Econ 711 – Midterm Exam, 4 November 2021

Question 1. Consuming Something. **(40 points)**

Yumei has preferences over $\mathbb{R}^2_+$ represented by the utilty function

$$u(x) \quad = \quad \max\left\{x_1^{\frac{1}{3}}x_2^{\frac{2}{3}},\;\; x_1^{\frac{2}{3}}x_2^{\frac{1}{3}}\right\}$$

Assume that prices and wealth are strictly positive.

(a) Are Yumei's preferences convex? Explain.

Since you can't use a calculator, feel free to use the fact that $2^{\frac{1}{3}}4^{\frac{2}{3}} \approx 3.17$ if that helps.

No, they're not convex.

Convex preferences require that if $x \succsim y$ and $x' \succsim y$, then $x^t = tx + (1-t)x' \succsim y$ for any $t \in [0,1]$. So consider $x = (2,4)$, $x' = (4,2)$, and $x^t = \frac{1}{2}(2,4) + \frac{1}{2}(4,2) = (3,3)$. Per the hint, $u(x) = u(x') \approx 3.17$, but $u(x^t) = 3$. So if we let $y = x$, we have $x \succsim y$, $x' \succsim y$, and $\frac{1}{2}x + \frac{1}{2}x' \prec y$, so preferences cannot be convex.

(b) Show that at any solution x^* to her consumer problem, if $p_1 \neq p_2$, Yumei demands strictly more of whichever good is cheaper.

Suppose $p_1 > p_2$. (The argument for $p_1 < p_2$ is identical.) First suppose that for some x^* that solves the consumer problem, $x_1^* > x_2^*$. Then Yumei could get the same utility for strictly less money by consuming (x_2^*, x_1^*) instead of (x_1^*, x_2^*). But then (x_2^*, x_1^*) would be another solution to her consumer problem, and cost strictly less than her budget; this would violate Walras' Law. So she must have $x_1^* \leq x_2^*$.

So now suppose $p_1 > p_2$ and $x_1^* = x_2^*$. Then Yumei can get strictly higher utility for less money, by consuming more of good 2 and less of good 1. For example, if she's currently consuming $x^* = (z,z)$, she's getting utility $u(x^*) = z$; if she instead consumed $x' = \left(\frac{2}{3}z, \frac{4}{3}z\right)$, this would cost strictly less (since $p_1 > p_2$) and would give her utility

$$u(x') \quad = \quad \left(\frac{2}{3}z\right)^{\frac{1}{3}}\left(\frac{4}{3}z\right)^{\frac{2}{3}} \quad = \quad \frac{2^{\frac{1}{3}}4^{\frac{2}{3}}}{3}z \quad \approx \quad \frac{3.17}{3}z \quad > \quad u(x^*)$$

meaning x^* could not be optimal.

(A slightly different approach would be to show that if $x_1^* = x_2^* = z$, Yumei could get more utility at lower cost by consuming a little more of good 2 and a little less of good 1, by calculating $\frac{d}{d\epsilon}u(z-\epsilon, z+\epsilon)$ and noting that this derivative is strictly positive at $\epsilon = 0$.)

(c) Solve the consumer problem and state $x^*(p,w)$.

If $p_1 > p_2$, we know that at the solution, $x_1 < x_2$ and therefore $x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} > x_1^{\frac{2}{3}}x_2^{\frac{1}{3}}$, so when $p_1 > p_2$, we can think of Yumei simply maximizing $x_1^{\frac{1}{3}}x_2^{\frac{2}{3}}$. This is Cobb-Douglas utility, so you may just remember the solution, but if not, we can note that the non-negativity constraints won't bind, take the log of the utility function, and set up the Lagrangian

$$\mathcal{L} \quad = \quad \frac{1}{3}\log x_1 + \frac{2}{3}\log x_2 + \lambda\left(w - p \cdot x\right)$$

The first-order conditions are $\frac{1}{3x_1} = \lambda p_1$ and $\frac{2}{3x_2} = \lambda p_2$, giving $p_1 x_1 = \frac{1}{3\lambda}$ and $p_2 x_2 = \frac{2}{3\lambda}$; Walras' Law then gives $p_1 x_1 + p_2 x_2 = \frac{1}{\lambda} = w$, or $\lambda = \frac{1}{w}$, letting us recover $x^* = \left(\frac{1}{3}\frac{w}{p_1}, \frac{2}{3}\frac{w}{p_2}\right)$.

If $p_1 < p_2$, symmetrically, $x^* = \left(\frac{2}{3}\frac{w}{p_1}, \frac{1}{3}\frac{w}{p_2}\right)$. If $p_1 = p_2$, then either of the two solutions give the same utility (but their convex combinations do not). So Marshallian demand is

$$x^*(p, w) \quad = \quad \begin{cases} \left(\frac{1}{3}\frac{w}{p_1}, \frac{2}{3}\frac{w}{p_2}\right) & if \quad p_1 > p_2 \\ \left(\frac{2}{3}\frac{w}{p_1}, \frac{1}{3}\frac{w}{p_2}\right) & if \quad p_1 < p_2 \\ \left\{\left(\frac{1}{3}\frac{w}{p_1}, \frac{2}{3}\frac{w}{p_2}\right), \; \left(\frac{2}{3}\frac{w}{p_1}, \frac{1}{3}\frac{w}{p_2}\right)\right\} & if \quad p_1 < p_2 \end{cases}$$

(d) Holding p_1 fixed, describe how Yumei's demand for good 1 changes as p_2 increases. Is good 1 a gross complement or a gross substitute for good 2?

While $p_2 < p_1$, x_1 remains constant as p_2 changes; and while $p_2 > p_1$, x_1 remains constant as p_2 changes. When p_2 goes from being less than p_1 to being more than p_1, however, x_1 jumps up from $\frac{1}{3}\frac{w}{p_1}$ to $\frac{2}{3}\frac{w}{p_1}$. Thus, x_1 is increasing in p_2, so good 1 is a gross substitute for good 2.

Now instead of just one consumer, suppose there were 20 consumers, each with preferences represented by this same utility function. From period to period, prices vary, as does each consumer's wealth level.

(e) If only prices and aggregate (total) demand were observed, would this data satisfy GARP? Why or why not?

Would the data be consistent with the choices of a single consumer with convex preferences? Explain.

Since preferences are homothetic, demand aggregates – as you showed on the homework, with multiple consumers having identical homothetic preferences, total demand looks like the demand of a single rational consumer, and must therefore satisfy GARP.

Afriat's Theorem tells us that any data that's rationalizable at all, is rationalizable by a utility function which is continuous, monotone and concave. Concave utility implies convex preferences. Thus, the data would be consistent with the choices of a single consumer with convex preferences, even though the actual consumers who generated the data did not have convex preferences!

Question 2. Producing Something. **(40 points)**

Hakeem runs a small farm that grows one kind of crop and can use either of two production technologies, one that is more dependent on labor and one that is more dependent on chemical fertilizer. Each technology uses its own technology-specific capital (specialized machinery). Hakeem's farm's production function is

$$f(k_1, k_2, f, \ell) \quad = \quad \max\left\{k_1^{\frac{1}{5}} f^{\frac{1}{5}} \ell^{\frac{2}{5}}, \;\; k_2^{\frac{1}{5}} f^{\frac{2}{5}} \ell^{\frac{1}{5}}\right\}$$

where k_i is technology-i-specific capital, f is fertilizer, and ℓ is labor. Suppose that k_1 and k_2 are fixed in the short term but can be changed in the long term, while f and ℓ can be freely adjusted in the short term. Let p be the output price, c the price of fertilizer, and w the price of labor. The two types of capital k_1 and k_2 have the same price r, but capital useful for one technology cannot be used for the other technology or swapped for the other type of capital in the short term.

(a) Show that the production function is not concave.

Is the production set $Y = \{(q, -z) : f(z) \geq q\}$ convex?

A concave production function would require $f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$ for any $t \in [0, 1]$. Consider $z = (1, 0, 1, 1)$, $z' = (0, 1, 1, 1)$, and $z^t = \frac{1}{2}z + \frac{1}{2}z' = (\frac{1}{2}, \frac{1}{2}, 1, 1)$. Plugging in, $f(z) = f(z') = 1$, but $f(z^t) = \left(\frac{1}{2}\right)^{\frac{1}{5}} < 1$, so the production function is not concave.

Similarly, the production set is not convex, because it contains the points $y = (1, -1, 0, -1, -1)$ and $y = (1, 0, -1, -1, -1)$, but not their convex combination $(1, -\frac{1}{2}, -\frac{1}{2}, -1, -1)$ since we just showed $f(\frac{1}{2}, \frac{1}{2}, 1, 1) < 1$.

(b) Suppose that right now, $\max\{k_1, k_2\} > 0$ and $\min\{k_1, k_2\} = 0$ (Hakeem has one type of machinery and not the other). Is Hakeem's short-term profit maximization problem supermodular? What effect would a decrease in the price of fertilizer c have on his demand for labor in the short term?

If $k_1 > 0 = k_2$, then Hakeem's short-term problem is

$$\max_{f,\ell}\left\{k_1^{\frac{1}{5}} f^{\frac{1}{5}} \ell^{\frac{2}{5}} - rk_1 - cf - w\ell\right\}$$

which is supermodular in (f, ℓ) (the only two choice variables in the short term) and has increasing differences in those choice variables and $-c$. Thus, a decrease in c would increase Hakeem's demand for both fertilizer and labor in the short term. (If $k_2 > 0 = k_1$, the argument is the same, just with the exponents on f and ℓ flipped.)

(c) Suppose that right now, $k_1 = k_2 = 1$ (Hakeem has some of each type of machinery). Is his short-term problem supermodular? Explain. If yes, what effect would an increase in c have on his use of labor in the short run? If no, explain how an increase in c could lead to either an increase or a decrease in labor used in the short term.

Now Hakeem's short-term problem is

$$\max_{f,\ell}\left\{\max\left\{f^{\frac{1}{5}}\ell^{\frac{2}{5}}, \; f^{\frac{2}{5}}\ell^{\frac{1}{5}}\right\} - rk_1 - cf - w\ell\right\}$$

This problem is *not* supermodular. One way to see this is to let $g(f, \ell)$ denote the maximand and check whether $\frac{\partial g}{\partial \ell}$ is increasing in f. Differentiating,

$$\frac{\partial g}{\partial \ell} = \begin{cases} \frac{2}{5} f^{\frac{1}{5}} \ell^{-\frac{3}{5}} - w & if \quad f < \ell \\ \\ \frac{1}{5} f^{\frac{2}{5}} \ell^{-\frac{4}{5}} - w & if \quad f > \ell \end{cases}$$

This is increasing in f within each range, but jumps downward when f crosses ℓ from below, so it's not everywhere increasing in f.

An increase in c that did not change which technology Hakeem used would reduce his use of both fertilizer and labor in the short run. However, an increase in c could cause Hakeem to switch from using technology 2 to technology 1, which could lead to an increase in his demand for labor. So ℓ could either increase or decrease in response to a change in c.

Suppose that Hakeem is maximizing his profits given current prices, and is doing this by using technology 2, so $k_2 > 0 = k_1$. For parts (d) and (e), you don't need to give formal proofs, just a clear explanation of the intuition for the results.

(d) If c decreases, he'll continue to use technology 2 in the long run. (This is true, and you may use it without proving it.) Will the decrease in c lead to an increase or decrease in Hakeem's use of labor in the short run? In the long run? Will the long-run effect be larger or smaller than the short-run effect?

Since Hakeem will keep using technology 2, it's as if his production function is just $k_2^{\frac{1}{5}} f^{\frac{2}{5}} \ell^{\frac{1}{5}}$ in both the short and long term, which is supermodular. When c decreases, in the short run, f and ℓ will both increase. In the long run, k_2 will also increase, which will further increase f and ℓ. So labor will go up more in the long-run than in the short-run.

(e) Will an increase in c lead to an increase or decrease in his use of labor in the short run? Can you tell the effect it will have in the long run? Explain.

In the short run, Hakeem is "locked into" technology 2, so his production function is supermodular; an increase in c leads to decreases in both f and ℓ. In the long run, if Hakeem sticks with technology 2, f and ℓ will further decrease when he adjusts k_2 downwards. However, the increase in c could lead him to switch to technology 1, in which case ℓ would likely increase in the long run (after decreasing in the short run).

Question 3. Gambling. **(20 points)**

Lucky is an expected-utility-maximizing decisionmaker with Bernoulli utility function

$$u(x) \quad = \quad \frac{x}{1+x}$$

(a) Is Lucky risk-averse, risk-neutral or risk-loving?

Does Lucky have increasing or decreasing absolute risk aversion?

Does Lucky have increasing or decreasing relative risk aversion?

Differentiating,

$$u'(x) \quad = \quad \frac{1}{1+x} - \frac{x}{(1+x)^2} \quad = \quad \frac{1+x-x}{(1+x)^2} \quad = \quad \frac{1}{(1+x)^2}$$

and

$$u''(x) \quad = \quad -\frac{2}{(1+x)^3} \quad < \quad 0$$

so Lucky is risk-averse.

His coefficient of absolute risk aversion,

$$A(x) \quad = \quad \frac{-u''(x)}{u'(x)} \quad = \quad \frac{\frac{2}{(1+x)^3}}{\frac{1}{(1+x)^2}} \quad = \quad \frac{2}{1+x}$$

is decreasing, so he has decreasing absolute risk aversion.

His coefficient of relative risk aversion,

$$R(x) \quad = \quad \frac{-u''(x)}{u'(x)}x \quad = \quad \frac{2x}{1+x}$$

is increasing, so he has increasing relative risk aversion. (To see $R(x)$ is increasing, you can calculate $R'(x) = \frac{1}{(1+x)^2}(2(1+x) - 2x) > 0$ or rewrite it as $R(x) = \frac{2}{\frac{1}{x}+1}$ and note the denominator is decreasing in x.)

Please answer parts (b), (c) and (d) **without plugging numbers into utility functions**. Lucky's preferences happen to make him indifferent between the following two lotteries:

lottery A		lottery B	
11	*with probability* 100%	9	*with probability* 50%
		14	*with probability* 50%

(b) Does Lucky prefer $A + 9$ or $B + 9$, i.e., a lottery giving 20 for sure or a lottery giving either 18 or 23 with equal probabilities? Explain.

Does Lucky prefer $10A$ or $10B$, i.e., a lottery giving 110 for sure or a lottery giving either 90 or 140 with equal probabilities? Explain.

Since Lucky has decreasing absolute risk aversion, increasing all payoffs by the same constant makes him more likely to prefer a risky gamble to a sure thing. Since $11 \sim \frac{1}{2}9 \oplus \frac{1}{2}14$, this means $20 \prec \frac{1}{2}18 \oplus \frac{1}{2}23$, or he strictly prefers $B+9$ to $A+9$.

Since Lucky has increasing relative risk aversion, increasing all payoffs proportionally makes him more likely to prefer a sure thing to a risky gamble. Since $11 \sim \frac{1}{2}9 \oplus \frac{1}{2}14$, this means $110 \succ \frac{1}{2}90 \oplus \frac{1}{2}140$, or he strictly prefers $10A$ to $10B$.

(c) Lucky's friend Bucky is also an expected-utility maximizer, and has Bernoulli utility function $v(x) = 1 - e^{-2.5x}$. What is Bucky's coefficient of absolute risk aversion? Does Bucky prefer lottery A or B? Explain.

Bucky's coefficient of absolute risk aversion is $-\frac{-(2.5)^2 e^{-2.5x}}{2.5e^{-2.5x}} = 2.5$. Lucky's is $\frac{2}{1+x} \leq 2 < 2.5$, so Bucky is more risk-averse than Lucky. Since Lucky is indifferent between A and B, Bucky strictly prefers A (the sure thing) to B (the risky gamble).

Consider the following three other lotteries C, D, and E:

lottery C			lottery D			lottery E		
8	$w.p.$	25%	9	$w.p.$	25%	9	$w.p.$	25%
10	$w.p.$	25%	11	$w.p.$	50%	12.5	$w.p.$	50%
14	$w.p.$	50%	14	$w.p.$	25%	14	$w.p.$	25%

(d) Does Lucky prefer lottery A or C? Explain.

Does Lucky prefer lottery A or D? Explain.

Does Lucky prefer lottery A or E? Explain.

Bucky prefers A to C. C is a mean-preserving spread around B – we can get C by starting with B, and then replacing the outcome 9 with $\frac{1}{2}8 \oplus \frac{1}{2}10$. So every risk-averse expected utility maximizer prefers B to C. Since Lucky is indifferent between A and B, he prefers A to C.

Bucky is indifferent between A and D. D can be created as a compound lottery giving B half the time and A half the time. By independence, if $A \sim B$, then $\frac{1}{2}A \oplus \frac{1}{2}A \sim \frac{1}{2}A \oplus \frac{1}{2}B$, or $A \sim D$.

Bucky prefers E to A. We can see this two ways. First, E first-order stochastically dominates D, since it just replaces the outcome 11 with the outcome 12.5; since $A \sim D$ and $D \prec E$, $A \prec E$. Or second, we can see E as a compound lottery giving B half the time and the sure thing 12.5 half the time. Since $A \sim B$ and $A \prec \delta_{E_A} = \delta_{12.5}$, by transitivity, $B \prec \delta_{12.5}$, and so by independence, $B \prec \frac{1}{2}B \oplus \frac{1}{2}\delta_{12.5} = E$, so $A \sim B \prec E$ and $A \prec E$.