus, the annual debate over the rankings seems inevitable.

To the extent that dominance hierarchies are transitive, we might wonder what sort of network-formation process would generate this outcome.[7] Perhaps some insight can be gained through close observation of animal populations. Given the origin of the phrase "pecking order," research by Chase (1982) seems especially relevant. After being placed together in a pen, previously unacquainted chickens will eventually establish a dominance hierarchy that is complete. Further, while these hierarchies sometimes contain cycles, transitivity is more common than chance would dictate (i.e., if dominance between each pair of chickens was determined by flipping a coin). Based on his observation of the network-formation process, Chase suggested that this result could be explained by "bystander effects." Dominance can be established directly when chicken i attacks (i.e., pecks, scratches, or jumps on) some other chicken k. However, if this attack was witnessed by bystander j, the attack may also induce dominance of i over j or dominance of j over k (or both). Thus, a single attack may generate a transitive subnetwork, increasing the degree to which the entire network will be transitive when it is complete.

Chase's description of this process led Fararo and Skvoretz (1986) to develop the following model of network formation. The process begins in period 0 with no pre-existing dominance relation. Each subsequent period then brings the following series of events:

1. Some actor i attacks some actor k, and this attack is witnessed by some bystander j. The triple (i, j, k) is drawn randomly from the set of all distinct triples such that k does not already dominate i.

2. If attacker i does not already dominate attackee k, then i dominates k (from this period forward) with probability π.

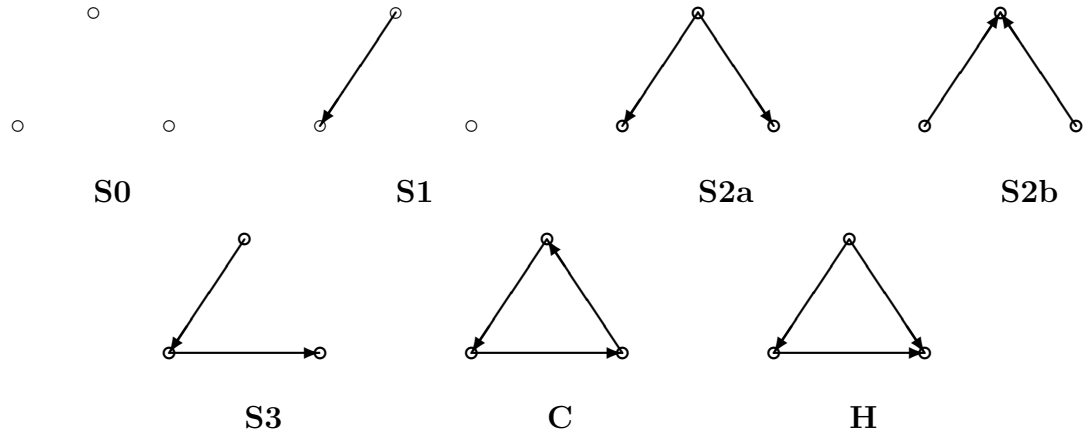
3. If dominance has not yet been established (in either direction) between attacker i and bystander j, then i dominates j (from this period forward) with probability θ.

4. If dominance has not yet been established (in either direction) between bystander j and attackee k, then j dominates k (from this period forward) with probability θ.

For simplicity, we assume that the two bystander effects (in steps 3 and 4) occur with the same probability θ, and that the three random draws (in steps 2, 3, and 4) are independent. Note that this network-formation process is monontonic (i.e., ties are added but never removed), preserves the asymmetry of the dominance relation (i.e., i dominates j only if j does not dominate i), and eventually results in a dominance relation that is complete.

[7] Given the mathematical definition of transitivity just stated, a social relation either is or is not transitive. However, sociologists are often assess the degree of transitivity in a network by determining the proportion of triads (subnetworks of three individuals) that are transitive.

This model can be specified as an absorbing Markov chain, with the states of the chain given by the possible configurations of the network. For simplicity, suppose that the group is composed of 3 actors.[8] Because we are concerned only with the structure of the network (rather than the identities of the actors), the 7 possible network configurations are depicted below. A directed edge from node i to node j indicates that i dominates j.



Given the assumptions of the model, the chain is intially in state **S0** (the empty graph), and will eventually reach one of the two absorbing states, **C** (for "cycle") or **H** (for "hierarchy").

Having considered the states of the chain, we can now tackle the transition matrix. Because this model is more complicated than others we have encountered, the derivation of the transition probabilities requires some thought. One approach is to construct, for each of the non-absorbing states, a probability tree diagram reflecting the sequence of steps within each period. Using this diagram, we can then determine the probability of moving from that non-absorbing state to each possible state in the next period. To illustrate, Figure 4.1 depicts the probability tree diagram for state **S3**. Summing over the various paths from state **S3** to state **H**, we find that this transition occurs with probability

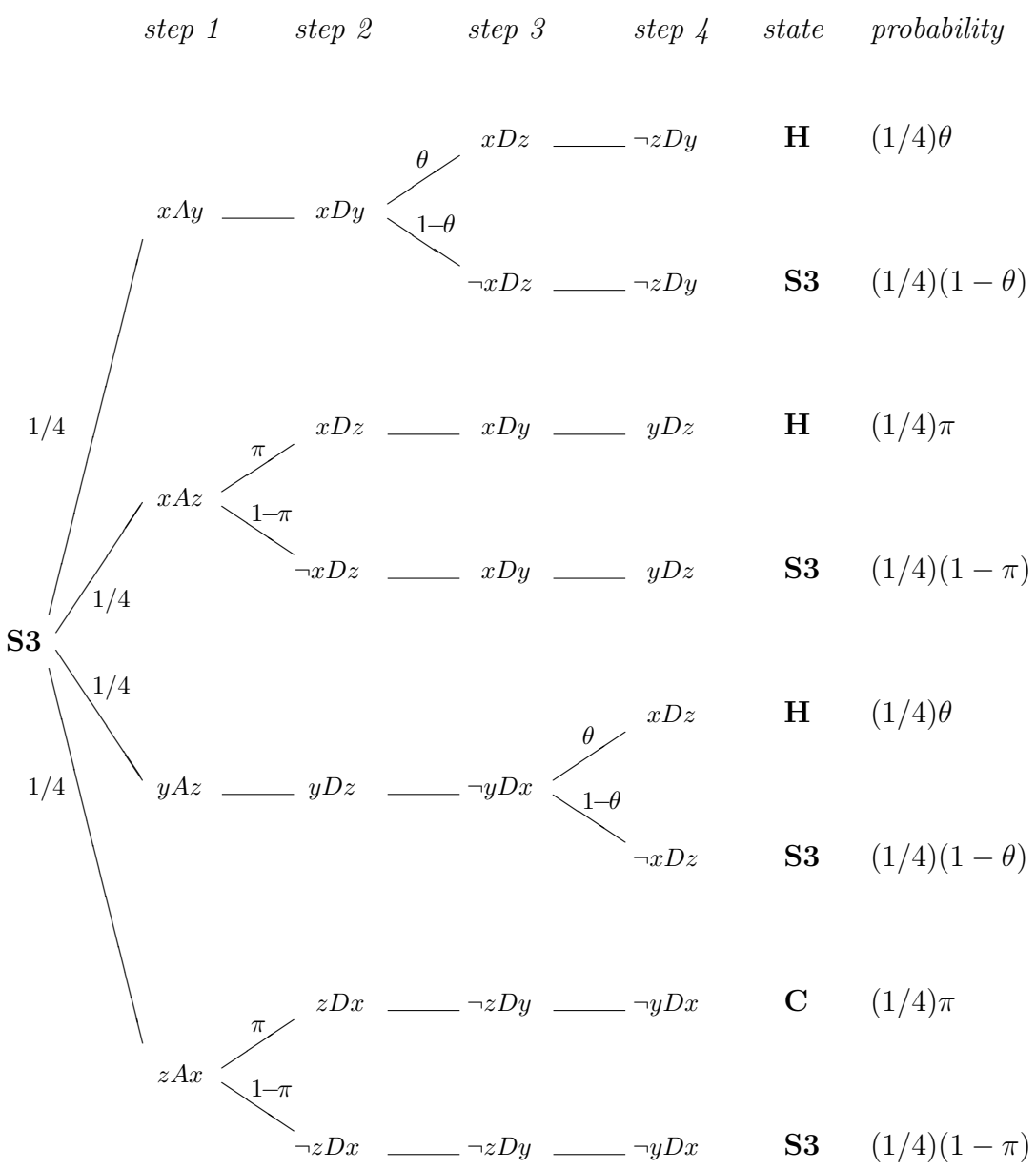
$$(1/4)\theta + (1/4)\pi + (1/4)\theta \;=\; (2\theta + \pi)/4,$$

and the transition probability from **S3** to **S3** or from **S3** to **C** can obtained in the same way. Note that it is not possible to transition from state **S3** to any of the other non-absorbing states.

It is straightforward (though admittedly tedious) to construct similar probability tree diagrams for each of the other non-absorbing states. (I'll leave this to the reader

[8]This assumption was originally adopted by Fararo and Skvoretz (1986) not only for simplicity, but also because many of Chase's observations were actually made on groups composed of 3 chickens.

Figure 4.1 Probability tree diagram for state S3



In this diagram, iAj indicates that i attacks j, iDj indicates that i dominates j, and $\neg iDj$ indicates that i does not dominate j. State **S3** is characterized by xDy and yDz (where the labels x, y, and z are assigned to match actors' positions in the network). Unlabeled edges indicate certain (probability 1) outcomes.

as an exercise.) After constructing the tree diagrams for **S2a** and **S2b**, you would find that all of the transition probabilities are identical for these states. That is, the rows of the transition matrix corresponding to **S2a** and **S2b** would be identical. This makes it possible to merge these two states into a single state.[9] Labeling this state **S2**, the model thus has 6 states, and the transition matrix is given below. Note that this matrix has been specified in canonical form, with absorbing states listed first.

	H	**C**	**S0**	**S1**	**S2**	**S3**
H	1	0	0	0	0	0
C	0	1	0	0	0	0
S0	$\pi\theta^2$	0	$(1-\pi)(1-\theta)^2$	$\pi(1-\theta)^2 + 2(1-\pi)\theta(1-\theta)$	$2\pi\theta(1-\theta)$	$(1-\pi)\theta^2$
S1	$\frac{\theta^2+4\pi\theta}{5}$	0	0	$\frac{(1-\theta)^2+4(1-\pi)(1-\theta)}{5}$	$\frac{2[(1-\pi)\theta+(\pi+\theta)(1-\theta)]}{5}$	$\frac{2[\pi(1-\theta)+(1-\pi)\theta]}{5}$
S2	$\frac{\theta+\pi}{2}$	0	0	0	$\frac{1-\theta+1-\pi}{2}$	0
S3	$\frac{2\theta+\pi}{4}$	$\frac{\pi}{4}$	0	0	0	$\frac{1-\theta+1-\pi}{2}$

Having developed the model, we can now assess Chase's argument about the bystander effect. In particular, we can examine how the strength of the direct effect (given by the parameter π) and the bystander effect (given by the parameter θ) influence the probability that the chain is eventually absorbed into the transitive hierarchy rather than the non-transitive cycle. To facilitate numerical experiments, I've written a Matlab function m-file placed in Appendix 4.5.

```
>> help fararoskvoretz

  P = fararoskvoretz(p,q)
  output P = transition matrix for network-formation model
  input p (= pi) = probability that attacker dominates attackee
  input q (= theta) = probability that bystander effects occur
```

Thus, using this m-file, we can easily substitute different numerical values of the parameters into the transition matrix given above.

To begin, let's suppose a moderately strong direct effect ($\pi = 0.5$) but no bystander effect ($\theta = 0$). In the usual way, we can determine the long-run outcome by raising the transition matrix to some high power.

[9] See Kemeny and Snell (1960) for a discussion of the "lumpability" conditions under which states of a chain can be merged in this way.

```
>> P = fararoskvoretz(.5,0)

P =
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
         0         0    0.5000    0.5000         0         0
         0         0         0    0.6000    0.2000    0.2000
    0.2500         0         0         0    0.7500         0
    0.1250    0.1250         0         0         0    0.7500

>> P^100

ans =
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
    0.7500    0.2500    0.0000    0.0000    0.0000    0.0000
    0.7500    0.2500         0    0.0000    0.0000    0.0000
    1.0000         0         0         0    0.0000         0
    0.5000    0.5000         0         0         0    0.0000
```

We are assuming that the chain always begins in state **S0**. Thus, from the 3rd row of the P^{100} matrix, we find there is a 75% chance that the chain is ultimately absorbed in the transitive hierarchy (state **H**), and hence a 25% chance that the chain is absorbed into the non-transitive cycle (state **c**). It is in this baseline case that we are essentially "flipping a coin" to determine the direction of dominance for each pair.

Holding constant the direct effect ($\pi = 0.5$), let's now consider what happens when we introduce a weak bystander effect ($\theta = 0.2$).

```
>> P = fararoskvoretz(.5,.2)

P =
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
    0.0200         0    0.3200    0.4800    0.1600    0.0200
    0.0880         0         0    0.4480    0.2640    0.2000
    0.3500         0         0         0    0.6500         0
    0.2250    0.1250         0         0         0    0.6500

>> P^100

ans =
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
    0.8982    0.1018    0.0000    0.0000    0.0000    0.0000
    0.8706    0.1294         0    0.0000    0.0000    0.0000
    1.0000         0         0         0    0.0000         0
    0.6429    0.3571         0         0         0    0.0000
```

Again assuming that the chain begins in state **S0**, there is now a greater chance (89.82%) that the network is ultimately transitive. Consistent with Chase's orginal argument, the bystander effect seems to promote transitivity.

While this is an interesting finding, there are many other combinations of parameter values to be considered. In their paper, Fararo and Skvoretz (1986, p 600, Table 3) performed a "grid search," computing the long-run probability of absorption in state **H** given initial state **S0** for every pair in

$$\{(\pi, \theta) \mid \pi, \theta \in \{0, 0.1, 0.2, \ldots, 1\}\}.$$

This is easily accomplished in Matlab using a nested loop.

```
>> table = []; for pi = 0:.1:1; row = []; for theta = 0:.1:1; P = fararoskvoretz(pi,theta); ...
X = P^1000; row = [row, X(3,1)]; end; table = [table; row]; end; table

table =

         0    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
    0.7500    0.9067    0.9372    0.9493    0.9552    0.9585    0.9602    0.9609    0.9609    0.9603    0.9591
    0.7500    0.8774    0.9137    0.9285    0.9355    0.9388    0.9400    0.9398    0.9385    0.9364    0.9333
    0.7500    0.8638    0.9037    0.9207    0.9284    0.9314    0.9318    0.9304    0.9278    0.9241    0.9192
    0.7500    0.8562    0.8996    0.9192    0.9281    0.9313    0.9313    0.9292    0.9255    0.9205    0.9143
    0.7500    0.8515    0.8982    0.9208    0.9315    0.9357    0.9360    0.9336    0.9295    0.9238    0.9167
    0.7500    0.8482    0.8981    0.9241    0.9372    0.9428    0.9440    0.9420    0.9380    0.9322    0.9250
    0.7500    0.8459    0.8987    0.9281    0.9440    0.9517    0.9542    0.9533    0.9499    0.9448    0.9382
    0.7500    0.8442    0.8997    0.9325    0.9514    0.9615    0.9659    0.9665    0.9645    0.9607    0.9556
    0.7500    0.8429    0.9009    0.9370    0.9590    0.9718    0.9785    0.9811    0.9811    0.9792    0.9763
    0.7500    0.8418    0.9021    0.9415    0.9667    0.9825    0.9917    0.9968    0.9991    0.9999    1.0000

>> % columns correspond to theta in {0, .1, ..., 1.0}; rows correspond to pi in {0, .1, ..., 1.0}
>> % entries give the probability that the process is eventually absorbed in state H
```

Note that the two results already computed appear in row 6 (corresponding to $\pi = 0.5$), columns 1 and 3 (corresponding to $\theta = 0$ and $\theta = 0.2$).

Intuition might suggest that a stronger bystander effect necessarily increases the likelihood of transitivity. However, we can see from our grid search that an increase in θ can actually *reduce* this likelihood. For instance, consider the 6th row of the table (so that π is held fixed at 0.5). As θ begins to rise from 0, the probability of absorption in **H** also rises. However, once θ rises to 0.6, this probability reaches a maximum, and any further increase in the bystander effect causes the probability of absorption in **H** to fall. It is also interesting to note that, holding constant the strength of the bystander effect, the likelihood of transitivity is lowest when the direct effect is moderate. For instance, consider the final column of the table (so that θ is held fixed at 1.0). If the direct effect is either absent ($\pi = 0$) or certain ($\pi = 1$) then state **H** is reached with certainty. But given intermediate values of π, there is some chance that the process will be absored in state **C**. Overall, analysis of the model suggests that the implications of the bystander effect are somewhat more nuanced than might have been expected from Chase's original (informal) argument.

4.4 Further reading

Most social psychology textbooks contain a chapter on conformity. See, e.g., Aronson (The Social Animal, 1988, Chap 1). The original Asch experiments were reported in Asch (1951); the absorbing chain model was developed by Cohen (Sociometry 1958). My presentation draws heavily on Bradley and Meeks (1986, Chap 9). See Kemeny and Snell (Mathematical Models in the Social Sciences, 1962, Chap 5) for a more elaborate treatment focusing on the parameter estimation.

Both Bradley and Meeks (1986, Chaps 8 and 9) and Kemeny et al (1968, pp 282-291) offer introductions to absorbing Markov chains, discussing the fundamental matrix and its interpretation.

The standard textbook for social network analysis is Wasserman and Faust (1994). Many researchers have proposed mathematical methods for ranking college football teams. See Callaghan et al (Am Math Monthly 2007) for one recent suggestion and further references. The research on pecking orders among chickens is reported in Chase (Am Soc Rev 1980, Science 1982). The absorbing chain model was developed by Fararo and Skvoretz (Am Soc Rev 1986).

4.5 Appendix

4.5.1 fararoskvoretz m-file

```
function P = fararoskvoretz(p,q)
% P = fararoskvoretz(p,q)
% output P = transition matrix for network-formation model
% input p (= pi) = probability that attacker dominates attackee
% input q (= theta) = probability that bystander effects occur

P= [1, 0, 0, 0, 0, 0; ...
    0, 1, 0, 0, 0, 0; ...
    p*q^2, 0, (1-p)*(1-q)^2, p*(1-q)^2+2*(1-p)*q*(1-q), 2*p*q*(1-q), (1-p)*q^2; ...
    (q^2+4*p*q)/5, 0, 0, ((1-q)^2+4*(1-p)*(1-q))/5, ...
     2*((1-p)*q+p*(1-q)+(1-q)*q)/5, 2*(p*(1-q)+(1-p)*q)/5; ...
    (q+p)/2, 0, 0, 0, (1-q+1-p)/2, 0; ...
    (2*q+p)/4, p/4, 0, 0, 0, (1-q+1-p)/2];
```