Econ 711 – Fall 2017 – First Half Final Exam – Solutions

1. **Aggregating Demand (15 points)**

Let $X = \mathbb{R}^4_+$. There are two consumers, with utility functions

$$u_1(x) = \left(x_1 + x_2^{4/5} + x_3^{3/5} + x_4^{2/5}\right)^{0.7} \quad \text{and}$$

$$u_2(x) = \left(x_1 + x_2^{2/5} + x_3^{3/5} + x_4^{4/5}\right)^{1.8}$$

(a) *Show that for each consumer i, at a given price level p, there is a wealth level $\underline{w}_i(p)$ such that the Marshallian demand for good 1 is positive if and only if $w_i > \underline{w}_i(p)$.*

(b) *Suppose that at various price levels p, and various individual wealth levels $w_1 > \underline{w}_1(p)$ and $w_2 > \underline{w}_2(p)$, the two consumers' Marshallian demand is observed. Will consumer 1's observed choices satisfy GARP? Will consumer 2's?*

(c) *Suppose that at these various price and wealth levels described in part (b), only the combined demand $x_1^* + x_2^*$ of the two consumers is observed. Will the aggregate demand observations satisfy GARP? Why or why not?*

For part (a), first note that any monotonic transformation of a consumer's utility function leaves their preferences, and therefore the solution to their consumer problem, unchanged. Thus, for consumer 1, we can rewrite the Marshallian problem as

$$\max_{x \geq 0} \left\{x_1 + x_2^{4/5} + x_3^{3/5} + x_4^{2/5}\right\} \quad \text{subject to} \quad p \cdot x \leq w$$

Note that for $i > 1$, $\frac{\partial u}{\partial x_i} \to \infty$ as $x_i \to 0$, so with positive wealth, consumer 1 will always consume a strictly positive amount of goods 2, 3, and 4.

Since preferences are LNS, we know that $p \cdot x = w$, or $x_1 = \frac{1}{p_1}(w - p_2 x_2 - p_3 x_3 - p_4 x_4)$. Let $(\tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ be the solution to the problem

$$\max_{x_2, x_3, x_4 \geq 0} \left\{\frac{1}{p_1}(w - p_2 x_2 - p_3 x_3 - p_4 x_4) + x_2^{4/5} + x_3^{3/5} + x_4^{2/5}\right\}$$

which would be the consumer's problem if there were no nonnegativity constraint $x_1 \geq 0$ on the first good. This "unconstrained" problem has a unique solution, since we can rewrite it as

$$\frac{w}{p_1} + \max_{x_2 \geq 0}\left\{\frac{-p_2}{p_1}x_2 + x_2^{4/5}\right\} + \max_{x_3 \geq 0}\left\{\frac{-p_3}{p_1}x_3 + x_3^{3/5}\right\} + \max_{x_4 \geq 0}\left\{\frac{-p_4}{p_1}x_4 + x_4^{2/5}\right\}$$

and each of the three small maximization problems is strictly concave and therefore has a unique maximizer.

If $w - p_2\tilde{x}_2 - p_3\tilde{x}_3 - p_4\tilde{x}_4 \geq 0$, then the solution to the ("constrained") consumer problem is the same as the unconstrained,

$$x^*(p, w) = \left(\frac{1}{p_1}(w - p_2\tilde{x}_2 - p_3\tilde{x}_3 - p_4\tilde{x}_4), \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right)$$

as the non-negativity constraint on x_1 does not bind. On the other hand, if $w - p_2\tilde{x}_2 - p_3\tilde{x}_3 - p_4\tilde{x}_4 < 0$, then w is not high enough for consumer 1 to afford his "unconstrained-optimal" quantities of the last three goods; since the marginal utility of each good is strictly decreasing, when this is the case, the consumer will spend all his income on the last three goods and set $x_1^* = 0$. The same argument holds for consumer 2.

For part (b), the answer is yes, and yes; any choices which are based on optimization according to rational preferences will satisfy GARP.

For part (c), the answer is yes again. The key is to note again that a monotonic transformation of a utility function does not change a consumer's Marshallian demand. Thus, these two consumers behave exactly like consumers with utility functions

$$\begin{aligned} u_1(x) &= x_1 + x_2^{4/5} + x_3^{3/5} + x_4^{2/5} \quad \text{and} \\ u_2(x) &= x_1 + x_2^{2/5} + x_3^{3/5} + x_4^{4/5} \end{aligned}$$

We showed earlier that this implies indirect utility functions $v_1(p, w_1)$ and $v_2(p, w_2)$ of the form $a_i(p) + b(p)w_i$, which is the sufficient condition for aggregate demand to be consistent with choices of a single rational consumer, which must therefore satisfy GARP. (It's important that the observations are all from price/wealth combinations where $w_i \geq \underline{w}_i(p)$ for both consumers, since without that, the indirect utility function would not be linear in wealth and the argument would break down.)

(To see that $v_i(p, w_i) = a_i(p) + b(p)w_i$, we can note that for $w_i \geq \underline{w}_i(p)$, the consumer optimally chooses

$$x^*(p, w) = \left(\frac{1}{p_1} \left(w - p_2\tilde{x}_2 - p_3\tilde{x}_3 - p_4\tilde{x}_4 \right), \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \right)$$

and therefore

$$v_1(p, w_1) = u_1(x^*(p, w_1)) = \frac{1}{p_1} \left(w - p_2\tilde{x}_2 - p_3\tilde{x}_3 - p_4\tilde{x}_4 \right) + \tilde{x}_2^{4/5} + \tilde{x}_3^{3/5} + \tilde{x}_4^{2/5}$$

which is $a_1(p) + b(p)w_1$ if we let $b(p) = \frac{1}{p_1}$ and

$$\begin{aligned} a_1(p) &= -\frac{p_2\tilde{x}_2 + p_3\tilde{x}_3 + p_4\tilde{x}_4}{p_1} + \tilde{x}_2^{4/5} + \tilde{x}_3^{3/5} + \tilde{x}_4^{2/5} \\ &= \max_{x_2, x_3, x_4 \geq 0} \left\{ \frac{1}{p_1} \left(-p_2x_2 - p_3x_3 - p_4x_4 \right) + x_2^{4/5} + x_3^{3/5} + x_4^{2/5} \right\} \end{aligned}$$

and similarly for v_2.)

2. **The Law of Demand (30 points)**

Throughout, assume that preferences are locally non-satiated, $X = \mathbb{R}^k_+$, $p \gg 0$, and $w > 0$.

(a) *(Compensated Law of Demand.) Suppose there is a price change from p to p'. Let $x \in x(p, w)$ and $x' \in x(p', w')$, where the new wealth w' is compensated such that the consumer is as happy as before, i.e., $v(p', w') = v(p, w)$. Show that $(p' - p) \cdot (x' - x) \leq 0$.*

(b) *("Overcompensated Law of Demand.") Suppose there is a price change from p to p'. Let $x \in x(p, w)$ and $x' \in x(p', w')$ where the new wealth w' is compensated such that the consumer can just afford the original bundle at the new prices, i.e., $w' = p' \cdot x$. Show that (i) $v(p', w') \geq v(p, w)$ and (ii) $(p' - p) \cdot (x' - x) \leq 0$.*

(c) *("Undercompensated Law of Demand.") Suppose there is a price change from p to p'. Let $x \in x(p, w)$ and $x' \in x(p', w')$, where the new wealth w' is compensated such that the cost of the new bundle, evaluated at the old prices, is the same as the initial wealth, $w = p \cdot x'$. Show that (i) $v(p', w') \leq v(p, w)$ and (ii) $(p' - p) \cdot (x' - x) \leq 0$.*

For part (a), we know that $v(p', w') = v(p, w)$; since $v(p, w) = u(x)$ and $v(p', w') = u(x')$, this means $u(x) = u(x')$. Since preferences are locally non-satiated, this means that since x is optimal at prices p and wealth w,

$$p \cdot x \quad \leq \quad p \cdot x'$$

(If $p \cdot x' < p \cdot x = w$, given LNS preferences, there would be another bundle close to x' which was strictly preferred to x' (and therefore x), contradicting x being optimal.) By the same logic,

$$p' \cdot x' \quad \leq \quad p' \cdot x$$

since x' was chosen at p'. Thus,

$$p' \cdot (x' - x) \quad \leq \quad 0 \quad \leq \quad p \cdot (x' - x)$$

and therefore

$$(p' - p) \cdot (x' - x) \quad \leq \quad 0$$

For part (b), since $x \in B(p', w')$ but x' is chosen instead, we know $u(x') \geq u(x)$, and therefore $v(p', w') = u(x') \geq u(x) = v(p, w)$.

In addition, since $u(x') \geq u(x)$ and x is chosen at prices (p, w), we know that $p \cdot x' \geq w$. (If not, then since preferences are LNS, there would be a bundle near x' still inside the budget set that dominated x.) Thus, $p \cdot x' \geq p \cdot x$, or $p \cdot (x' - x) \geq 0$. On the other hand, $p' \cdot x = w' = p' \cdot x'$, or $p' \cdot (x' - x) = 0$; so

$$(p' - p) \cdot (x' - x) \leq 0$$

For part (c), it's basically the same argument as part (b), but with (p', w', x') and (p, w, x) switched. First of all, since $w = p \cdot x'$, x was chosen when x' was available, establishing $u(x) \geq u(x')$. Since $v(p, w) = u(x)$ and $v(p', w') = u(x')$, this proves $v(p, w) \geq v(p', w')$.

Second, since $u(x) \geq u(x')$ and x' is chosen at (p', w'), it must be that $p' \cdot x \geq w = p' \cdot x'$, or $p' \cdot (x' - x) \leq 0$. We know that $p \cdot x' = w = p \cdot x$, or $p \cdot (x' - x) = 0$; so $(p' - p) \cdot (x' - x) \leq 0$, completing the proof.

3. **Topkis and the Consumer Problem (15 points)**

Let $X = \mathbb{R}^2_+$; fix a target utility level $\bar{u}$, and for $x_1 \geq 0$, define

$$x_2(x_1) \quad = \quad \min\{x_2 \, : \, u(x_1, x_2) \geq \bar{u}\}$$

or as $+\infty$ if the set on the right is empty. We can write the Hicksian expenditure minimization problem as

$$\min_{x \geq 0} p \cdot x \quad \text{subject to} \quad u(x) \geq \bar{u}$$

$$= \quad \min_{x_1, x_2 \geq 0} p \cdot x \quad \text{subject to} \quad x_2 \geq x_2(x_1)$$

$$= \quad \min_{x_1 \geq 0} \{p_1 x_1 + p_2 x_2(x_1)\}$$

Since we're used to maximizing rather than minimizing to apply Topkis, we can write this as

$$\max_{x_1 \geq 0} \{-p_1 x_1 - p_2 x_2(x_1)\}$$

(a) *Show that when $X = \mathbb{R}^2_+$ and Hicksian demand is single-valued, this approach yields an alternate proof that Hicksian demand is downward-sloping in own price.*

(b) *Show that when $X = \mathbb{R}^2_+$ and Hicksian demand is single-valued, this approach yields an alternate proof that with only two goods, the goods must be substitutes.*

(c) *We could extend this approach to $X = \mathbb{R}^3_+$ by defining*

$$x_3(x_1, x_2) \quad = \quad \min\{x_3 \, : \, u(x_1, x_2, x_3) \geq \bar{u}\}$$

and solving

$$\max_{x_1, x_2 \geq 0} \{-p_1 x_1 - p_2 x_2 - p_3 x_3(x_1, x_2)\}$$

Would the Topkis-based proof that h_1 is decreasing in p_1 go through? Why or why not?

For part (a), the objective function $g(x_1, p_1) = -p_1 x_1 - p_2 x_2(x_1)$ has increasing differences in x_1 and $-p_1$, so Topkis' Theorem applies, telling us that x_1^* (in this case, Hicksian demand) is decreasing in p_1 if the problem has a unique solution.

For part (b), we can calculate $\frac{\partial g}{\partial p_2} = -x_2(x_1)$, and note that as long as x_2 is weakly decreasing in x_1, $\frac{\partial g}{\partial p_2}$ is increasing in x_1 and therefore g has increasing differences in x_1 (the decision variable) and p_2. Thus, x_1^* is increasing in p_2, so the goods must be substitutes.

To show that x_2 is weakly decreasing in x_1, it's easiest to assume that $u(x_1, x_2)$ is weakly increasing in x_1. In that case, the set $\{x_2 : u(x_1, x_2) \geq \bar{u}\}$ is getting larger as x_1 increases, so its minimum is falling; so $x_2(x_1)$ must be weakly decreasing in x_1.

(Even if we do not assume that u is everywhere weakly increasing in x_1, it must be weakly increasing in x_1 at the optimum, i.e., at the solution to the consumer's expenditure minimization problem. If u were strictly decreasing in x_1 at a point x with $u(x) \geq \bar{u}$, then $x' = (x_1 - \epsilon, x_2)$ would also satisfy $u(x') \geq \bar{u}$ for ϵ sufficiently small, and would be strictly cheaper than x. Thus, even if u were non-monotone, it would still be weakly increasing in x_1 at the "relevant" points. However, we only proved Topkis' Theorem for objective functions which satisfy single crossing globally, not just at select points, so the "simple" proof I had in mind would require the assumption that u is weakly increasing in x_1.)

For part (c), the main complication would be that we would now be maximizing over two choice variables, and we would need to ensure that the objective function was supermodular. Given the objective function, this would require that $-p_3 \frac{\partial^2 x_3(x_1,x_2)}{\partial x_1 \partial x_2} \geq 0$ everywhere; or else that $-p_3 \frac{\partial^2 x_3(x_1,x_2)}{\partial x_1 \partial x_2} \leq 0$ everywhere, since then we could flip the sign one of the two choice variables. If one of these two conditions could be shown – which would depend on the properties of the utility function – then the proof would extend.

4. **Monotone Selection Theorems (20 points)**

   (a) *For a one-dimensional choice set $X \subseteq \mathbb{R}$, prove the extension to Topkis' Theorem we stated (without proof) in class, commonly known as the Monotone Selection Theorem:*

   Theorem. *Let $X \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$, let $g : X \times T \to \mathbb{R}$, and let*

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x,t)$$

   Suppose that g has **strictly increasing differences** *– that is, that $x' > x$ implies $g(x',t) - g(x,t)$ is* strictly *increasing in t. Then for any $t' > t$ and any selection $x \in x^*(t)$ and $x' \in x^*(t')$, it must be that $x' \geq x$.*

   (b) *Prove the following multi-dimensional version of the Monotone Selection Theorem:*

   Theorem. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be a product set, with $X_i \subseteq \mathbb{R}$ for each i. Let $T \subseteq \mathbb{R}$, let $g : X \times T \to \mathbb{R}$, and let*

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x,t)$$

   Suppose that g is supermodular in X, has increasing differences in X and t, and has strictly *increasing differences in x_1 and t. Then for any $t' > t$ and any $x \in x^*(t)$ and $x' \in x^*(t')$, it must be that $x'_1 \geq x_1$.*

For part (a), it's easiest to prove this by contradiction. Let $t' > t$, suppose g has strictly increasing differences, let $x \in x^*(t)$ and $x' \in x^*(t')$, and suppose $x' < x$.

Since $x' \in x^*(t')$, we know that $g(x',t') \geq g(x,t')$, or

$$g(x,t') - g(x',t') \quad \leq \quad 0$$

Similarly, since $x \in x^*(t)$, we know $g(x,t) \geq g(x',t)$, or

$$g(x,t) - g(x',t) \quad \geq \quad 0$$

But since we're assuming $x > x'$ and g has strictly increasing differences, this means that since $t' > t$

$$g(x,t') - g(x',t') \quad > \quad g(x,t) - g(x',t) \quad \geq \quad 0$$

which gives us a contradiction since we had already seen this must be weakly negative. Thus, it must be that $x' \geq x$ to avoid the contradiction.

For part (b), it's again easiest to use proof by contradiction. Let $t' > t$, let $x \in x^*(t)$ and $x' \in x^*(t')$, and suppose $x'_1 < x_1$.

Since $x \in x^*(t)$, we know that $g(x,t) \geq g(y,t)$ for any $y \in X$, so

$$g(x,t) - g(x \wedge x', t) \quad \geq \quad 0$$

Next, we want to show that if g has increasing differences in X and t, and strictly increasing differences in x_1 and t, then

$$g(x,t') - g(x \wedge x', t') \quad > \quad g(x,t) - g(x \wedge x', t)$$

Let's let $y = x \wedge x'$, and define z^j to be a vector consisting of the first j elements of x followed by the last j elements of y, that is,

$$\begin{aligned}
z^m &= (x_1, x_2, x_3, \ldots, x_{m-1}, x_m) \\
z^{m-1} &= (x_1, x_2, x_3, \ldots, x_{m-1}, y_m) \\
z^{m-2} &= (x_1, x_2, x_3, \ldots, y_{m-1}, y_m) \\
&\vdots \\
z^2 &= (x_1, x_2, y_3, \ldots, y_{m-1}, y_m) \\
z^1 &= (x_1, y_2, y_3, \ldots, y_{m-1}, y_m) \\
z^0 &= (y_1, y_2, y_3, \ldots, y_{m-1}, y_m)
\end{aligned}$$

Then we can write

$$\begin{aligned}
g(x,t) - g(y,t) \quad = \quad g(z^m,t) - g(z^0,t) \quad &= \quad g(z^m,t) - g(z^{m-1},t) \\
&\quad + g(z^{m-1},t) - g(z^{m-2},t) \\
&\quad\quad \vdots \\
&\quad + g(z^2,t) - g(z^1,t) \\
&\quad + g(z^1,t) - g(z^0,t)
\end{aligned}$$

Next, note that since $y = x \wedge x'$, and the i^{th} element of $x \wedge x'$ is the min of x_i and x'_i and is therefore weakly lower than x_i, each difference $g(z^j,t) - g(z^{j-1},t)$ is the effect on g of a weak increase in the j^{th} element of its argument, holding the other $m-1$ elements constant. This means that if g has increasing differences in each x_j and t, then each of these differences is weakly increasing in t. Finally, note that in the last one, $g(z^1,t) - g(z^0,t)$, $x_1 > y_1$, because by assumption $x_1 > x'_1 = \min\{x_1, x'_1\}$; so if g has strictly increasing differences in x_1 and t, this last difference is *strictly* increasing in t. This means that for $t' > t$, we've proved that

$$g(x,t') - g(x \wedge x', t') \quad > \quad g(x,t) - g(x \wedge x', t) \quad \geq \quad 0$$

and therefore

$$g(x,t') - g(x \wedge x', t') \quad > \quad 0$$

To complete the proof, we note that since x' is optimal at t',

$$g(x',t') - g(x \vee x', t') \quad \geq \quad 0$$

and so adding the two,

$$g(x,t') - g(x \wedge x', t') + g(x',t') - g(x \vee x', t') \quad > \quad 0$$

or

$$g(x,t') + g(x',t') \quad > \quad g(x \wedge x', t') + g(x \vee x', t')$$

which contradicts g being supermodular in x.

5. **A Two-Factory Firm (20 points)**

Consider a company with one output, and production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$. *Suppose* f *is such that the firm's cost minimization problem always has a unique solution.*

(a) *Recall that* f *is* homothetic *if it is a monotonic transformation of a function which is homogeneous of degree* 1, *that is, if* $f(z) = h(g(z))$ *with* $g(\lambda z) = \lambda g(z)$ *and* h *strictly increasing.*

i. *Show that if* f *is homothetic, then the firm's conditional factor demand*

$$z(q,w) \quad = \quad \arg\min_{z \in \mathbb{R}^m_+} w \cdot z \quad \text{subject to} \quad f(z) \geq q$$

is increasing in q *for every input good.*

ii. *Show therefore that if* f *is homothetic, the firm's cost function*

$$c(q,w) \quad = \quad \min_{z \in \mathbb{R}^m_+} w \cdot z \quad \text{subject to} \quad f(z) \geq q$$

has increasing differences in q *and* w_i *for every input price* w_i.

(b) *Now suppose the firm sells its output in City A, but has two plants that can produce it, one in City A and one in Town B. The two plants use the same technology, represented by the homothetic production function* f, *but the inputs for each plant are purchased locally, and their prices vary in the two locations. Town B input prices tend to be lower, but the firm must also pay a transportation cost* t *to transport each unit of the good from Town B to City A. Thus, the cost of producing* q^A *units in the City A plant is*

$$c^A \quad = \quad c(q^A, w^A)$$

and the cost of producting q^B *units in the Town B plant is*

$$c^B \quad = \quad c(q^B, w^B) + tq^B$$

i. *Show that if the firm is a price taker in both input and output markets, its production in the City A plant is independent of Town B input prices and the transportation cost.*

ii. *Suppose now that the firm is a price taker in input markets but not in the output market, and therefore chooses output levels to maximize the objective function*

$$(q^A + q^B)P(q^A + q^B) - c(q^A, w^A) - c(q^B, w^B) - tq^B$$

where $P(\cdot)$ *is strictly decreasing. Suppose that* $P(q)$ *is differentiable and concave in* q, *and that the firm's problem has a unique solution at each price level. Prove what happens to* q^A *and* q^B *as* w^B_i *(the price of one of the input goods in Town B) rises.*

For part (a)i, consider the firm's cost minimization problem

$$\begin{aligned}
z(q,w) &= \arg\min_{z\in\mathbb{R}^m_+} w\cdot z \quad \text{subject to} \quad f(z)\geq q \\
&= \arg\min_{z\in\mathbb{R}^m_+} w\cdot z \quad \text{subject to} \quad h(g(z))\geq q \\
&= \arg\min_{z\in\mathbb{R}^m_+} w\cdot z \quad \text{subject to} \quad g(z)\geq h^{-1}(q) \\
&= \arg\min_{z\in\mathbb{R}^m_+} w\cdot z \quad \text{subject to} \quad g\left(\frac{z}{h^{-1}(q)}\right)\geq 1
\end{aligned}$$

Changing variables to $\tilde{z} = \frac{z}{h^{-1}(q)}$,

$$\begin{aligned}
\tilde{z}(q,w) &= \arg\min_{\tilde{z}\in\mathbb{R}^m_+} w\cdot \left(h^{-1}(q)\tilde{z}\right) \quad \text{subject to} \quad g(\tilde{z})\geq 1 \\
&= \arg\min_{\tilde{z}\in\mathbb{R}^m_+} h^{-1}(q)\,(w\cdot\tilde{z}) \quad \text{subject to} \quad g(\tilde{z})\geq 1
\end{aligned}$$

whose solution does not depend on q. Thus,

$$\tilde{z}(q,w) = \frac{1}{h^{-1}(q)} z(q,w)$$

is the same for all q; since h is strictly increasing, h^{-1} is strictly increasing, so

$$z(q,w) = h^{-1}(q)\tilde{z}(w)$$

is increasing in q in every dimension.

For part (a)ii, we can use Shepard's Lemma, which gives

$$\frac{\partial c(q,w)}{\partial w_i} = z_i(q,w)$$

which we just showed was increasing in q, so c has increasing differences in q and w_i for each i.

For part (b)i, if the firm is a price taker, we can write its problem as

$$\begin{aligned}
&\max_{q^A,q^B\geq 0} \left\{p(q^A+q^B) - c(q^A,w^A) - c(q^B,w^B) - tq^B\right\} \\
&= \max_{q^A\geq 0}\left\{pq^A - c(q^A,w^A)\right\} + \max_{q^B\geq 0}\left\{pq^B - c(q^B,w^B) - tq^B\right\}
\end{aligned}$$

so the firm's optimal production from each plant is independent of the problem it faces at the other plant, hence q^A is independent of w^B and t.

For part (b)ii, let $g(q^A, q^B, w_i^B)$ denote the firm's objective function, holding the other input prices (including t) fixed. Note that

$$\frac{\partial g}{\partial q^A} = P(q^A + q^B) + (q^A + q^B)P'(q^A + q^B) - c'(q^A, w^A)$$

and therefore

$$\frac{\partial^2 g}{\partial q^A \partial q^B} = P'(q^A + q^B) + P'(q^A + q^B) + (q^A + q^B)P''(q^A + q^B)$$

Since P is decreasing and concave, $P' \leq 0$ and $P'' \leq 0$, so this is everywhere weakly negative. Thus, to make the problem supermodular, we'll "flip the sign" of q^A, and consider the firm choosing $-q^A$ and q^B, knowing the objective function is now supermodular.

(If P is not twice-differentiable, this still works, because $\frac{\partial g}{\partial q^A}$ given above is decreasing in q^B, since P is decreasing, P' is negative, and P' is decreasing since P is concave.)

Now,

$$\frac{\partial g}{\partial q^B} = P(q^A + q^B) + (q^A + q^B)P'(q^A + q^B) - \frac{\partial c(q^B, w^B)}{\partial q^B}$$

and so

$$\frac{\partial^2 g}{\partial q^B \partial w_i^B} = -\frac{\partial^2 c(q^B, w^B)}{\partial q^B \partial w_i^B}$$

We showed in part a.ii above that the cost function for each plant has increasing differences in quantity and input prices, so $\frac{\partial^2 c(q^B, w^B)}{\partial q^B \partial w_i^B} \geq 0$ and therefore $\frac{\partial^2 g}{\partial q^B \partial w_i^B} \leq 0$, and we will therefore want to think of the parameter in question as $-w_i^B$.

From above, $\frac{\partial g}{\partial(-q^A)} = -\frac{\partial g}{\partial q^A}$ does not depend on w_i^B, we can say that $\frac{\partial g}{\partial(-q^A)}$ is weakly increasing in $-w_i^B$.

So if we think of the firm's objective function as a function of the choice variables $-q^A$ and q^B and the parameter $-w_i^B$, the problem is supermodular and has increasing differences, so Topkis' Theorem applies. Topkis tells us that $(-q^A, q^B)$ is increasing in $-w_i^B$. Since the firm's problem has a unique solution, this means that as w_i^B rises, $-w_i^B$ falls, so $-q^A$ and q^B fall – meaning that q^A goes up and q^B goes down in response to an increase in w_i^B.