Chapter 8: Growth Accounting / Solow Residual [1]

1 Growth accounting

Let's assume a standard Cobb-Douglas aggregate production function:
$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$$

so output (Y_t) is a function of productivity (A_t), physical capital (K_t), and labor (L_t) at time t; $0 \le \alpha \le 1$. Unlike the Solow model as we typically set it up with labor-augmenting technological progress, productivity here is neutral in the sense that it just scales up output and does not enter like a factor $(K_t$ or $L_t)$ would. How do we write output growth $(\%\Delta Y_t)$ as a function of the growth rates of its components $(\%\Delta A_t, \%\Delta K_t, \%\Delta L_t)$? The approximation $\Delta \log(x) \approx \%\Delta x$ (for $x \approx 1$) can be used to go from the Cobb-Douglas production function to an expression involving only growth rates. Think about how you would do this. The first step should be to take logs of the production function, then take differences. Once we have everything in $\Delta \log$ terms, the approximation can be applied to yield the answer we're looking for, output growth as a function of component growth rates. Let's do this.

(1) Take logs of the production function (step by step):
$$\log(Y_t) = \log(A_t K_t^\alpha L_t^{1-\alpha})$$

$$\log(Y_t) = \log(A_t) + \log(K_t^\alpha) + \log(L_t^{1-\alpha})$$

$$\log(Y_t) = \log(A_t) + \alpha \log(K_t) + (1-\alpha)\log(L_t)$$

(2) Repeat this for time $t+1$ and take differences (again, going though every step):
$$\log(Y_{t+1}) = \log(A_{t+1}) + \alpha \log(K_{t+1}) + (1-\alpha)\log(L_{t+1})$$

$$\log(Y_{t+1}) - \log(Y_t) = [\log(A_{t+1}) - \log(A_t)] + \alpha[\log(K_{t+1}) - \log(K_t)] + (1-\alpha)[\log(L_{t+1}) - \log(L_t)]$$

$$\Delta \log(Y_t) = \Delta \log(A_t) + \alpha \Delta \log(K_t) + (1-\alpha)\Delta \log(L_t)$$

Notice that we could have skipped from everything in log to everything in $\Delta \log$ by writing Δ in front of each term. This is a useful property of the difference operator Δ (it's a linear operator).

(3) Finally, use the approximation $\Delta \log(x) \approx \%\Delta x$:
$$\%\Delta Y_t = \%\Delta A_t + \alpha\%\Delta K_t + (1-\alpha)\%\Delta L_t$$

which was our goal.

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[1] Econ 302, 11/8/2009; UW-Madison. TAs Lihan Liu and Scott Swisher.

2   What is the "Solow residual"?

Why is the last equation useful? Think about how you'd measure productivity growth $\%\Delta A_t$. You can gather data on the growth rates of capital, labor, and output; measure how much stuff was produced using GDP, survey firms and find out how much capital and labor they used in any given quarter. Therefore, you can observe $\%\Delta Y_t$, $\%\Delta K_t$, and $\%\Delta L_t$ in an actual economy. The problem is that $\%\Delta A_t$ is in a sense "what is left over" after accounting for growth in everything else; we can't go and ask firms what $\%\Delta A_t$ is. This is why $\%\Delta A_t$ is called the Solow residual. Let's write $\%\Delta A_t$ in terms of what we can measure:

$$\%\Delta A_t = \%\Delta Y_t - [\alpha \%\Delta K_t + (1-\alpha)\%\Delta L_t]$$

This equation is the only feasible way to compute $\%\Delta A_t$. In words, productivity growth is what remains in output growth after subtracting out growth in the factors of production (capital and labor). Productivity growth is the part of output growth that we can't explain using growth in capital and labor. This is all assuming that our aggregate production function is correct.
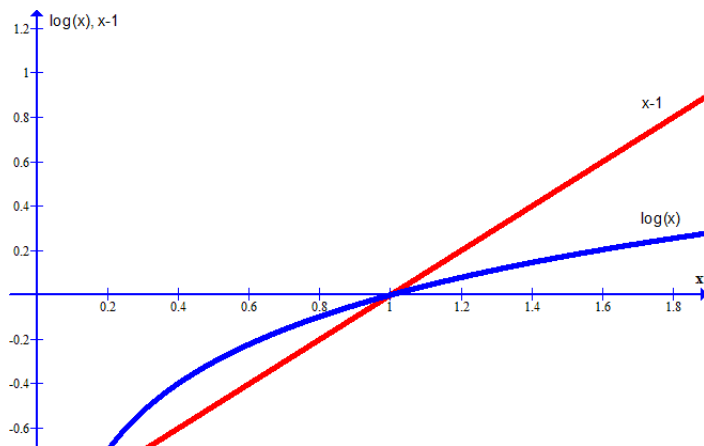
3   Why does this work? (optional)

Provided that x is close to one, the approximation $\Delta \log(x) \approx \%\Delta x$ is reasonably accurate. Let's think more about what this means; I want to rewrite $\%\Delta Y_t$ and $\Delta \log(Y_t)$ in a particular way.

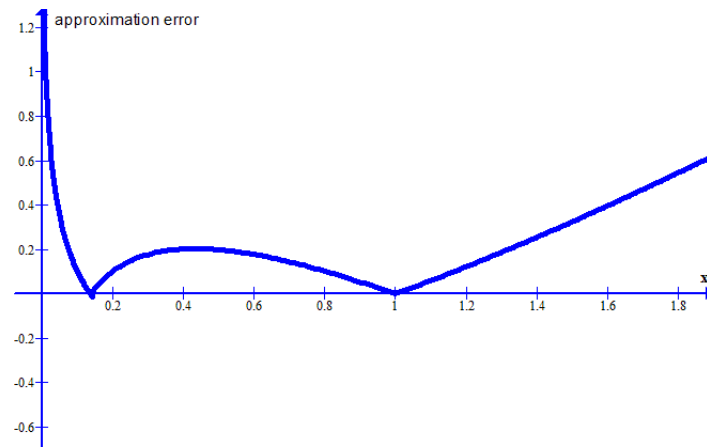$$\%\Delta Y_t = \frac{Y_{t+1} - Y_t}{Y_t} = \frac{Y_{t+1}}{Y_t} - 1$$

$$\Delta \log(Y_t) = \log(Y_{t+1}) - \log(Y_t) = \log(\frac{Y_{t+1}}{Y_t})$$

Let's define $x \equiv \frac{Y_{t+1}}{Y_t}$, so are $x - 1$ and $\log(x)$ close near $x = 1$? Graph $1 - x$ and $\log(x)$ versus x near $x = 1$:

The two curves are relatively close around $x = 1$, so the approximation is good. $x \approx 1$ means that $\frac{Y_{t+1}}{Y_t} \approx 1$, which should hold if we are dealing with data on national income; GDP, capital, and labor only grow by a few percentage points per year, so Y_{t+1} is not too different from Y_t.

Finally, let's graph approximation error $\epsilon \equiv |\log(x) - (x - 1)|$ versus x near $x = 1$:

Again, the error is small if you are close enough to $x = 1$. This is why we can approximate percentage change in a variable as the change in the log of that variable.