Question 1. Sun-Dried Tomatoes

I run a small business making fancy jars of sun-dried tomatoes. My production function is

$$f(z) \quad = \quad S^{0.3} z_1^{0.3} z_2^{0.4}$$

where z_1 and z_2 are tomato seeds and fertilizer – ordinary inputs whose levels I control – and S is the amount of sunlight my factory gets, which depends on the weather (and is out of my control). Let p denote the price of sun-dried tomatoes, and w_1 and w_2 the prices of seeds and fertilizer; assume I'm a price taker in all of these markets.

(a) *State my profit maximization problem.*

What happens to my output level, and my use of the two inputs I control, if the price of sun-dried tomatoes goes up? Explain.

What happens to my output level, and my use of the two inputs I control, if weather patterns change and my factory gets less direct sunlight? Explain.

My profit maximization problem is

$$\max_{z_1, z_2 \geq 0} \left\{ p S^{0.3} z_1^{0.3} z_2^{0.4} - w_1 z_1 - w_2 z_2 \right\}$$

Letting g denote the objective function (and "$x \uparrow y$" to mean "x is increasing in y"),

$$\frac{\partial g}{\partial z_1} \quad = \quad 0.3 p S^{0.3} z_1^{-0.7} z_2^{0.4} - w_1 \quad \uparrow \quad z_2, p, S, -w_1, (-w_2)$$

$$\frac{\partial g}{\partial z_2} \quad = \quad 0.4 p S^{0.3} z_1^{0.3} z_2^{-0.6} - w_2 \quad \uparrow \quad z_1, p, S, (-w_1), -w_2$$

(where parentheses indicate a derivative does not depend on a parameter at all, so we're free to say it's weakly increasing or decreasing as needed); so g is supermodular in the choice variables (z_1, z_2), and has increasing differences in the choice variables and the parameters $(p, S, -w_1, -w_2)$.

If the price p of sun-dried tomatoes goes up, then, Topkis' Theorem says I'll increase my use of both inputs z_1 and z_2, which increases my output level.

And similarly, an increase in S increases my use of both inputs, which (along with the increase in S itself) leads to more output as well.

(b) *I learn of an opportunity to move my factory to a different location that's more expensive but has more exposure to sunlight. Am I more likely to do this if the price of fertilizer is high or low? Explain.*

If we now consider S as a choice variable, we can think of "choosing" different possible levels of S from some set $\mathbb{S}$, each with some associated cost $c(S)$, and write the problem as

$$\max_{S \in \mathbb{S}, z_1, z_2 \geq 0} \left\{ p S^{0.3} z_1^{0.3} z_2^{0.4} - w_1 z_1 - w_2 z_2 - c(S) \right\}$$

The new problem is supermodular in (S, z_1, z_2), and has increasing differences in the choice variables and $(p, -w_1, -w_2)$. Thus, a lower price of fertilizer implies a higher choice of S (along with higher choices of z_1 and z_2) – I'm more likely to change locations when the price of fertilizer is low.

I choose to remain in my old location, and instead find some inexpensive heat lamps, which allow me to supplement natural sunlight with artificial light, changing my production function to

$$f(z) \quad = \quad (S+\sqrt{z_0})^{0.3}\, z_1^{0.3} z_2^{0.4}$$

where z_0 is electricity.

(c) *Show that if the price of electricity falls, I use more of each input, and produce more.*

My new problem is

$$\max_{z_0,z_1,z_2 \geq 0} \left\{ p\,(S+\sqrt{z_0})^{0.3}\, z_1^{0.3} z_2^{0.4} - w_0 z_0 - w_1 z_1 - w_2 z_2 \right\}$$

with

$$\begin{aligned}
\frac{\partial g}{\partial z_0} &= 0.3p\tfrac{(S+\sqrt{z_0})^{-0.7}}{2\sqrt{z_0}} z_1^{0.3} z_2^{0.4} - w_0 && \uparrow \quad z_1, z_2, p, -w_0, (-w_1), (-w_2) \\
\frac{\partial g}{\partial z_1} &= 0.3p(S+\sqrt{z_0})^{0.3} z_1^{-0.7} z_2^{0.4} - w_1 && \uparrow \quad z_0, z_2, p, (-w_0), -w_1, (-w_2) \\
\frac{\partial g}{\partial z_2} &= 0.4p(S+\sqrt{z_0})^{0.3} z_1^{0.3} z_2^{-0.6} - w_2 && \uparrow \quad z_0, z_1, p, (-w_0), (-w_1), -w_2
\end{aligned}$$

so g is supermodular in the choice variables (z_0, z_1, z_2), with increasing differences in the choice variables and $(p, -w_0, -w_1, -w_2)$. Thus, a decrease in w_0 leads to increases in all three inputs (z_0, z_1, z_2), and therefore greater production.

(d) *Rather than choosing z_0 given S, we can think of my problem as choosing the total level of light, $L = S + \sqrt{z_0}$, and setting $z_0 = (L-S)^2$, making my problem*

$$\max_{L \geq S, z_1 \geq 0, z_2 \geq 0} \left\{ pL^{0.3} z_1^{0.3} z_2^{0.4} - w_0 (L-S)^2 - w_1 z_1 - w_2 z_2 \right\}$$

Show that an increase in S raises my profits, and increases my total use of light L and my use of seeds and fertilizer.

An increase in S must raise my profits, since I could (if I wanted) just use the same levels of each input and produce more output.

Thinking about the problem as maximizing over (L, z_1, z_2),

$$\begin{aligned}
\frac{\partial g}{\partial L} &= 0.3pL^{-0.7} z_1^{0.3} z_2^{0.4} - 2w_0(L-S) && \uparrow \quad z_1, z_2, S \\
\frac{\partial g}{\partial z_1} &= 0.3pL^{0.3} z_1^{-0.7} z_2^{0.4} - w_1 && \uparrow \quad L, z_2, (S) \\
\frac{\partial g}{\partial z_2} &= 0.4pL^{0.3} z_1^{0.3} z_2^{-0.6} - w_2 && \uparrow \quad L, z_1, (S)
\end{aligned}$$

so the problem is supermodular in the choice variables (L, z_1, z_2), with increasing differences in these choice variables and S; so an increase in S increases my levels of L, z_1, and z_2.

(e) *Over two "periods," my business operates under two different combinations of sunlight and market prices, and generates the following data:*

S	w_0	w_1	w_2	p	z_0	z_1	z_2	output	profits
8	3	12	16	40	4	10	10	10	108
6	2	12	16	40	9	9	9	9	90

An econ PhD student observes market prices (w_0, w_1, w_2, p)*, my input purchases* (z_0, z_1, z_2)*, and my output level for the two periods, but does not know my production function and does not realize that sunlight is one of my inputs. Can the student rationalize the data as profit-maximizing behavior for* some *production function, or will he or she (wrongly) conclude that I was running my business irrationally? Explain.*

Ignoring sunshine and considering only my other inputs and my output level, the data would appear to violate the Weak Axiom of Profit Maximization. Letting $p^2 = (2, 12, 16, 40)$ be the prices at the second observation, $y^1 = (-4, -10, -10, 10)$ the first observed production plan and $y^2 = (-9, -9, -9, 9)$ the second production plan,

$$p^2 \cdot y^1 = -8 - 120 - 160 + 400 = 112$$

$$p^2 \cdot y^2 = -18 - 108 - 144 + 360 = 90$$

so $p^2 \cdot y^2 < p^2 \cdot y^1$. Thus, it would appear I could have earned higher profits in the second period by choosing y^1 instead of y^2; the grad student would conclude that there was no production set that would rationalize the data.

(In fact, we could see more quickly that the data is not rationalizable, by noting that when w_0 fell from observation 1 to observation 2, my profit went down, despite z_0 being one of my inputs. Just from that, we know that if I had just stuck with my production plan from the first period, my profits would have gone up instead.)

Question 2. A Little Quasilinearity

Consider a consumer with preferences represented by the utility function

$$u(x) \quad = \quad x_1 + U(x_2, x_3)$$

Suppose that U is increasing in both arguments, continuous, and supermodular in (x_2, x_3). Assume the non-negativity constraint on x_1 does not bind (so you can ignore it), and the consumer problem has a unique solution.

(a) *"Plug in" the budget constraint and write the consumer problem as an unconstrained maximization problem over x_2 and x_3. Are goods 2 and 3 normal or inferior? Is good 1 normal or inferior?*

Since preferences are locally non-satiated, we know the budget constraint will always bind; knowing that the non-negativity constraint on x_1 doesn't bind, we can plug in $x_1 = \frac{w - p_2 x_2 - p_3 x_3}{p_1}$ and write the consumer's problem as

$$\max_{x_2, x_3 \geq 0} \left\{ \frac{w - p_2 x_2 - p_3 x_3}{p_1} + U(x_2, x_3) \right\}$$

The values of x_2 and x_3 that solve this problem are the solutions to

$$\max_{x_2, x_3 \geq 0} \left\{ \frac{-p_2 x_2 - p_3 x_3}{p_1} + U(x_2, x_3) \right\}$$

which do not depend on w, so goods 2 and 3 are neither normal nor inferior. This means that $x_1^* = \frac{1}{p_1}(w - p_2 x_2^* - p_3 x_3^*)$ is increasing in w, so good 1 is normal.

(b) *Show that goods 2 and 3 are gross complements.*

We just noted that x_2^* and x_3^* are the solutions to

$$\max_{x_2, x_3 \geq 0} \left\{ \frac{-p_2 x_2 - p_3 x_3}{p_1} + U(x_2, x_3) \right\}$$

Since U is supermodular, this problem is supermodular in (x_2, x_3), and has increasing differences in (x_2, x_3) and $(-p_2, -p_3)$. Also note that this problem is being solved over $\mathbb{R}^2_+$, a product set, so we can apply Topkis' Theorem, so x_2 and x_3 both fall when either p_2 or p_3 rises. This means goods 2 and 3 are gross complements – the demand for each is decreasing in the price of the other.

(c) *Consider the consumer's expenditure minimization problem. Show that good 1 is a (Hicksian) substitute for the other two goods.*

The expenditure minimization problem is

$$\min_{x_1, x_2, x_3} \{p_1 x_1 + p_2 x_2 + p_3 x_3\} \qquad \text{subject to} \qquad x_1 + U(x_2, x_3) \quad \geq \quad u$$

Since u is continuous, the no-excess-utility property holds, so $x_1 = u - U(x_2, x_3)$; so we can rewrite expenditure minimization as

$$\min_{x_2, x_3} \{p_1 \left(u - U(x_2, x_3)\right) + p_2 x_2 + p_3 x_3\}$$

Since we're used to applying Topkis to maximization rather than minimization problems, let's maximize the negative, or

$$\max_{x_2,x_3}\{-p_1u+p_1U(x_2,x_3)-p_2x_2-p_3x_3\}$$

This problem is supermodular in (x_2,x_3), with increasing differences in (x_2,x_3) and $(-p_2,-p_3)$, so an increase in p_2 leads to a decrease in the Hicksian demand for goods 2 and 3. Since $u(x_1,x_2,x_3)$ is increasing in all three arguments and Hicksian demand must maintain constant utility as prices change (or since $x_1=u-U(x_2,x_2)$ and we just said the Hicksian demand for goods 2 and 3 goes down), the Hicksian demand for good 1 must go up when p_2 (or p_3) rises; so good 1 is a substitute for the other two goods.

(Alternatively, we could note that Hicksian demand at target utility level u is equal to Marshallian demand at wealth $e(p,u)$,

$$h_2(p,u) \quad = \quad x_2(p,e(p,u)) \qquad \text{and} \qquad h_3(p,u) \quad = \quad x_3(p,e(p,u))$$

Since we saw in part (a) that $x_2(p,w)$ and $x_3(p,w)$ don't depend on w, we can write this as

$$h_2(p,u) \quad = \quad x_2(p) \qquad \text{and} \qquad h_3(p,u) \quad = \quad x_3(p)$$

Since we just saw that Marshallian demand for goods 2 and 3 are decreasing in p_2 and p_3, this tells us that Hicksian demand for both goods is as well; then since utility is increasing in all three goods, the Hicksian demand for good 1 must rise to maintain the same level of utility.)

(d) *Given your answer to the last part of (a), does (c) imply that good 1 must also be a gross substitute for the other goods? Explain.*

Since good 1 is a normal good, this does *not* establish that good 1 is a gross substitute for the others. An increase in p_2 has two effects on the Marshallian demand for good 1, substitution and wealth effects. We just saw that the substitution effect is positive. However, since good 1 is a normal good, the wealth effect is negative (an increase in p_2 makes the consumer effectively poorer, which makes them demand less of good 1); it's not obvious which effect will dominate.

(In the differentiable case, the Slutsky equation gives

$$\frac{\partial x_1}{\partial p_2} \quad = \quad \frac{\partial h_1}{\partial p_2}-\frac{\partial x_1}{\partial w}x_2$$

Part (c) establishes that $\frac{\partial h_1}{\partial p_2}\geq 0$; but part (a) establishes that $\frac{\partial x_1}{\partial w}\geq 0$, so the two effects go in opposite directions, and the sign of $\frac{\partial x_1}{\partial p_2}$ is unclear.)

Question 3. A Little Separability

A consumer has preferences over five goods – peanut butter, jelly, bacon, lettuce, and tomato – represented by the utility function

$$u(x) \quad = \quad \min\{x_1, x_2\}^{0.8} + \min\{x_3, x_4, x_5\}^{0.9}$$

Assume $p \gg 0$ *and* $w > 0$.

(a) *Argue that the consumer will optimally consume all five goods.*

Since the derivative of $z^{0.8}$ or $z^{0.9}$ goes to $+\infty$ as $z \to 0$, if a consumer is not consuming both of goods 1 and 2, or not consuming all three of goods 3, 4, and 5, there's basically infinite returns to spending just a tiny bit of money on each of them; so it's always optimal to consume all five goods in positive quantities.

(b) *Are preferences homothetic? Do preferences over* (x_3, x_4, x_5) *depend on* (x_1, x_2)*? Do preferences over* (x_1, x_2) *depend on* (x_3, x_4, x_5)*?*

Preferences are not homothetic. (You didn't need to offer an example, but for intuition, note that $(2, 2, 0, 0, 0) \succ (0, 0, 1, 1, 1)$, since $2^{0.8} > 1^{0.9}$. If we scale both bundles by a constant $\lambda > 0$, we see $(2\lambda, 2\lambda, 0, 0, 0) \succ (0, 0, \lambda, \lambda, \lambda)$ if and only if $(2\lambda)^{0.8} > \lambda^{0.9}$, or $2^{0.8} > \lambda^{0.1}$, so for λ sufficiently large, the ranking flips; homothetic preferences would require $(2\lambda, 2\lambda, 0, 0, 0) \succ (0, 0, \lambda, \lambda, \lambda)$ for all λ.)

Preferences over (x_3, x_4, x_5) do not depend on (x_1, x_2), and vice versa – both of these follow from the additive separability of the utility function.

(c) *Define*

$$v_1(p, m_1) \quad = \quad \max_{x_1, x_2 \geq 0} \left\{\min\{x_1, x_2\}^{0.8}\right\} \qquad \textit{subject to} \quad p_1 x_1 + p_2 x_2 \quad \leq \quad m_1$$

$$v_2(p, m_2) \quad = \quad \max_{x_3, x_4, x_5 \geq 0} \left\{\min\{x_3, x_4, x_5\}^{0.9}\right\} \quad \textit{subject to} \quad p_3 x_3 + p_4 x_4 + p_5 x_5 \quad \leq \quad m_2$$

as the "utility value," respectively, of spending m_1 *optimally on goods 1 and 2 combined, or spending* m_2 *optimally on goods 3, 4 and 5 combined. Calculate the consumer's demand for goods* 1 *and* 2*, conditional on spending a total of* m_1 *on the two goods combined; calculate the demand for goods* 3*,* 4 *and* 5 *conditional on spending a total of* m_2 *on the three goods combined; and calculate* $v_1(p, m_1)$ *and* $v_2(p, m_2)$*.*

The consumer solves the first problem by spending all of m_1 and consuming equal quantities of goods 1 and 2, giving $x_1 = x_2 = \frac{m_1}{p_1+p_2}$; plugging these into $\min\{x_1, x_2\}^{0.8}$ gives $v_1(p, m_1) = \left(\frac{m_1}{p_1+p_2}\right)^{0.8}$.

Similarly, the consumer solves the second problem by spending all of m_2 and consuming equal quantities of goods 3, 4 and 5, giving $x_3 = x_4 = x_5 = \frac{m_2}{p_3+p_4+p_5}$ and $v_2(p, m_2) = \left(\frac{m_2}{p_3+p_4+p_5}\right)^{0.9}$.

(d) *Consider the consumer's problem of allocating a budget w among the two sets of goods,*

$$\max_{m_1, m_2 \geq 0} \{v_1(p, m_1) + v_2(p, m_2)\} \quad \textit{subject to} \quad m_1 + m_2 \quad \leq \quad w$$

and let (m_1^*, m_2^*) be the solution. Show that m_1^* is decreasing in p_1 and p_2 and increasing in p_3, p_4 and p_5, and that m_1^* and m_2^* are both increasing in w. (You do not need to give closed-form solutions for m_1^* and m_2^*.)

The consumer's budget allocation problem is

$$\max_{m_1, m_2 \geq 0} \left\{ \left(\frac{m_1}{p_1 + p_2} \right)^{0.8} + \left(\frac{m_2}{p_3 + p_4 + p_5} \right)^{0.9} \right\} \qquad \text{subject to} \qquad m_1 + m_2 \leq w$$

Since v_1 is strictly increasing in m_1 (or since v_2 is strictly increasing in m_2), the consumer will allocate the whole budget, meaning $m_1 + m_2 = w$; so we can set $m_2 = w - m_1$ and solve

$$\max_{m_1 \geq 0} \left\{ \left(\frac{m_1}{p_1 + p_2} \right)^{0.8} + \left(\frac{w - m_1}{p_3 + p_4 + p_5} \right)^{0.9} \right\}$$

This is conveniently concave in m_1, and the solution is given by the first-order condition,

$$\frac{0.8 m_1^{-0.2}}{(p_1 + p_2)^{0.8}} - \frac{0.9(w - m_1)^{-0.1}}{(p_3 + p_4 + p_5)^{0.9}} \quad = \quad 0$$

which rearranges to

$$\frac{0.8(p_3 + p_4 + p_5)^{0.9}}{0.9(p_1 + p_2)^{0.8}} \quad = \quad \frac{m_1^{0.2}}{(w - m_1)^{0.1}}$$

Since the right-hand side is strictly increasing in m_1, this defines a unique value for m_1^* (and therefore for $m_2^* = w - m_1^*$). Since the left-hand side is increasing in p_3, p_4 and p_5 and decreasing in p_1 and p_2, so is m_1^*. If w increases and prices stay the same, the left-hand side of this equation is unchanged, so the right-hand side must stay the same as well. If m_1 went up and $m_2 = w - m_1$ went down, the right-hand side would go up; if m_1 went down (and $m_2 = w - m_1$ therefore went up), the right-hand side would go down; and since $m_1 + (w - m_1) = w$, went up, the numerator and denominator can't both go down. Thus, m_1 and $w - m_1$ must both go up when w goes up.

(e) *Use parts (c) and (d) to show that goods 1 and 2 are gross substitutes for good 5;*

that goods 3 and 4 are gross complements for good 5;

and that all five goods are normal.

If p_5 goes up, then m_1 goes up (part (d)), so $x_1 = x_2 = \frac{m_1}{p_1 + p_2}$ goes up (part (c)). Since x_1 and x_2 are increasing in p_5, goods 1 and 2 are gross substitutes for good 5.

Similarly, if p_5 goes up, then m_2 goes down, so $x_3 = x_4 = \frac{m_2}{p_3 + p_4 + p_5}$ goes down (both because m_2 falls and because p_5 rises). Since x_3 and x_4 are decreasing in p_5, goods 3 and 4 are gross complements for good 5.

If w goes up, then m_1 and m_2 both go up; so $x_1 = x_2 = \frac{m_1}{p_1 + p_2}$ and $x_3 = x_4 = x_5 = \frac{m_2}{p_3 + p_4 + p_5}$ all go up, meaning all five goods are normal.