# *Chapter 7: Solow Model I* [1]

## 1 The Solow-Swan Model

### 1.1 Per-worker Quantities

Cobb-Douglas production function: $Y = F(K, L) = K^\alpha L^{1-\alpha}$
Output per worker: $y \equiv \frac{Y}{L}$; capital per worker: $k \equiv \frac{K}{L}$; consumption per worker: $c \equiv \frac{C}{L}$; investment per worker: $i \equiv \frac{I}{L}$.

$$Y = K^\alpha L^{1-\alpha} \Rightarrow \frac{Y}{L} = (\frac{K}{L})^\alpha \Rightarrow y = k^\alpha$$

$$C = (1-s)Y \Rightarrow \frac{C}{L} = (1-s)\frac{Y}{L} \Rightarrow c = (1-s)y$$

$$I = sY \Rightarrow \frac{I}{L} = s\frac{Y}{L} \Rightarrow i = sy$$

### 1.2 Assumptions

1. No technological progress: $A = \bar{A}$
2. No population/labor force growth: $L = \bar{L}$
3. Exogenous, constant savings rate: $s = \bar{s}$
4. Exogenous, constant depreciation rate: $\delta = \bar{\delta}$
5. No government sector: $G = T = 0$
6. No international sector: $X = M = 0$

$$G = NX = 0 \Rightarrow Y = C + I$$

### 1.3 Law of Motion for the Capital Stock

"Capital gain": investment, $i = sf(k) = sk^\alpha$
"Capital loss": depreciated capital, $\delta k$
Law of motion (discrete time): $\Delta k = i - \delta k = sk^\alpha - \delta k$

$$i < \delta k \Rightarrow \Delta k < 0$$

$$i = \delta k \Rightarrow \Delta k = 0$$

$$i > \delta k \Rightarrow \Delta k > 0$$

[1] Econ 302, Week 9, 10/30/2009; UW-Madison. TAs Lihan Liu and Scott Swisher.

## 1.4 Steady-state (Equilibrium)

*Steady-state in the Solow model*: in long-run equilibrium, capital per worker (the capital-labor ratio) is constant.

*Steady-state condition*: the following equation defines a steady-state in the Solow model.

General case:

$$sf(k_{ss}) = \delta k_{ss} \;\Rightarrow\; \frac{k_{ss}}{f(k_{ss})} = \frac{s}{\delta} \tag{1}$$

Cobb-Douglas case:

$$sk_{ss}^{\alpha} = \delta k_{ss} \;\Rightarrow\; k_{ss} = (\frac{s}{\delta})^{\frac{1}{1-\alpha}} \tag{2}$$

If this steady-state condition holds, the flows in to (investment) and out of (depreciation) $k$ are constant.

$$k < k_{ss} \;\Rightarrow\; sf(k) > \delta k \;\Rightarrow\; \Delta k > 0$$

$$k = k_{ss} \;\Rightarrow\; sf(k) = \delta k \;\Rightarrow\; \Delta k = 0$$

$$k > k_{ss} \;\Rightarrow\; sf(k) < \delta k \;\Rightarrow\; \Delta k < 0$$

Steady-state quantities associated with $k_{ss}$: $y_{ss}$, $c_{ss}$, $i_{ss}$ ($k^*$ : $y^*$, $c^*$, $i^*$).

## 1.5 Policy and the Golden Rule $k_{gr}$

The Solow model predicts that countries with higher rates of savings and investment will have higher levels of capital and output/income per worker in the long-run, *ceteris paribus*.

How to increase $k_{ss}$, and therefore $y_{ss}$?

1. Increase $s$: $s \uparrow \Rightarrow \; k_{ss} \uparrow \Rightarrow \; y_{ss} \uparrow$
2. Decrease $\delta$: $\delta \downarrow \Rightarrow \; k_{ss} \uparrow \Rightarrow \; y_{ss} \uparrow$

*"Golden rule" capital-labor ratio*: The level of capital per worker $k_{gr}$ that maximizes $c_{ss} = f(k) - \delta k$.

First-order condition with respect to $k$:

$$\frac{\partial y}{\partial k} = \delta \;\Rightarrow\; f'(k_{gr}) = \alpha k_{gr}^{\alpha-1} = \delta$$

$$k_{gr}^{1-\alpha} = \frac{\alpha}{\delta} \;\Rightarrow\; k_{gr} = (\frac{\alpha}{\delta})^{\frac{1}{1-\alpha}}$$

## 2 Exercise: Solow Model

Consider the Solow growth model without population growth or technological change. The parameters of the model are given by $s = 0.2$ (savings rate) and $\delta = 0.05$ (depreciation rate). Let $k$ denote capital per worker; $y$ output per worker; $c$ consumption per worker; $i$ investment per worker.

a) Rewrite production function $Y = K^{\frac{1}{3}}L^{\frac{2}{3}}$ in per-worker terms.

b) Find the steady-state level of the capital stock, $k_{ss}$.

c) What is the "golden rule" level of $k$ for this economy? Recall that the golden rule level of the capital stock $k_{gr}$ maximizes consumption per worker in steady-state. Report your answer to two decimal places.

d) Let's say that a benevolent social planner wishes to obtain $k = k_{gr}$ in steady-state. What is the associated savings rate $s_{gr}$ that must be imposed by the social planner to support $k_{gr}$?

e) Compare your result in the previous part with the assumed savings rate $s$. To obtain $k_{gr}$, do citizens need to save more or less?

f) Plot the following on a single graph: $y = f(k)$, $\delta k$, $sf(k)$, and $s_{gr}f(k)$. Does the savings curve pivot up or down, relative to its initial position, when the planner's $s_{gr}$ is implemented?

g) Discuss two to three economic policies that could help the social planner implement $s_{gr}$ in a real-world situation.