Very Rough Notes: General Equilibrium

September 2014 John Kennan

Edgeworth Box: Production Example. Suppose there are two goods, x, y
Technologies.

$$\begin{aligned} x &= \sqrt{KL} \\ \sqrt{y} &= \sqrt{K} + \sqrt{L} \end{aligned}$$

or

$$\begin{aligned} x &= \sqrt{K_x L_x} \\ \frac{1}{y} &= \frac{1}{K_y} + \frac{1}{L_y} \end{aligned}$$

Suppose the endowment is $\{K = 100, L = 100\}$

If $K_x = 36, L_x = 64$, then $x = 6 \times 8 = 48$, and $K_y = 64, L_y = 36$ and $\frac{1}{y} = \frac{1}{64} + \frac{1}{36} = \frac{100}{64 \times 36}$

For the first technology,

If $K_x = 36, L_x = 64$, then $x = 6 \times 8 = 96$, and $K_y = 64, L_y = 36$ and $\sqrt{y} = 8 + 6$ so $y = 196$

If $K_x = 64, L_x = 36$, then $x = 8 \times 6 = 48$, and $K_y = 36, L_y = 64$ and $\sqrt{y} = 8 + 6$ so $y = 196$

But if the inputs are divided equally, then the outputs are $x = 50, y = 200$

If the whole endowment is used to produce x, then $x = 100, y = 0$

If the whole endowment is used to produce y, then $x = 0, y = 400$

Let $z_1 = \sqrt{K_y}, z_2 = \sqrt{L_y}$

Then $K_x = \omega_K - z_1^2$, $L_x = \omega_L - z_2^2$

So $x^2 = (\omega_K - z_1^2)(\omega_L - z_2^2)$

This should be maximal, subject to $\sqrt{y} = z_1 + z_2$

The first-order condition is

$$-2z_1 \left(\omega_L - z_2^2\right) + \left(\omega_K - z_1^2\right) 2z_2 = 0$$

so if $\omega_K = \omega_L$ the solution is $z_1 = z_2$, meaning that the contract curve in the Edgeworth box it the diagonal, and the production possibility set is a straight line, given by

$$y = 4(\omega - x)$$

Marginal rate of technical substitution

$$MRTS = -\frac{dk}{dl}_{dq=0}$$

In the Cobb-Douglas case

$$\log(q) = \theta \log(k) + (1 - \theta) \log(l)$$

so

$$d \log(q) = \frac{\theta}{k} dk + \left(\frac{1-\theta}{l}\right) dl$$

and

$$MRTS = \frac{1-\theta}{\theta} \frac{k}{l}$$

Marginal rate of transformation

Text, Example 13.1.

$$\begin{aligned} x &= \sqrt{k_x l_x} \\ y^4 &= k_y l_y^3 \\ k_x + k_y &= 100 \\ l_x + l_y &= 100 \end{aligned}$$

Equating the MRTS for the two goods implies

$$\frac{k_x}{l_x} = \frac{3k_y}{l_y}$$

Let $\alpha = \frac{l_x}{\omega_l}$, and $\kappa = \frac{k}{l}$. Then $\alpha\kappa_x + (1-\alpha)\kappa_y = \omega_K$, and since $\kappa_x = 3\kappa_y$, it follows that

$$(1+2\alpha)\kappa_y = \frac{\omega_K}{\omega_L}$$

Equilibrium. If everyone has the same preferences, equilibrium requires that the production possibility curve and the indifference curve are tangent to each other.

Exchange. There are m individuals, indexed by i, and n goods.

Each individual has an endowment $\omega \in \mathbb{R}^n$(the text uses $\bar{x}$ for the endowment).

The individual's budget constraint is

$$p \cdot x^i = p \cdot \omega^i$$

Efficiency in an Exchange Economy. Suppose there are two people and two goods.

Preferences

$$\begin{aligned} u^1(x) &= \sqrt{x_1^1 x_2^1} \\ u^2(x) &= \sqrt{x_1^2} + \sqrt{x_2^2} \end{aligned}$$

Suppose the endowment is $\{\omega_1 = 100, \omega_2 = 100\}$

If $x_1^1 = 36, x_2^1 = 64$, then $u^1 = 6 \times 8 = 48$, and $x_1^2 = 64, x_2^2 = 36$,and $u^2 = 8 + 6 = 14$

If $x_1^1 = 64, x_2^1 = 36$, then $u^1 = 8 \times 6 = 48$, and $x_1^2 = 36, x_2^2 = 64$,and $u^2 = 6 + 8 = 14$

But if the inputs are divided equally, then the utilities are $u^1 = 50, u^2 = 2\sqrt{50} > 14$

If the whole endowment is used allocated to person One, then $u^1 = 100, u^2 = 0$.

If the whole endowment is used allocated to person Two, then $u^2 = 0, u^2 = 20$.

In this example, the contract curve is the 45^o line (by symmetry).

In general, efficiency requires that the marginal rates of substitution are equal for all consumers

Allocations. Think of a big table of numbers. Each row refers to a particular good. Each consumer has two columns, one for the endowment vector, and the other for the consumption (net demand) vector.

One of the goods is labor, and each consumer is endowed with some amount of time; more generally, there might be different kinds of labor.

Each firm has a single column, specifying the inputs and outputs of the firm (where the inputs are negative numbers, and the outputs are positive). The entries in this column must be feasible (the specified output quantities can be produced using the input quantities).

For each good, total consumption can't exceed the sum of endowment and total production.

Feasibility requires

$$\sum_i x_\ell^i = \sum_i \omega_\ell^i + \sum_j y_\ell^y$$

for all commodities ℓ.

General Equilibrium.

$$\sum_{i=1}^m X^i\left(p^*, p^* \cdot \omega^i\right) = \sum_{i=1}^m \omega^i$$

(agents optimize, and markets clear)

Walras Law. For *any* price vector p

$$p \cdot \sum_{i=1}^m X^i\left(p, p \cdot \omega^i\right) = p \cdot \sum_{i=1}^m \omega^i$$

Efficiency (Pareto Optimality). An allocation is efficient (Pareto Optimal) if no one can be made better off without making someone worse off.

Efficiency in Exchange. Suppose there are two consumers with different marginal rates of substitution between two goods, x and z.

This means

$$\frac{MU_x^1}{MU_z^1} < \frac{MU_x^2}{MU_z^2}$$

If One's consumption bundle is modified so that

$$\frac{MU_x^1}{MU_z^1} = -\frac{\Delta z^1}{\Delta x^1}$$

then One is indifferent between the new consumption bundle and the old one. And if Two's consumption bundle is modified by the same amounts in the opposite direction (so that total consumption of these two people remains constant for each good), then

$$\begin{aligned}\frac{\Delta z^2}{\Delta x^2} &= \frac{\Delta z^1}{\Delta x^1} \\ &= -\frac{MU_x^1}{MU_z^1}\end{aligned}$$

so

$$\begin{aligned}-\frac{\Delta z^2}{\Delta x^2} &= \frac{MU_x^1}{MU_z^1} \\ &< \frac{MU_x^2}{MU_z^2}\end{aligned}$$

If $\Delta x^2 > 0$ then

$$-MU_z^2 \Delta z^2 < MU_x^2 \Delta x^2$$

But this means

$$\Delta U^2 = MU_x^2 \Delta x^2 + MU_z^2 \Delta z^2 > 0$$

so that the modification of the two consumption plans leaves One equally well off (i.e. at the same utility level), while Two is better off.

For example, if the utility functions are $u^1 = xz$, and $u^2 = \sqrt{x} + \sqrt{z}$, then the marginal rates of substitution are

$$\begin{aligned} m^1 &= -\frac{z}{x} \\ m^2 &= -\sqrt{\frac{z}{x}} \end{aligned}$$

so if $(x^1, z^1) = (36, 64)$ and $(x^2, z^2) = (64, 36)$, then

$$\begin{aligned} m^1 &= -\frac{16}{9} \\ m^2 &= -\frac{6}{8} \end{aligned}$$

and if $(x^1, z^1) = (64, 36)$ and $(x^2, z^2) = (36, 64)$, then

$$\begin{aligned} m^1 &= -\frac{9}{16} \\ m^2 &= -\frac{8}{6} \end{aligned}$$

Efficiency Conditions. For any pair of goods, the marginal rates of substitution must be the same for everyone who consumes positive quantities of these two goods.

For any pair of factors of production, the marginal rates of technical substitution must be the same for every product that uses positive quantities of these two factors.

For any pair of goods, the marginal rate of transformation must be the same as the marginal rate of substitution in consumption.

Planning. Suppose you are given dictatorial power over everything in the economy. You are interested only in making things better for everyone. To that end, you want to make sure that you at least achieve an efficient allocation. How would you do this?

First Welfare Theorem. Any competitive equilibrium allocation is Pareto Optimal.

If there is an alternative allocation that is a Pareto improvement, the value of aggregate consumption at the equilibrium prices is strictly larger in this alternative allocation (someone is doing strictly better, so the value of this person's consumption bundle might be strictly greater, or it would have been chosen before; and no one is doing worse, and if they could have achieved this by spending less money, then local nonsatiation implies that they could have done better). The value of consumption is the value of net production plus the value of the endowment. But the value of net production can't be higher in the alternative plan, because if it were, some producer was not maximizing profit. And since the value of the endowment is unchanged, this gives a contradiction.

This is an elementary result. The second theorem is deeper.

Second Welfare Theorem. Any Pareto Optimal allocation can be implemented as a competitive equilibrium, given some redistribution of the endowments.

This can be illustrated using an Edgeworth Box for an exchange economy.