Econ 711 – Fall 2017 – First Half Final Exam

Due online Sunday November 5 at 2 p.m.

Unlike the homework assignments, this exam is meant to be done **on your own**. *Please do not discuss these problems with anyone – in or out of the class – before 2 p.m. Sunday. Thank you, and good luck!*

1. **Aggregating Demand (15 points)**

   Let $X = \mathbb{R}^4_+$. There are two consumers, with utility functions

   $$\begin{aligned} u_1(x) &= \left(x_1 + x_2^{4/5} + x_3^{3/5} + x_4^{2/5}\right)^{0.7} \quad \text{and} \\ u_2(x) &= \left(x_1 + x_2^{2/5} + x_3^{3/5} + x_4^{4/5}\right)^{1.8} \end{aligned}$$

   (a) Show that for each consumer i, at a given price level p, there is a wealth level $\underline{w}_i(p)$ such that the Marshallian demand for good 1 is positive if and only if $w_i > \underline{w}_i(p)$.

   (b) Suppose that at various price levels p, and various individual wealth levels $w_1 > \underline{w}_1(p)$ and $w_2 > \underline{w}_2(p)$, the two consumers' Marshallian demand is observed. Will consumer 1's observed choices satisfy GARP? Will consumer 2's?

   (c) Suppose that at these various price and wealth levels described in part (b), only the combined demand $x_1^* + x_2^*$ of the two consumers is observed. Will the aggregate demand observations satisfy GARP? Why or why not?

2. **The Law of Demand (30 points)**

   Throughout, assume that preferences are locally non-satiated, $X = \mathbb{R}^k_+$, $p \gg 0$, and $w > 0$.

   (a) (Compensated Law of Demand.) Suppose there is a price change from p to p'. Let $x \in x(p,w)$ and $x' \in x(p',w')$, where the new wealth w' is compensated such that the consumer is as happy as before, i.e., $v(p',w') = v(p,w)$. Show that $(p'-p)\cdot(x'-x) \leq 0$.

   (b) ("Overcompensated Law of Demand.") Suppose there is a price change from p to p'. Let $x \in x(p,w)$ and $x' \in x(p',w')$ where the new wealth w' is compensated such that the consumer can just afford the original bundle at the new prices, i.e., $w' = p' \cdot x$. Show that (i) $v(p',w') \geq v(p,w)$ and (ii) $(p'-p)\cdot(x'-x) \leq 0$.

   (c) ("Undercompensated Law of Demand.") Suppose there is a price change from p to p'. Let $x \in x(p,w)$ and $x' \in x(p',w')$, where the new wealth w' is compensated such that the cost of the new bundle, evaluated at the old prices, is the same as the initial wealth, $w = p \cdot x'$. Show that (i) $v(p',w') \leq v(p,w)$ and (ii) $(p'-p)\cdot(x'-x) \leq 0$.

3. **Topkis and the Consumer Problem (15 points)**

I said in class that I didn't introduce the Monotone Comparative Statics approach in consumer theory because the consumer problem is solved over a budget set or upper contour set, and these sets are not closed under meet and join and therefore do not fit with the Topkis approach. However, we can sometimes transform problems in a way that makes them compatible with the MCS approach. For example, a firm's general profit maximization problem $\max_{y \in Y} p \cdot y$ does not fit, since Y is not a product set; but if we focus on single-output firms and rephrase the problem as an unconstrained choice over input vectors, MCS works. Here, we consider an analogous strategy for applying MCS to the consumer problem.

Let $X = \mathbb{R}^2_+$; fix a target utility level $\bar{u}$, and for $x_1 \geq 0$, define

$$x_2(x_1) \quad = \quad \min\{x_2 \ : \ u(x_1, x_2) \geq \bar{u}\}$$

or as $+\infty$ if the set on the right is empty. We can write the Hicksian expenditure minimization problem as

$$\begin{aligned} & \min_{x \geq 0} p \cdot x \quad \text{subject to} \quad u(x) \geq \bar{u} \\ = \quad & \min_{x_1, x_2 \geq 0} p \cdot x \quad \text{subject to} \quad x_2 \geq x_2(x_1) \\ = \quad & \min_{x_1 \geq 0} \{p_1 x_1 + p_2 x_2(x_1)\} \end{aligned}$$

Since we're used to maximizing rather than minimizing to apply Topkis, we can write this as

$$\max_{x_1 \geq 0} \{-p_1 x_1 - p_2 x_2(x_1)\}$$

(a) Show that when $X = \mathbb{R}^2_+$ and Hicksian demand is single-valued, this approach yields an alternate proof that Hicksian demand is downward-sloping in own price.

(b) Show that when $X = \mathbb{R}^2_+$ and Hicksian demand is single-valued, this approach yields an alternate proof that with only two goods, the goods must be substitutes.

(c) We could extend this approach to $X = \mathbb{R}^3_+$ by defining

$$x_3(x_1, x_2) \quad = \quad \min\{x_3 \ : \ u(x_1, x_2, x_3) \geq \bar{u}\}$$

and solving

$$\max_{x_1, x_2 \geq 0} \{-p_1 x_1 - p_2 x_2 - p_3 x_3(x_1, x_2)\}$$

Would the Topkis-based proof that h_1 is decreasing in p_1 go through? Why or why not?

4. **Monotone Selection Theorems (20 points)**

(a) For a one-dimensional choice set $X \subseteq \mathbb{R}$, prove the extension to Topkis' Theorem we stated (without proof) in class, commonly known as the Monotone Selection Theorem:

Theorem. *Let $X \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$, let $g : X \times T \to \mathbb{R}$, and let*

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x, t)$$

Suppose that g has **strictly increasing differences** *– that is, that $x' > x$ implies $g(x', t) - g(x, t)$ is* strictly *increasing in t. Then for any $t' > t$ and any selection $x \in x^*(t)$ and $x' \in x^*(t')$, it must be that $x' \geq x$.*

(b) Prove the following multi-dimensional version of the Monotone Selection Theorem:

Theorem. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be a product set, with $X_i \subseteq \mathbb{R}$ for each i. Let $T \subseteq \mathbb{R}$, let $g : X \times T \to \mathbb{R}$, and let*

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x, t)$$

Suppose that g is supermodular in X, has increasing differences in X and t, and has strictly *increasing differences in x_1 and t. Then for any $t' > t$ and any $x \in x^*(t)$ and $x' \in x^*(t')$, it must be that $x'_1 \geq x_1$.*

5. **A Two-Factory Firm (20 points)**

Consider a company with one output, and production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$. Suppose f is such that the firm's cost minimization problem always has a unique solution.

(a) Recall that f is *homothetic* if it is a monotonic transformation of a function which is homogeneous of degree 1, that is, if $f(z) = h(g(z))$ with $g(\lambda z) = \lambda g(z)$ and h strictly increasing.

i. Show that if f is homothetic, then the firm's conditional factor demand

$$z(q, w) \quad = \quad \arg\min_{z \in \mathbb{R}^m_+} w \cdot z \quad \text{subject to} \quad f(z) \geq q$$

is increasing in q for every input good.

ii. Show therefore that if f is homothetic, the firm's cost function

$$c(q, w) \quad = \quad \min_{z \in \mathbb{R}^m_+} w \cdot z \quad \text{subject to} \quad f(z) \geq q$$

has increasing differences in q and w_i for every input price w_i.

(b) Now suppose the firm sells its output in City A, but has two plants that can produce it, one in City A and one in Town B. The two plants use the same technology, represented by the homothetic production function f, but the inputs for each plant are purchased locally, and their prices vary in the two locations. Town B input prices tend to be lower, but the firm must also pay a transportation cost t to transport each unit of the good from Town B to City A. Thus, the cost of producing q^A units in the City A plant is

$$c^A \quad = \quad c(q^A, w^A)$$

and the cost of producting q^B units in the Town B plant is

$$c^B \quad = \quad c(q^B, w^B) + tq^B$$

i. Show that if the firm is a price taker in both input and output markets, its production in the City A plant is independent of Town B input prices and the transportation cost.

ii. Suppose now that the firm is a price taker in input markets but not in the output market, and therefore chooses output levels to maximize the objective function

$$(q^A + q^B)P(q^A + q^B) - c(q^A, w^A) - c(q^B, w^B) - tq^B$$

where $P(\cdot)$ is strictly decreasing. Suppose that $P(q)$ is differentiable and concave in q, and that the firm's problem has a unique solution at each price level. Prove what happens to q^A and q^B as w^B_i (the price of one of the input goods in Town B) rises.