1. **Impatience and Time (In)Consistency (30 points)**

Consider a simple model with three periods: you can consume now, you can consume tomorrow, and whatever is left of your budget after that, you get to enjoy in retirement in the third period. Your utility function (evaluted in period 1*) is*

$$U_1(c_1, c_2, c_3) \quad = \quad \log c_1 + \beta\delta \log c_2 + \beta\delta^2 c_3$$

(Note that this is linear, not log, in c_3*.) Normalize the price of good* 3 *to* 1*, so that your budget constraint is*

$$p_1 c_1 + p_2 c_2 + c_3 \quad \leq \quad w$$

and assume that w *is high enough that the non-negativity constraint on* c_3 *won't bind.*

(a) *Solve the consumer's problem.*

We could solve this with a Lagrangian; but since preferences are LNS, the budget constraint will hold with equality, and since we're told the nonnegativity constraint on c_3 won't bind, we can simply rewrite the budget constraint as

$$c_3 \quad = \quad w - p_1 c_1 - p_2 c_2$$

and solve

$$\max_{c_1, c_2 \geq 0} \left\{ \log c_1 + \beta\delta \log c_2 + \beta\delta^2 (w - p_1 c_1 - p_2 c_2) \right\}$$

This has FOC

$$\frac{1}{c_1} - \beta\delta^2 p_1 \quad = \quad 0 \qquad \text{and} \qquad \frac{\beta\delta}{c_2} - \beta\delta^2 p_2 \quad = \quad 0$$

giving $c_1 = \frac{1}{\beta\delta^2 p_1}$ and $c_2 = \frac{1}{\delta p_2}$; plugging these back into the budget constraint to find c_3 gives

$$(c_1^*, c_2^*, c_3^*) \quad = \quad \left(\frac{1}{\beta\delta^2 p_1}, \; \frac{1}{\delta p_2}, \; w - \frac{1}{\beta\delta^2} - \frac{1}{\delta} \right)$$

(b) *Let* $c^* = (c_1^*, c_2^*, c_3^*)$ *be your optimal consumption plan as of period* 1*. Now suppose you've already consumed* c_1^* *and moved on to period* 2*, at which point your utility function is now*

$$U_2(c_2, c_3) \quad = \quad \log c_2 + \beta\delta c_3$$

Solve this new consumer problem, with budget $w' = w - p_1 c_1^*$*. Is your new plan to consume more or less in period two than your original plan?*

Again rewriting the budget constraint as $c_3 = w' - p_2 c_2$, we can solve

$$\max_{c_2 \geq 0} \left\{ \log c_2 + \beta\delta (w - p_2 c_2) \right\}$$

giving FOC

$$\frac{1}{c_2} - \beta\delta p_2 \quad = \quad 0$$

and therefore

$$(c_2, c_3) \quad = \quad \left(\frac{1}{\beta\delta p_2}, \; w' - \frac{1}{\beta\delta} \right)$$

where $w' = w - p_1 c_1^*$. Since $\beta < 1$,

$$c_2 = \frac{1}{\beta\delta p_2} > \frac{1}{\delta p_2} = c_2^*$$

so you will now consume more than your original plan c_2^*.

(c) *How does your new choice of c_2 depend on your budget $w - p_1 c_1$? Knowing "future you" will deviate from your original plan c^*, is there an incentive to consume more or less than c_1^* in period 1 to compensate?*

For the particular utility function given, there are no wealth effects for either c_1 or c_2, and so the choice of c_2 does not depend on the second-period budget $w' = w - p_1 c_1$. Thus, there's no reason to distort your period-one consumption c_1, since it won't alter your subsequent choice of c_2 (and the sub-optimal choice of c_2 doesn't change your optimal choice of c_1).

(d) *How much utility would you sacrifice for a technology which forced you to "stick to your plan" and consume c^*? (That is, using U_1 to evaluate your utility, how much utility do you expect to "lose" because you know you'll re-optimize consumption in period 2?)*

The loss in utility is

$$\begin{aligned}
u(c_1^*, c_2^*, c_3^*) - u(c_1^*, c_2', c_3') &= \log c_1^* + \beta\delta \log\left(\frac{1}{\delta p_2}\right) + \beta\delta^2 \left(w - p_1 c_1^* - p_2 \frac{1}{\delta p_2}\right) \\
&\quad - \left[\log c_1^* + \beta\delta \log\left(\frac{1}{\beta\delta p_2}\right) + \beta\delta^2 \left(w - p_1 c_1^* - p_2 \frac{1}{\beta\delta p_2}\right)\right] \\
&= \beta\delta \log\left(\frac{1}{\delta p_2}\right) - \beta\delta^2 \frac{1}{\delta} - \beta\delta \log\left(\frac{1}{\beta\delta p_2}\right) + \beta\delta^2 \frac{1}{\beta\delta} \\
&= -\beta\delta \log(\delta p_2) + \beta\delta \log(\beta\delta p_2) - \beta\delta + \delta \\
&= \beta\delta \log \frac{\beta\delta p_2}{\delta p_2} - \beta\delta + \delta \quad = \quad \beta\delta \log \beta + \delta(1 - \beta)
\end{aligned}$$

so this is how much you would be willing to pay for a commit device forcing you to consume c_2^*.

2. **Alchemy and Mining (40 points)**

Alchemy is the millennia-old junk science of trying to turn lead into gold.

A firm has two different technologies for producing gold: the first one uses labor and pickaxes, and the second one uses lead and magic. Let $z = (z_1, z_2, z_3, z_4) = (\textit{labor, pickaxes, lead, magic})$*; the amount of gold produced is*

$$f(z) = \left(z_1^{0.7\alpha} z_2^{0.3\alpha} + z_3^{0.4\alpha} z_4^{0.6\alpha}\right)^\beta$$

where $\alpha, \beta > 0$.

The firm is a price-taker in the markets for gold and magic, but buys the other inputs locally and has some market power in the other input markets. For $i = 1, 2, 3$, *let* $W_i(z_i)$ *be the price per unit of input* i, *as a function of the quantity demanded. Assume* $W_i(\cdot)$ *is weakly increasing and differentiable for each* i. *Let* p *be the price of gold, and* w_4 *the price of magic. The firm's objective function is*

$$pf(z) - W_1(z_1)z_1 - W_2(z_2)z_2 - W_3(z_3)z_3 - w_4 z_4$$

Assume all prices are positive and finite at all input levels, and that the firm's problem has a unique solution.

(a) *For what values of* α *and* β *does the firm's technology have increasing returns to scale?*

For $\lambda > 0$,

$$\begin{aligned} f(\lambda z) &= \left((\lambda z_1)^{0.7\alpha}(\lambda z_2)^{0.3\alpha} + (\lambda z_3)^{0.4\alpha}(\lambda z_4)^{0.6\alpha}\right)^\beta \\ &= \left(\lambda^\alpha z_1^{0.7\alpha} z_2^{0.3\alpha} + \lambda^\alpha z_3^{0.4\alpha} z_4^{0.6\alpha}\right)^\beta \\ &= (\lambda^\alpha)^\beta \left(z_1^{0.7\alpha} z_2^{0.3\alpha} + z_3^{0.4\alpha} z_4^{0.6\alpha}\right)^\beta \\ &= \lambda^{\alpha\beta} f(z) \end{aligned}$$

so f has increasing returns if $\alpha\beta \geq 1$.

(b) *Show that if* $\alpha < 1$, *the firm uses all four inputs. Are there cases with* $\alpha > 1$ *where the firm would use all four inputs? Explain (in words).*

At $z_1 = z_2 = \epsilon$ (and $z_3, z_4 > 0$), the marginal change in output from increasing both z_1 and z_2 is proportional to $\alpha\epsilon^{\alpha-1}$, which goes to infinity as $\epsilon \to 0$. Thus, at $z_1 = z_2 = 0$, at any positive but finite price levels, there's some level of production using the first two inputs which would be profitable.

If the firm were a price taker in all input markets, then it would never use all four inputs when $\alpha > 1$. However, in this case, the firm might choose to use all four if the pair of inputs that was cheapest initially got much more expensive as more of the input was demanded. For example, suppose $W_1(z_1) = w_1$ and $W_2(z_2) = w_2$ were both constants, w_4 was very low, and $W_3(z_3)$ started out very low but jumped to a very high level at a particular level z_3^*. It might be optimal to produce using the second technology up to that level, and then supplement that with the first technology.

(c) *Show that if $\beta > 1$, an increase in the price of gold leads to increases of the use of all four inputs. (HINT: Topkis.)*

Suppose $\beta > 1$; we'll show the firm's problem is supermodular in z, and has increasing differences in (z,p). Let $g(z,p)$ be the firm's problem, as defined above.
Consider first

$$\frac{\partial g}{\partial z_1} = p\beta\left(z_1^{0.7\alpha}z_2^{0.3\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}\right)^{\beta-1} z_2^{0.3\alpha}0.7\alpha z_1^{0.7\alpha-1} - W_1'(z_1)z_1 - W_1(z_1)$$

For $\beta > 1$, this is increasing in z_2, z_3, and z_4, as well as p. The arguments for z_2, z_3, and z_4 are identical; so g has increasing differences in each pair (z_i, z_j), so it's supermodular in z; and it has increasing differences in (z_i, p) for each i.
By Topkis, then, an increase in p leads to a higher set of optimizers; since the firm's problem has a unique solution, this solution must therefore be weakly bigger in each dimension.

(d) *What is the impact of a decrease in the price of magic on the firm's use of labor if $\beta > 1$? What if $\beta < 1$?*

If $\beta > 1$, the firm's problem is supermodular in z, and has increasing differences in $(z,(p,-w_4))$, since $\frac{\partial g}{\partial z_i}$ does not depend on w_4 for $i = \{1,2,3\}$, and can therefore be thought of as weakly decreasing in w_4; and

$$\frac{\partial g}{\partial z_4} = p\beta\left(z_1^{0.7\alpha}z_2^{0.3\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}\right)^{\beta-1} z_3^{0.4\alpha}0.6\alpha z_4^{0.6\alpha-1} - w_4$$

is decreasing in w_4. By Topkis, then, if w_4 falls, all four input levels go up, so the firm's use of labor increases.
If $\beta < 1$, however, the situation is different. Now,

$$\frac{\partial g}{\partial z_1} = p\beta\left(z_1^{0.7\alpha}z_2^{0.3\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}\right)^{\beta-1} z_2^{0.3\alpha}0.7\alpha z_1^{0.7\alpha-1} - W_1'(z_1)z_1 - W_1(z_1)$$

is *decreasing* is z_3 and z_4, and it's not immediately obvious whether it's increasing or decreasing in z_2. It turns out to still be increasing in z_2, however; to see this, we can write $z_2^{0.3\alpha}$ as $z_2^{0.3\alpha(1-\beta)}z_2^{0.3\alpha\beta} = (\frac{1}{z_2})^{0.3\alpha(\beta-1)}z_2^{0.3\alpha\beta}$, and rewrite $\frac{\partial g}{\partial z_1}$ as

$$\begin{aligned}\frac{\partial g}{\partial z_1} &= p\beta\left(z_1^{0.7\alpha}z_2^{0.3\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}\right)^{\beta-1}\left(\frac{1}{z_2^{0.3\alpha}}\right)^{\beta-1} z_2^{0.3\alpha\beta}0.7\alpha z_1^{0.7\alpha-1} - W_1'(z_1)z_1 - W_1(z_1)\\ &= p\beta\left(z_1^{0.7\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}z_2^{-0.3\alpha}\right)^{\beta-1} z_2^{0.3\alpha\beta}0.7\alpha z_1^{0.7\alpha-1} - W_1'(z_1)z_1 - W_1(z_1)\end{aligned}$$

Since $\beta < 1$, the expression $\left(z_1^{0.7\alpha} + z_3^{0.4\alpha}z_4^{0.6\alpha}z_2^{-0.3\alpha}\right)^{\beta-1}$ is increasing in z_2 (since the term inside the parentheses is decreasing in z_2, but it's raised to a negative power), so the whole thing is increasing in z_2. Thus,

$$\frac{\partial^2 g}{\partial z_1\partial z_2} > 0 > \frac{\partial^2 g}{\partial z_1\partial z_3}, \quad \frac{\partial^2 g}{\partial z_1\partial z_4}$$

and we can likewise show that $\frac{\partial^2 g}{\partial z_2\partial z_3}$ and $\frac{\partial^2 g}{\partial z_2\partial z_4} < 0$ and $\frac{\partial^2 g}{\partial z_3\partial z_4} > 0$.

Thus, g is supermodular in $(z_1, z_2, -z_3, -z_4)$ when $\beta < 1$. Since $\frac{\partial g}{\partial z_i}$ is decreasing in w_4 for $i = 4$ and independent of w_4 otherwise, g has increasing differences in $(z_1, z_2, -z_3, -z_4)$ and w_4. Thus, when $\beta < 1$, a decrease in w_4 leads to increases in z_3 and z_4 but decreases in z_1 and z_2, hence less use of labor.

(e) *Suppose the firm's use of lead is fixed in the short term, while the other input levels are adjustable. If $\beta < 1$ and the price of magic falls, is the change in the firm's use of labor larger in the short term or the long term? Explain.*

This is LeChatelier's Principle – the long-term change will always be larger than the short-term change.

With $\beta < 1$, we just saw that g is supermodular in $(z_1, z_2, -z_3, -z_4)$, with increasing differences in this and w_4. Thus, in the long run, when w_4 goes down, we know z_1 and z_2 will go down and z_3 and z_4 will go up. With z_3 fixed in the short term, however, we can think of it as a parameter. In the short term, then, the firm's problem is supermodular in $(z_1, z_2, -z_4)$ with increasing differences in $(z_1, z_2, -z_4)$ and w_4, so when w_4 goes down, z_1 goes down. And thinking of z_3 as a parameter, the firm's problem is supermodular in $(z_1, z_2, -z_4)$, with increasing differences in this and $-z_3$; so in the long term, when z_3 goes up, z_1 goes down again, making the long-term change bigger than the short-term change.

3. **Quasilinear Revealed Preference**

Consider a quasilinear utility function on the choice set $\mathbb{R} \times \mathbb{R}^m_+$. Let y denote the consumption of the first (numeraire) good, and z the remaining goods, so that $x = (y, z)$ and

$$u(y, z) \quad = \quad y + U(z_1, \ldots, z_m)$$

with $z \geq 0$ but no non-negativity constraint on y.

Fix the price of the y good at 1 throughout the problem, and let price vectors p refer to the prices of the other m goods. Recall that with quasilinear utility, Marshallian demand for the last m goods does not depend on w, and the indirect utility function can therefore be written as

$$v(p, w) \quad = \quad V(p) + w$$

You may assume that V is differentiable if you wish.

(a) *Show that V is convex in p.*

Preferences are locally nonsatiated, so the budget constraint will hold with equality, or $y = w - p \cdot z$ at any optimal bundle. Since there's no nonnegativity constraint on y, we can therefore restate the consumer's problem as

$$v(p, w) \quad = \quad \max_{z \geq 0} \{w - p \cdot z + U(z)\}$$

For each z, the expression $w - p \cdot z + U(z)$ is a linear function of p; so thinking about the consumer's problem as a function of p, it's the maximum a collection of linear functions, and is therefore convex. Since $v(p, w) = V(p) + w$, this means V is convex in p as well.

More formally, let p and p' be two price vectors, and $p^t = tp + (1-t)p'$. Let $x^t = (y^t, z^t)$ be a solution to the consumer's problem at prices p^t, so that

$$v(p^t, w) \quad = \quad V(p^t) + w \quad = \quad u(w - p^t \cdot z^t, z^t) \quad = \quad w - p^t \cdot z^t + U(z^t)$$

so that $V(p^t) = -p^t \cdot z^t + U(z^t)$.

Now, at prices p, the consumer could still choose to consume $z = z^t$, leaving $w - p \cdot z^t$ to spend on the y good; since the consumer maximizes between this and other possible options,

$$v(p, w) \quad = \quad V(p) + w \quad \geq \quad w - p \cdot z^t + U(z^t) \qquad \longrightarrow \qquad V(p) \quad \geq \quad -p \cdot z^t + U(z^t)$$

By the same logic,

$$V(p') \quad \geq \quad -p' \cdot z^t + U(z^t)$$

Putting these together,

$$\begin{aligned} tV(p) + (1-t)V(p') &\geq t\left(-p \cdot z^t + U(z^t)\right) + (1-t)\left(-p' \cdot z^t + U(z^t)\right) \\ &= -(tp + (1-t)p') \cdot z^t + U(z^t) \\ &= V(tp + (1-t)p') \end{aligned}$$

proving V convex.

(For a shorter proof, one could also be clever and note that

$$u \quad = \quad v(p, e(p,u)) \quad \longrightarrow \quad u \quad = \quad V(p) + w \quad \longrightarrow \quad e(p,u) \quad = \quad u - V(p)$$

We know that the expenditure function is concave in p; this implies that $V(p)$ must be convex.)

(b) *Show that the Law of Demand holds for the last m goods, i.e., that*

$$(p^1 - p^0) \cdot (z^1 - z^0) \quad \leq \quad 0$$

By revealed preference, if z^0 is chosen at p^0, consuming z^0 (and $y^0 = w - p^0 \cdot x^0$) must give higher utility than consuming x^1 (and $y^1 = w - p^0 \cdot z^1$), so

$$w - p^0 \cdot z^0 + U(z^0) \quad \geq \quad w - p^0 \cdot z^1 + U(z^1)$$

By the same logic, consuming z^1 must be at least as good as consuming z^0 at prices p^1, or

$$w - p^1 \cdot z^1 + U(z^1) \quad \geq \quad w - p^1 \cdot z^0 + U(z^0)$$

Adding these together,

$$\begin{aligned} 2w - p^0 \cdot z^0 - p^1 \cdot z^1 + U(z^0) + U(z^1) &\geq 2w - p^0 \cdot z^1 - p^1 \cdot z^0 + U(z^1) + U(z^0) \\ -p^0 \cdot z^0 - p^1 \cdot z^1 &\geq -p^0 \cdot z^1 - p^1 \cdot z^0 \\ p^0 \cdot (z^1 - z^0) &\geq p^1 \cdot (z^1 - z^0) \\ 0 &\geq (p^1 - p^0) \cdot (z^1 - z^0) \end{aligned}$$

(c) *Show that*

$$V(p^1) - V(p^0) \quad \geq \quad -z^0 \cdot (p^1 - p^0)$$

This question comes from Varian (2012), "Revealed Preference and its Applications," *The Economic Journal* 122(560). In words, it boils down to: if you change prices, but give me enough money so I can still afford my old consumption bundle, you can't have made me worse off.

Consider the consumption bundle $x^0 = (w - p^0 \cdot z^0, z^0)$ which solves the consumer's problem at p^0. At prices p^1, this bundle would cost

$$(w - p^0 \cdot z^0) + p^1 \cdot z^0 \quad = \quad w + z^0 \cdot (p^1 - p^0)$$

So at prices p^1 and wealth $w + z^0 \cdot (p^1 - p^0)$, the consumer could still afford the bundle x^0, and therefore

$$v(p^1, w + z^0 \cdot (p^1 - p^0)) \quad \geq \quad u(x^0) \quad = \quad v(p^0, w)$$

or

$$V(p^1) + w + z^0 \cdot (p^1 - p^0) \quad \geq \quad V(p^0) + w$$

or

$$V(p^1) - V(p^0) \quad \geq \quad -z^0 \cdot (p^1 - p^0)$$

For an alternate proof, if V is differentiable, this condition can be shown from convexity of V (shown in part (a)) and Roy's Identity. For any convex function f,

$$f(z') - f(z) \quad \geq \quad \nabla f(z) \cdot (z' - z)$$

where ∇ is the gradient. (This is the observation that a convex function lies above its tangent plane.) Since $\frac{\partial v(p,w)}{\partial w} = 1$, Roy's Identity simply says that $z_i(p) = -\frac{\partial v(p,w)}{\partial p_i} = -\frac{\partial V(p,w)}{\partial p_i}$, and therefore $\nabla V(p^0) = -z^0$, so we immediately get

$$V(p^1) - V(p^0) \quad \geq \quad \nabla V(p_0) \cdot (p^1 - p^0) \quad = \quad -z^0 \cdot (p^1 - p^0)$$