1. **Aggregate Hicksian Demand (30 points)**

Suppose there are $n \geq 2$ consumers, and consumer $i \in \{1, 2, \ldots, n\}$ has indirect utility

$$v_i(p, w_i) \quad = \quad a_i(p) + \frac{w_i}{c(p)}$$

This is the same formulation you've seen before, just with $c(p) = 1/b(p)$, because this will simplify algebra later in the problem. You've seen that this implies the combined Marshallian demand $\sum_i x^i(p, w_i)$ of the n consumers is the same as the demand of a single "representative consumer" with indirect utility

$$V(p, W) \quad = \quad A(p) + \frac{W}{c(p)}$$

where $A(p) = \sum_i a_i(p)$ and $W = \sum_i w_i$.

(a) *Calculate the expenditure function $e_i(p, u_i)$ for consumer i, and the expenditure function $E(p, U)$ for the representative consumer, and show that $\sum_i e_i(p, u_i) = E(p, \sum_i u_i)$.*

Since we know that for any valid indirect utility function and expenditure function,

$$v(p, e(p, u)) \quad = \quad u$$

we can plug in and find

$$u_i \; = \; v_i(p, e_i(p, u_i)) \; = \; a_i(p) + \frac{e_i(p, u_i)}{c(p)} \qquad \longrightarrow \qquad e_i(p, u_i) \; = \; c(p)\,(u_i - a_i(p))$$

and similarly

$$E(p, U) \quad = \quad c(p)(U - A(p))$$

Then

$$\sum_i e_i(p, u_i) \;=\; \sum_i c(p)(u_i - a_i(p)) \;=\; c(p)\left(\sum_i u_i - \sum_i a_i(p)\right)$$

$$= \; c(p)\left(\sum_i u_i - A(p)\right) \; = \; E\left(p, \sum_i u_i\right)$$

(b) *Calculate consumer i's Hicksian demand $h_1^i(p, u_i)$ for good 1, and the representative consumer's $H_1(p, U)$, and show that $\sum_i h_1^i(p, u_i) = H_1(p, \sum_i u_i)$.*

We can calculate i's Hicksian demand for good 1 as

$$h_1^i(p, u_i) = \frac{\partial e^i}{\partial p_1}(p, u_i) = \frac{\partial c(p)}{\partial p_1} u_i - \frac{\partial c(p)}{\partial p_1} a_i(p) - c(p) \frac{\partial a_i(p)}{\partial p_1}$$

and similarly

$$H_1(p, U) = \frac{\partial E}{\partial p_1}(p, U) = \frac{\partial c(p)}{\partial p_1} U - \frac{\partial c(p)}{\partial p_1} A(p) - c(p) \frac{\partial A(p)}{\partial p_1}$$

We can then calculate

$$\begin{aligned} \sum_i h_1^i(p, u_i) &= \frac{\partial c(p)}{\partial p_1} \sum_i u_i - \frac{\partial c(p)}{\partial p_1} \sum_i a_i(p) - c(p) \frac{\partial (\sum_i a_i(p))}{\partial p_1} \\ &= H_1\left(p, \textstyle\sum_i u_i\right) \end{aligned}$$

(c) *Show that if the price of good 1 falls from p_1^0 to p_1^1, the sum of the Compensating Variation of the n consumers is the same as the CV of the representative consumer.*

Compensating Variation is calculated as

$$CV = \int_{p_1^1}^{p_1^0} h_1(p, u^0) dp_1$$

where $u^0 = v(p^0, w)$, so if we let $u_i^0 = v_i(p^0, w_i)$,

$$\sum_i CV_i = \sum_i \int_{p_1^1}^{p_1^0} h_1^i(p, u_i^0) dp_1 = \int_{p_1^1}^{p_1^0} \left(\sum_i h_1^i(p, u_i^0) \right) dp_1 = \int_{p_1^1}^{p_1^0} H_1^i \left(p, \sum_i u_i^0 \right) dp_1$$

Now, since $V(p^0, \sum_i w_i) = \sum_i v_i(p^0, w_i) = \sum_i u_i^0$,

$$\int_{p_1^1}^{p_1^0} H_1^i \left(p, \sum_i u_i^0 \right) dp_1 = CV_{\text{representative consumer}}$$

giving the result.

Alternatively, we can use the definition

$$CV = e(p^0, u^0) - e(p^1, u^0)$$

where $u^0 = v(p^0, w)$. Given the answer to 1(a) above,

$$\begin{aligned}
\sum_i CV_i &= \sum_i \left(c(p^0)(u_i^0 - a_i(p^0)) - c(p^1)(u_i^1 - a_i(p^1))\right) \\
&= c(p^0)\left(\sum_i u_i^0 - A(p^0)\right) - c(p^1)\left(\sum_i u_i^0 - A(p^1)\right) \\
&= E\left(p^0, \sum_i u_i^0\right) - E\left(p^1, \sum_i u_i^0\right)
\end{aligned}$$

Since

$$U^0 \quad = \quad \sum_i u_i^0 \quad = \quad \sum_i v_i(p^0, w_i) \quad = \quad V\left(p^0, \sum_i w_i\right)$$

is the initial indirect utility of the representative consumer, $E(p^0, U^0) - E(p^1, U^0) = \sum_i CV_i$ is the CV of the representative consumer.

2. **Zero Demand (20 points)**

Suppose preferences are locally non-satiated. Show that if Marshallian demand for good i is 0 at prices p, it can't become positive when the price of good i rises.

(Formally, if $p'_i > p_i$ and $p'_j = p_j$ for all $j \neq i$, show that if there's any *$x \in x(p,w)$ such that $x_i = 0$, then at* every *$x' \in x(p',w)$, $x'_i = 0$.)*

This is easiest to prove by contradiction. Suppose this were false – that there were some $x \in x(p,w)$ with $x_i = 0$, and some $x' \in x(p',w)$ with $x'_i > 0$. Since preferences are LNS, Walras' Law implies $p \cdot x = p' \cdot x' = w$.

The key thing to notice is that since $x_i = 0$ and $p'_j = p_j$ for every $j \neq i$, $p' \cdot x = p \cdot x = w$; and since $x'_i > 0$ and $p'_i > p_i$, $p \cdot x' < p' \cdot x' = w$. We can then argue it a few different ways:

- First, we can argue directly: since $x \in B(p',w)$ and $x' \in x(p',w)$, $x' \succsim x$; then since $x' \in B(p,w)$ and $x \in x(p,w)$, $x' \in x(p,w)$ as well; but since $p \cdot x' < w$, this would contradict Walras' Law.
- Alternatively, we could make basically the same argument using WARP: since x and x' are both in both $B(p,w)$ and $B(p',w)$, with $x \in C(B(p,w))$ and $x' \in C(B(p',w))$, we must have $x' \in C(B(p,w))$, which would again contradict Walras' Law.
- Finally, we could make basically the same argument using GARP: since $x \in x(p,w)$ and $p \cdot x' \leq w$, $x \succsim^D x'$ and therefore $x \succsim^R x'$; but since $x' \in x(p',w)$ and $p' \cdot x < w$, $x' \succ^D x$, which would contradict GARP.

I would give partial credit for someone who tried to argue using the Slutsky Equation. One could note that

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - x_i \frac{\partial x_i}{\partial w}$$

and that $\frac{\partial h_i}{\partial p_i} \leq 0$ (always) and the second term is zero when $x_i = 0$, so $\frac{\partial x_i}{\partial p_i} \leq 0$ whenever $x_i = 0$. However, this is not worth full credit, because (a) the problem didn't specify that demand was single-valued or differentiable, and (b) having derivative zero doesn't imply something is not increasing. (The function $f(z) = z^2$ has derivative zero whenever it's equal to zero, and yet is strictly increasing on $\Re_+$.)

3. **The Slutsky Equation (20 points)**

Suppose demand is single-valued and differentiable. Write the Slutsky equation for the change in demand for good 1 when the price of good 2 changes. For each of the following cases, explain whether you can predict the sign of $\frac{\partial x_1}{\partial p_2}$*, and why or why not:*

(a) *goods 1 and 2 are complements, good 1 is a normal good and good 2 is an inferior good*

(b) *goods 1 and 2 are substitutes, and goods 1 and 2 are both normal goods*

(c) *there are only two goods and good 2 is a Giffen good*

The Slutsky equation is

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial h_1}{\partial p_2} - x_2 \frac{\partial x_1}{\partial w}$$

For the three cases:

(a) If 1 and 2 are complements, $\frac{\partial h_1}{\partial p_2} < 0$, and if good 1 is normal, $\frac{\partial x_1}{\partial w} > 0$, so $\frac{\partial x_1}{\partial p_2} < 0$

(b) If 1 and 2 are substitutes, $\frac{\partial h_1}{\partial p_2} > 0$, and if 1 is normal, $\frac{\partial x_1}{\partial w} > 0$; so $\frac{\partial x_1}{\partial p_2}$ could be either positive or negative, depending on which effect (substitution or wealth) dominates

(c) It's not apparent from the Slutsky equation, but when good 2 is a Giffen good, $\frac{\partial x_1}{\partial p_2} < 0$. This is because by $\frac{\partial x_2}{\partial p_2} > 0$ (the definition of a Giffen good), and (assuming preferences are LNS) $p_1 x_1 + p_2 x_2 = w$ must continue to hold as p_2 increases.

(With two goods, they must be substitutes, so $\frac{\partial h_1}{\partial p_2} > 0$; if good 2 is a Giffen good, then it is inferior, and with only two goods, this means good 1 must be normal, so $\frac{\partial x_1}{\partial w} > 0$ as well, which is why the result isn't obvious from the Slutsky equation.)

(Please note that the solution to this part was originally incorrect.)

4. **Intertemporal Choice (30 points)**

Consider a simple model of intertemporal choice, within our static utility-maximization framework. The "goods" 1 through k represent consumption in each of k different time periods, and for $x \in \Re^k_+$,

$$u(x) \quad = \quad \sum_{i=1}^{k} \beta^{i-1} v(x_i)$$

where $\beta \in (0,1)$ and $v : \Re_+ \to \Re$ is strictly increasing, strictly concave, differentiable, and satisfies an "Inada condition" $\lim_{z \to 0} v'(z) = +\infty$. The current "price" of good i is

$$p_i \quad = \quad \frac{1}{(1+r)^{i-1}}$$

reflecting the fact that the consumption good costs the same in each period, but that money saved today grows at an interest rate r until it is used to purchase the consumption good in period i.

(a) *If $w > 0$, show that Marshallian demand for every good is strictly positive, $x(p,w) \gg 0$.*

Briefly, when $\frac{\partial u}{\partial x_i} = \infty$ whenever $x_i = 0$, it will never be optimal to consume none of good i given positive wealth, because the first small amount gives unboundedly large marginal utility per dollar.

More formally, we know the Kuhn-Tucker conditions are

$$\frac{\partial u}{\partial x_i}(x) \quad = \quad \lambda p_i - \mu_i$$

With positive wealth and LNS preferences, we'll definitely consume *something*, so there's some good j where $x_j > 0$ and therefore $\mu_j = 0$, so

$$\lambda \quad = \quad \frac{1}{p_j} \frac{\partial u}{\partial x_j}(x) \quad = \quad \frac{1}{p_j} \beta^{j-1} v'(x_j)$$

which, since $x_j > 0$, is finite. Thus, if there was any good consumed at zero quantity, it would require

$$\frac{\partial u}{\partial x_i}(x) \quad = \quad \beta^{i-1} v'(0) \quad = \quad \lambda p_i - \mu_i$$

which is impossible since $v'(0) = \infty$ but $\lambda p_i - \mu_i$ is finite.

(b) *Show that if $\beta(1+r) > 1$, consumption increases from period to period ($x_{i+1} > x_i$), while if $\beta(1+r) < 1$, consumption decreases from period to period.*

Knowing that $x \gg 0$, we therefore know that $\mu_i = 0$ for all i, so the Kuhn-Tucker conditions are now

$$\frac{\partial u}{\partial x_i}(x) \quad = \quad \lambda p_i \qquad \longrightarrow \qquad \beta^{i-1} v'(x_i) \quad = \quad \frac{1}{(1+r)^{i-1}} \lambda$$

or

$$\left(\beta(1+r)\right)^{i-1} v'(x_i) \quad = \quad \lambda$$

for every i. If $\beta(1+r) > 1$, then $\left(\beta(1+r)\right)^{i-1}$ is increasing in i, and $v'(x_i)$ must therefore decrease in i; since v is strictly concave, this means x_i must increase in i. If $\beta(1+r) < 1$, then $\left(\beta(1+r)\right)^{i-1}$ is decreasing in i, and $v'(x_i)$ must therefore increase in i, meaning x_i must decrease in i.

(Alternatively, one could divide consecutive Kuhn-Tucker conditions and find

$$\beta(1+r) \quad = \quad \frac{v'(x_i)}{v'(x_{i+1})}$$

and reach the same conclusion: if $\beta(1+r) > 1$, this requires $v'(x_i) > v'(x_{i+1})$ and therefore $x_i < x_{i+1}$, and vice versa.

(c) *Now suppose one period has passed, and the consumer has consumed $x_1(p, w)$ according to plan. Show that his choices are* time-consistent*: that if he were to take his remaining budget $\bar{w} = (1+r)(w - p_1 x_1)$ and the current prices $p'_i = \frac{1}{(1+r)^{i-2}}$ and solve his forward-looking consumer problem*

$$\max \sum_{i=2}^{k} \beta^{i-2} v(x_i) \qquad \textit{subject to} \qquad \sum_{i=2}^{k} p'_i x_i \leq \bar{w}$$

his new choices would match the original solution to his consumer problem.

One way to do this is to consider the original problem as two nested optimization problems: in the inner problem, optimally choose $(x_2, \ldots, x_k)$ given a fixed choice of x_1, and in the outer problem, choose x_1 optimally. That is, we can restate the original consumer problem as

$$\max_{x_1 \in [0, \frac{w}{p_1}]} \left\{ v(x_1) + \left(\max_{x_2, \ldots, x_k} \sum_{i=2}^{k} \beta^{i-1} v(x_i) \quad \text{subject to} \quad \sum_{i=2}^{k} p_i x_i \leq w - p_1 x_1 \right) \right\}$$

If we let y denote the last $k-1$ goods, and $y(x_1)$ as the solution to the inner problem given the outer problem, then $x^* = (x_1^*, x_2^*, \ldots, x_k^*) = (x_1^*, y(x_1^*))$. (At the time all of the

x_i are being chosen, the last $k-1$ must be the optimal choice given the first one, or the whole bundle would not be optimal.) Thus, we know that

$$(x_2^*, \ldots, x_k^*) \quad \text{solves} \quad \max_{x_2,\ldots,x_k} \sum_{i=2}^{k} \beta^{i-1} v(x_i) \quad \text{subject to} \quad \sum_{i=2}^{k} p_i x_i \leq w - p_1 x_1^*$$

Now, let $(z_2, \ldots, z_k)$ be the solution to the "one-period-later problem" described in the question. By definition,

$$(z_2, \ldots, z_k) \quad \text{solves} \quad \max_{z_2,\ldots,z_k} \sum_{i=2}^{k} \beta^{i-2} v(z_i) \quad \text{subject to} \quad \sum_{i=2}^{k} p_i' z_i \leq (1+r)(w - p_1 x_1^*)$$

where $p_i' = \frac{1}{(1+r)^{i-2}}$ and $p_i = \frac{1}{(1+r)^{i-1}}$. But we can rewrite the latter problem as

$$\frac{1}{\beta} \max_{z_2,\ldots,z_k} \sum_{i=2}^{k} \beta^{i-1} v(z_i) \quad \text{subject to} \quad \sum_{i=2}^{k} p_i z_i \leq w - p_1 x_1^*$$

making it clear that it's exactly equivalent to the original "inner" problem of choosing $\{x_2, \ldots, x_k\}$ given x_1, and therefore has the same solution.

Alternatively, one could focus on the first-order (Kuhn-Tucker) conditions the solutions to each must satisfy. As noted above, the first problem is characterized by the conditions

$$(\beta(1+r))^{i-1} v'(x_i) \quad = \quad \lambda$$

for each $i = 1, 2, \ldots, k$, along with the budget constraint $\sum_i p_i x_i = w$. The new problem is characterized by

$$(\beta(1+r))^{i-2} v'(x_i) \quad = \quad \lambda$$

for each $i = 2, \ldots, k$, along with the new budget constraint. The two are equivalent (up to having a different value of λ), so if the budget constraints are the same, the two problems will have the same solution. We can write the two budget constraints as

$$\sum_{i=2}^{k} \frac{x_i}{(1+r)^{i-1}} \quad \leq \quad w - x_1 \qquad \text{and} \qquad \sum_{i=2}^{k} \frac{x_i}{(1+r)^{i-2}} \quad \leq \quad (1+r)(w - x_1)$$

respectively, showing they really are identical, so the two problems have the same solution.