Econ 711 – Midterm Exam, 27 October 2020

Question 1. A Consumer Problem.

Consider the utility function

$$u(x) \quad = \quad \min\{x_1, x_2\}^{\alpha}(x_3 + x_4)^{1-\alpha}$$

with $\alpha \in (0,1)$. *Assume prices are all strictly positive and distinct* ($p_i \neq p_j$ *for each* i, j).

(a) *Solve the consumer's problem, and calculate Marshallian demand* $x(p,w)$ *and the indirect utility function* $v(p,w)$. *Is good 1 normal or inferior?*

(HINT: look for ways to simplify the problem before you solve it!)

As the hint suggests, you save a lot of trouble if you simplify first. At strictly positive prices, it is always optimal to consume equal amounts of the first two goods; and if $p_3 \neq p_4$, it is always optimal to consume only the cheaper of the last two goods. Instead of thinking about the four goods individually, then, we can think of "aggregate goods" X_1, which consists of equal amounts of x_1 and x_2 and is available at price $P_1 = p_1 + p_2$; and X_2, which consists of the cheaper of x_3 and x_4, and is available at price $P_2 = \min\{p_3, p_4\}$. Taking the log of u, and then noting that the marginal utility of X_1 or X_2 is infinite at 0 and the non-negativity constraints therefore won't hold with equality (and therefore $\mu_1 = \mu_2 = 0$ and we can ignore the non-negativity constraints), the consumer problem can then be solved via the Lagrangian

$$\mathcal{L} \quad = \quad \alpha \log X_1 + (1-\alpha)\log X_2 + \lambda(w - P_1 X_1 - P_2 X_2)$$

This gives first-order conditions

$$\frac{\alpha}{X_1} \quad = \quad \lambda P_1 \qquad \text{and} \qquad \frac{1-\alpha}{X_2} \quad = \quad \lambda P_2$$

Since u is locally non-satiated and therefore Walras' Law holds, we can then use

$$P_1 X_1 + P_2 X_2 \quad = \quad \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} \quad = \quad w$$

to get $\lambda = \frac{1}{w}$, and therefore $X_1 = \frac{\alpha w}{P_1}$ and $X_2 = \frac{(1-\alpha)w}{P_2}$. (If you remembered that this was the solution to utility maximization with Cobb-Douglas utility, so much the better, you saved time.) Going back to thinking about the four goods, then, we find

$$x_1(p,w) \quad = \quad x_2(p,w) \quad = \quad \tfrac{\alpha w}{p_1 + p_2}$$

and

$$(x_3(p,w),\, x_4(p,w)) \quad = \quad \begin{cases} \left(\frac{(1-\alpha)w}{p_3}, 0\right) & if \quad p_3 < p_4 \\[2ex] \left(0, \frac{(1-\alpha)w}{p_4}\right) & if \quad p_3 > p_4 \end{cases}$$

Plugging these into the original utility function, we get

$$v(p,w) \quad = \quad u(x(p,w)) \quad = \quad \left(\frac{\alpha w}{p_1+p_2}\right)^{\alpha}\left(\frac{(1-\alpha)w}{\min\{p_3,p_4\}}\right)^{1-\alpha} \quad = \quad \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}w}{(p_1+p_2)^{\alpha}\min\{p_3,p_4\}^{1-\alpha}}$$

Finally, since $x_1(p,w)$ is increasing in w, good 1 is a normal good.

(b) *Use $v(p,w)$ to find the expenditure function $e(p,u)$, and use $e(p,u)$ to find $h_1(p,u)$, the Hicksian demand for the first good. Which goods are complements for good 1, and which goods are substitutes for it?*

Using $v(p,e(p,u)) = u$,

$$v(p,e(p,u)) \quad = \quad \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}e(p,u)}{(p_1+p_2)^{\alpha}\min\{p_3,p_4\}^{1-\alpha}} \quad = \quad u$$

and therefore

$$e(p,u) \quad = \quad \frac{(p_1+p_2)^{\alpha}\min\{p_3,p_4\}^{1-\alpha}u}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

Using Shephard's Lemma,

$$h_1(p,u) \quad = \quad \frac{\partial e}{\partial p_1}(p,u) \quad = \quad \frac{\alpha(p_1+p_2)^{\alpha-1}\min\{p_3,p_4\}^{1-\alpha}u}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

Since $\alpha \in (0,1)$, this is decreasing in p_2, and increasing in $\min\{p_3,p_4\}$, therefore weakly increasing in both p_3 and p_4. Thus, good 2 is a complement to good 1, while goods 3 and 4 are substitutes for good 1.

Now suppose that instead of a wealth endowment, the consumer has a positive endowment of goods 3 and 4, $e = (0,0,e_3,e_4)$, which she can consume or sell in any quantity. We say the consumer is a net buyer *of a good if she consumes more than her endowment of it, and a* net seller *if she consumes less than her endowment.*

(HINT: if you think of the consumer as first selling her whole endowment and then deciding how to spend the proceeds, you shouldn't need to re-solve the consumer problem to answer parts (c) and (d).)

(c) *If $p_3 > p_4$, is the consumer a net buyer or net seller of good 3? Is the demand for good 1 increasing or decreasing in p_3? Is this due to a substitution effect, a wealth effect, or both? Explain.*

As the hint suggests, there's no need to re-solve the consumer's problem, we can simply plug $w = p_3e_3 + p_4e_4$ into the Marshallian demand we calculated earlier. When $p_3 > p_4$, the consumer consumes zero of good 3, so she is a net seller (she sells her whole endowment of good 3). The demand for good 1 is $\frac{\alpha(p_3e_3+p_4e_4)}{p_1+p_2}$, which is increasing in p_3. This is due to a wealth effect – the consumer effectively has a larger budget $w = p_3e_3 + p_4e_4$ as p_3 increases, and spends the same fraction of this larger budget on good 1.

(Since you found earlier that goods 1 and 3 are substitutes, it's tempting to think of this as a mix of substitution and wealth effects. However, when $p_3 > p_4$, the consumer wasn't planning to consume any of good 3 anyway, so the substitution effect is zero and this is purely a wealth effect. To see this another way, since we found $h_1(p, u)$ was a constant times $(p_1 + p_2)^{\alpha-1} \min\{p_3, p_4\}u$, h_1 does not change with p_3 when $p_3 > p_4$, since $\min\{p_3, p_4\}$ stays the same; so the substitution effect is zero and the increase in x_1 is due purely to a wealth effect.)

(d) *If $p_3 < p_4$, when is the consumer a net buyer of good 3, and when are they a net seller? In each case, is the demand for good 1 increasing or decreasing in p_3? Is this due to a substitution effect, a wealth effect, or both? Explain.*

When $p_3 < p_4$, the consumer does demand some of good 3, specifically $x_3 = \frac{(1-\alpha)(p_3 e_3 + p_4 e_4)}{p_3}$ of it. The consumer is a net buyer when $\frac{(1-\alpha)(p_3 e_3 + p_4 e_4)}{p_3} > e_3$, and a net seller when $\frac{(1-\alpha)(p_3 e_3 + p_4 e_4)}{p_3} < e_3$.

The demand for good 1, $x_1 = \frac{\alpha(p_3 e_3 + p_4 e_4)}{p_1 + p_2}$, is always increasing in p_3. When the consumer is a net seller of good 3, this is a combination of substitution and wealth effects – good 3 is relatively more expensive, so the consumer aims to get more of their utility from goods 1 and 2 (note that h_1 is strictly increasing in p_3 when $p_3 < p_4$); and the consumer is effectively richer, having more money to spend on goods other than good 3. When the consumer is a net buyer of good 3, the wealth effect is negative (an increase in p_3 makes the consumer effectively poorer), but this is overwhelmed by the substitution effect since the combined effect is positive.

Question 2. Rationalizing Firm Behavior.

Consider the following "dataset" of market prices and a firm's observed production:

prices p	production y
$p^1 = (1, 1, 1)$	$y^1 = (10, -3, -4)$
$p^2 = (2, 1, 1)$	$y^2 = (15, -6, -8)$
$p^3 = (1, 1, 2)$	$y^3 = (8, -5, -1)$

(a) *Is the data consistent with the behavior of a profit-maximizing firm? Why or why not?*

If yes, give a production set Y that rationalizes the data.

If no, give a different value for the observation y^3 which would make the data rationalizable, and give a production set Y that rationalizes the revised data.

Yes, the data is consistent with a profit-maximizing firm, because it satisfies the Weak Axiom of Profit Maximization (WAPM). To see this, we can make a table of $p^i \cdot y^j$, and observe that $p^i \cdot y^i \geq p^i \cdot y^j$ for each (i, j) (the largest element of each row is the diagonal):

prices	$y^1 = (10, -3, -4)$	$y^2 = (15, -6, -8)$	$y^3 = (8, -5, -1)$
$p^1 = (1, 1, 1)$	**3**	1	2
$p^2 = (2, 1, 1)$	13	**16**	10
$p^3 = (1, 1, 2)$	-1	-7	**1**

Candidates for production sets Y rationalizing the data are the inner bound and outer bound,

$$Y^I = \{(10,-3,-4),(15,-6,-8),(8,-5,-1)\} \qquad \text{and}$$

$$Y^O = \{y \;:\; (1,1,1)\cdot y \le 3,\ (2,1,1)\cdot y \le 16,\ \text{and} \quad (1,1,2)\cdot y \le 1\}$$

or any other set Y such that $Y \supseteq Y^I$ and $Y \subseteq Y^O$.

(b) *Is the original data consistent with a profit-maximizing firm whose production set Y is* convex*? Why or why not?*

If yes, give a convex production set Y that rationalizes the data.

If no, give a different value for the observation y^3 that would make the data consistent with convex Y, and give a convex production set that would rationalize the revised data.

Yes, the data's consistent with a firm whose production set Y is convex. We saw that *any* data that's rationalizable, is rationalizable with convex Y. The easiest way to see this is that any data that's rationalizable can be rationalized by the outer bound Y^O, and the outer bound is always convex. (This is because it's the intersection of a bunch of half-spaces $\{y : p^i \cdot y \le p^i \cdot y^i\}$, each of which is convex.)

As noted, the outer bound Y^O, as defined above, is a convex set that rationalizes the data. Any other convex set inside Y^O and containing Y^I would work as well.

(c) *Interpret each observed production plan as $y = (q, -z_1, -z_2)$, with $q = f(z_1, z_2)$, and suppose that each observed y^i was the firm's unique optimal plan at those prices. Is the original data consistent with a profit-maximizing single-output firm whose production function f is* supermodular*?*

If yes, explain why.

If no, explain why not, and give a different value for the observation y^3 that would make the data consistent with supermodular f.

No, the data is not consistent with a supermodular production function. This is because Topkis' Theorem is violated. The firm's problem is

$$\max_{z_1,z_2} \{p_1 f(z_1, z_2) - p_2 z_2 - p_3 z_3\}$$

If f were supermodular, the firm's objective function would be supermodular in (z_1, z_2), with increasing differences in (z_1, z_2) and $-p_3$, and so Topkis would imply that an increase in p_3 must lead to decreases in both input levels. However, from observation 1 to observation 3, p_3 went up and z_3 went down, but z_2 went up (from 3 to 5), violating Topkis.

To make the data consistent with supermodular f, we could replace y^3 with a production plan where $(z_1, z_2) \le (3, 4)$, but which still satisfies WAPM. One candidate, for example, would be $y^3 = (5, -2, -1)$. This leaves $p^3 \cdot y^3 = 1$, making $p^3 \cdot y^3 \ge p^3 \cdot y^j$ for $j \in \{1, 2\}$; but it's still less profitable than the chosen production plans at the other prices. ($p^1 \cdot y^3 = 2 < p^1 \cdot y^1$, and $p^2 \cdot y^3 = 7 < p^2 \cdot y^2$.) There are other candidates as well, of course.

Question 3. Lotteries and the Very Risk-Averse.

Let X be a finite subset of $\mathbb{R}_+$. Given a lottery L over X, let L_* denote the worst-case outcome it allows, i.e., the lowest outcome that L puts positive probability on. Consider the minmax preferences *where lotteries are evaluated purely based on their worst-case outcomes: $L \succsim_{mnx} L'$ if and only if $L_* \geq L'_*$.*

(a) *Is $\succsim_{mnx}$ complete? Is it transitive? Does it satisfy continuity? Does it satisfy independence? (For each, explain why or why not, don't just say "yes" or "no.")*

Can $\succsim_{mnx}$ be represented by an expected utility function $U(L) = \sum_{i: x_i \in X} p_i u(x_i)$? Explain.

Yes, $\succsim_{mnx}$ is complete. For any two lotteries L and L', since L_* and L'_* are two positive real numbers, either $L_* \geq L'_*$ or $L'_* \geq L_*$ (or both), so either $L \succsim_{mnx} L'$ or $L' \succsim_{mnx} L$ (or both).

Yes, $\succsim_{mnx}$ is transitive. If $L \succsim_{mnx} L'$ and $L' \succsim_{mnx} L''$, then $L_* \geq L'_* \geq L''_*$, so $L_* \geq L''_*$ and therefore $L \succsim_{mnx} L''$.

No, $\succsim_{mnx}$ does not satisfy continuity. If we let δ_x denote the degenerate lottery giving prize x for sure, then for any $p > 0$,

$$\delta_{100} \quad \succ_{mnx} \quad p\delta_{50} \oplus (1-p)\delta_{150}$$

but this is reversed at $p = 0$. Thus, the set of p for which this holds is not closed, since it includes all positive p but not zero; or the preference for $p(50) \oplus (1-p)(150)$ jumps discontinuously at $p = 0$.

Yes, $\succsim_{mnx}$ satisfies independence. If $L \succsim_{mnx} L'$, then $L_* \geq L'_*$. Consider the lotteries $pL \oplus (1-p)L''$ and $pL' \oplus (1-p)L''$. The worst outcome from the first is $\min\{L_*, L''_*\}$, and the worst outcome from $pL' \oplus (1-p)L''$ is $\min\{L'_*, L''_*\}$. Since $L_* \geq L'_*$ implies $\min\{L_*, L''_*\} \geq \min\{L'_*, L''_*\}$, $L \succsim_{mnx} L'$ implies $pL \oplus (1-p)L'' \succsim_{mnx} pL' \oplus (1-p)L''$, which is independence.

Even though $\succsim_{mnx}$ satisfies the other three conditions, since it fails continuity, it cannot have an expected utility representation. (We saw that complete and transitive preferences over lotteries have an expected utility representation if and only if they satisfy continuity and independence.)

(b) *Consider the CARA utility function $u(x) = 1 - e^{-cx}$. Show that for any two lotteries over X, if $L \succ_{mnx} L'$, then $U(L) > U(L')$ under CARA utility for c sufficiently high. (Thus, in a rough sense, CARA preferences approach minmax preferences as $c \to \infty$.)*

Note that $u(x) = -e^{-cx}$ represents the same preferences as $1 - e^{-cx}$, so we can drop the constant 1 in u for simplicity. Think of L as $p_1 x_1 \oplus p_2 x_2 \oplus \ldots \oplus p_k x_k$, where $x_1 < x_2 < \ldots < x_k$ (the prizes are in increasing order) and $p_1, p_2, \ldots, p_k > 0$ (only prizes with a positive probability are listed). Similarly, think of $L' = p'_1 x'_1 \oplus \ldots \oplus p'_m x'_m$ with the same conditions. Now,

$$\begin{array}{rcl} U(L) & > & U(L') \\ & \updownarrow & \\ -p_1 e^{-cx_1} - p_2 e^{-cx_2} - \cdots - p_k e^{-cx_k} & > & -p'_1 e^{-cx'_1} - p'_2 e^{-cx'_2} - \cdots - p'_k e^{-cx'_k} \end{array}$$

If $L \succ_{mnx} L'$, then $x_1 > x'_1$, and so x'_1 is the smallest prize out of either lottery; multiplying every term by $e^{cx'_1}$ then gives

$$\begin{array}{rcl} U(L) & > & U(L') \\ & \updownarrow & \\ -p_1 e^{-c(x_1 - x'_1)} - p_2 e^{-c(x_2 - x'_1)} - \cdots - p_k e^{-c(x_k - x'_1)} & > & -p'_1 - p'_2 e^{-c(x'_2 - x'_1)} - \cdots - p'_k e^{-c(x'_k - x'_1)} \end{array}$$

where all the $(x_j - x'_1)$ and $(x'_j - x'_j)$ terms are strictly positive. This means as c gets large, every term on both sides of the last inequality vanishes except for $-p'_1$ on the right, meaning $U(L) > U(L')$ for c sufficiently large.

(c) *Consider the two lotteries*

$$L \quad = \quad \frac{2}{5}(\$10) \oplus \frac{3}{5}(\$100) \qquad \text{and} \qquad L' \quad = \quad \frac{1}{5}(\$10) \oplus \frac{4}{5}(\$20)$$

Minmax preferences rank these equally ($L \sim_{mnx} L'$), but CARA utility does not. Which lottery is preferred based on CARA utility for c sufficiently large?

Describe, as completely as you can, the preferences over lotteries that correspond to the limit of CARA utility as $c \to \infty$.

Well,

$$\begin{aligned} U(L) - U(L') &= \frac{2}{5}\left(-e^{-10c}\right) + \frac{3}{5}\left(-e^{-100c}\right) - \frac{1}{5}\left(-e^{-10c}\right) - \frac{4}{5}\left(-e^{-20c}\right) \\ &= \frac{1}{5}\left(-e^{-10c}\right) + \frac{3}{5}\left(-e^{-100c}\right) - \frac{4}{5}\left(-e^{-20c}\right) \\ &= e^{-10c}\left[-\frac{1}{5} + \frac{3}{5}\left(-e^{-90c}\right) - \frac{4}{5}\left(-e^{-10c}\right)\right] \end{aligned}$$

As c gets large, the last two terms inside the square brackets vanish, so $U(L) - U(L') < 0$ for c sufficiently large.

More generally, if we let $\succsim$ refer to the preferences represented by CARA utility in the limit as $c \to \infty$, we already know that $L \succ L'$ if its worst outcome is strictly better (since $L \succ_{mnx} L'$); this last example shows that among lotteries with the same worst outcome, $L \succ L'$ if they have the same worst outcome but L gives it with lower probability.

If two lotteries have the same worst outcome with the same probability, then their expected utilities each have $p_1(-e^{-cx_1})$ as the "leading term" (the term that vanishes slowest as c grows), but these will cancel on both sides, leaving $p_2(-e^{-cx_2})$ and $p'_2(-e^{-cx'_2})$ as the new "leading terms". Thus, the lottery with the better second-worst outcome is preferred; and if both lotteries have the second-worst outcome, the one that gives it with the lower probability is preferred.

By the same pattern, among lotteries with the same lowest two prizes and the same probabilities for each of them, the one with the better third-worst prize is preferred; if they have the same third-worst prize, the one that gives it with the lower probability is preferred; and so on.