Lecture 8: Properties of Preferences

1 Where we are

- We kicked off consumer theory by thinking about preferences over general choice sets, and showed the more-or-less equivalence between complete and transitive preferences and choice rules satisfying the Weak Axiom of Revealed Preferences
- In fact, you'll flesh out the formal results on the homework due next Monday

- And we defined what it means for a utility function to represent preferences, and showed if $X \subseteq \mathbb{R}^k$ and preferences are complete, transitive, continuous, and monotone, then they're represented by a continuous utility function
- (We can generalize this beyond monotone preferences, that just made the proof more manageable)

- Today, we'll focus on further assumptions people often make about preferences, and what they imply about the corresponding utility funcions

- First... any questions?

2 Utility

- We proved last time that for a pretty broad class of preferences, a utility representation exists
- A couple of things to note about that

- First of all, we are *not* interpreting the utility function as having any meaning,

  other than being a convenient way to represent an ordinal preference relation
- That is, it's tempting to think of utility as having meaning –

  happiness, or contentment, or something like that
- It's even tempting to imagine measuring it –

  sticking a couple probes in your brain and noticing that when you eat an apple, the meter goes up to 7, but when you eat ice cream, it goes to 12
- That is *not* what we're doing
- For right now, the cardinal values of utility are not meant to signify *anything* –

  all a utility function is, is a way to represent ordinal preferences
- That is, we're thinking of a utility function as an "as if" model –

  people behave *as if* they were maximizing this particular function, given their constraints –

  and that's it

- Relatedly, for any utility function that represents your preferences,
  there are a trillion other ones that would work just as well
- If $\succsim$ is represented by $u : X \rightarrow \mathbb{R}$,
  it's also represented by $\phi \circ u$ for any strictly increasing function $\phi : \mathbb{R} \rightarrow \mathbb{R}$
- So if u represents $\succsim$, so does $u + 5$, and $30u$, and $5u^3 - 15u^2 + 15u - 5$
- Later on, when we get into choice under uncertainty,
  we'll need to take the utility function more seriously;
  but for choice theory, we don't

- And relatedly, once we have a numerical representation of preferences,
  it's tempting to add them up across individuals
  and think about a benevolent dictator trying to maximize that sum
- But since there isn't a unique utility representation for each person, this is problematic
- What if we choose to represent my preferences by a utility function with a really wide range,
  and everyone elses with a utility function with range $[0, 0.001]$?
- These are all valid representations of preferences,
  but they would suggest that social utility is maximized by making me as happy as possible

- For the moment, we're not claiming utility functions are valid for interpersonal comparisons;
  just that they're a numerical way to represent an individual's preferences

2.1 Up next: properties of preferences

- Up next, we'll look at different properties that preferences might satisfy,

  and the implications they have for the utility representation

- In fact, once you're reading papers and writing papers,

  you'll typically just start with the utility function,

  so why look at conditions on underlying preferences?

- Again, I'm thinking of utility functions as an "as-if" model,

  and preferences being the true underlying primitive that people actually "have"

- So it's natural to wonder, *if* you're going to make particular assumptions about utility functions to make the model work,

  what conditions on preferences that implies,

  to see whether they're reasonable

- So, on to additional assumptions/restrictions that are sometimes placed on preferences,

  for two reasons:

  - more restrictions on preferences can give stronger theoretical results
  - more restrictions on preferences can give us more structure to do "better" empirical work

3 Monotone Preferences and such

- One we already saw:

  preferences are **monotone** if $x \gg y$ implies $x \succ y$

  (more of every good makes you strictly better off)

- we can also show that monotonicity plus continuity implies that $x \geq y$ implies $x \succsim y$:

  - if $x \geq y$, then $x + \frac{1}{n}e \gg y$ for every $n > 0$
  - by monotonicity, $x + \frac{1}{n}e \succ y$, and $x + \frac{1}{n}e \to x$,

    so by continuity, $x \succsim y$

- so if u represents monotone preferences, it must be at least weakly increasing in each good

- preferences are **strongly monotone** if $x > y$ implies $x \succ y$

  (more of any one good makes you strictly better off)

- if $\succsim$ is strongly monotone, any utility function u must be strictly increasing in each good

- Preferences are **locally non-satiated** if there's always a nearby point that's better

  Formally, preferences are LNS if $\forall x \in X$ and $\varepsilon > 0$,

  there exists some $x' \in X$ such that $\|x' - x\| \leq \varepsilon$ and $x' \succ x$

- (So wherever you are in X, there's always a nearby point that's strictly better!)

- Local non-satiation is an important property when we introduce wealth and prices: it's the weakest assumption that guarantees consumers will always spend their entire budget, which ends up simplifying the consumer problem
- Not surprisingly, if preferences $\succsim$ are represented by a utility function u, then $\succsim$ is LNS if and only if u has no local maxima

- LNS is a very popular assumption, because it's a fairly weak assumption that buys a lot
- Without it, rational choice theory places no restrictions on behavior, and isn't falsifiable – because whatever I see you doing, you might just be indifferent among everything and choosing randomly
- LNS is enough of an assumption for the theory to have some bite – to make some predictions which could, in principle, be violated by data
- And in fact, as we'll see, LNS alone does lead to some fairly sharp predictions

4 Convex Preferences

- Preferences are **convex** if the set of points weakly better than a given point y is convex
- formally, assuming the choice set X is convex,

  preferences are convex if for any $t \in (0,1)$, $x \succsim y$ and $x' \succsim y$ implies $tx + (1-t)x' \succsim y$
- (Also means that if $x \sim x'$, then any convex combination $tx + (1-t)x'$ is at least as good)
- preferences are **strictly convex** if $x \succsim y$ and $x' \succsim y$ imply $tx + (1-t)x' \succ y$

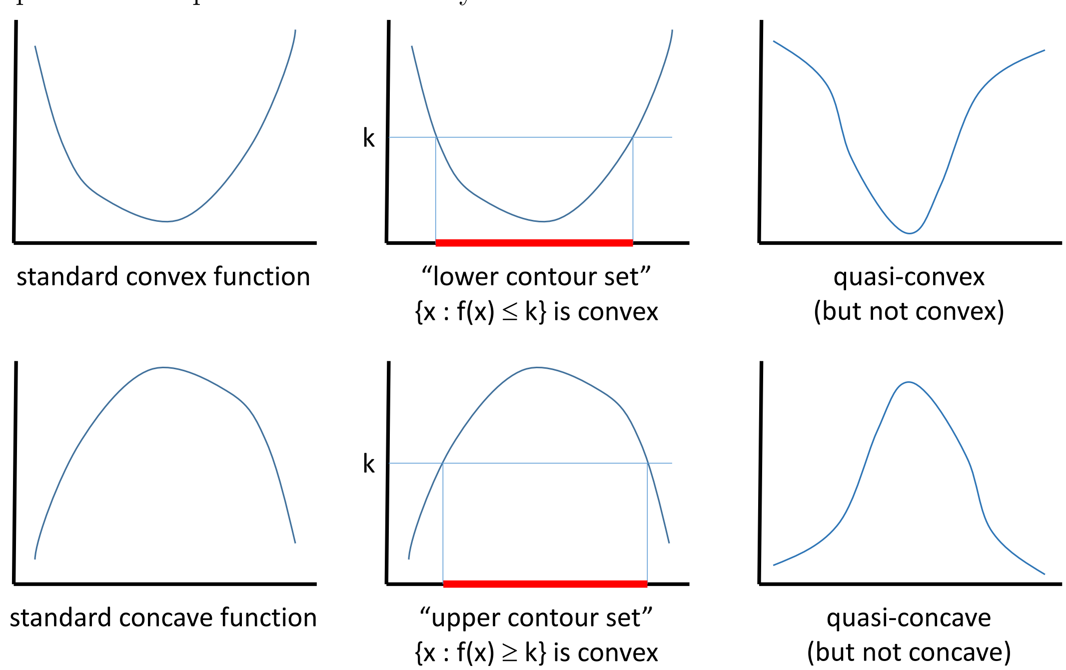
  for any $t \in (0,1)$ and $x' \neq x$

- To relate convex preferences to properties of a utility function,

  we need a couple new definitions
- A function $u : \mathbb{R}^k \to \mathbb{R}$ is **quasi-concave** if its upper contour sets are convex,

  that is, if the set $\{x : u(x) \geq z\}$ is a convex set for each constant z
- And a function is **quasi-convex** if its *lower* contour sets are convex,

  that is, if the set $\{x : u(x) \leq z\}$ is a convex set for each constant z
- How are quasi-concave and quasi-convex related to concave and convex?
  - q-convex and q-concave are basically the "ordinal" versions of convex and concave



- Back to convex preferences
- If $\succsim$ is convex, then any utility function representing $\succsim$ is quasi-concave
- (If $\succsim$ is convex,

  the set $\{x : x \succsim y\}$ is a convex set,

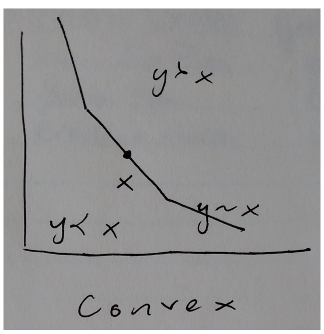
  so the set $\{x : u(x) \geq u(y)\}$ is a convex set,

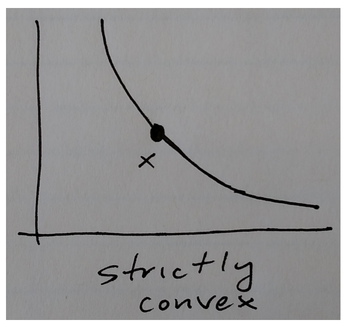
  which is exactly quasi-concavity)

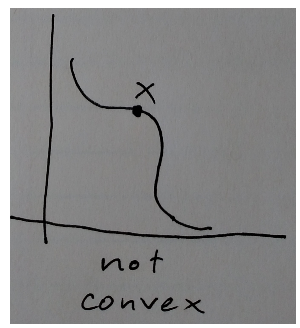
- If $\succsim$ is *strictly* convex, then u is strictly quasi-concave,

  meaning its upper contour sets are strictly convex –

  meaning, if x and x' are in the set, then $tx + (1-t)x'$ is in the interior of the set.[1]







- convex preferences are sometimes justified by the idea "people have a preference for variety"
- if x is pretty good, and x' is pretty good, then some mixture of x and x' should be better
- whether you buy this argument might depend on the level it's applied at
  - if X is "things you could eat for dinner tonight",

    then you might like steak or sushi but not a mixture
  - if X is food consumption over a month,

    maybe you prefer some steak and some sushi to just eating one or the other
  - this assumption is often used with highly aggregated goods – where the dimensions are "food", "clothing", etc. – or with consumption at different times

- convex prefs are popular in part because they're related to the utility function being concave,

  and if you're trying to maximize a function, it's very convenient for it to be concave;

  it's a powerful assumption – it often buys a lot – but it often seems like a leap of faith

[1]Strict quasiconvexity is a nice property because quasiconvexity is weaker than convexity, but still sufficient for the first-order conditions of a problem to indicate a minimum; likewise, strict quasi-concavity is sufficient for the FOC to indicate a maximum.

5 Separability

- Recall we're in $\mathbb{R}^k$, where each "dimension" is the quantity of a single good we could consume
- Depending on what these goods are,

  there are often times when some dimensions will intuitively seem independent of others
- For example, suppose $X = \mathbb{R}^9_+$, where $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ and...
  - x_1 is the amount of sushi you eat each month
  - x_2 is the amount of steak
  - x_3 is the amount of chicken
  - x_4 is the amount of cheese curds
  - x_5 is the number of Honda Accords you own
  - x_6 is the number of BMW 325i's you own
  - x_7 the number of motorcycles
  - x_8 the number and quality of condos, and x_9 the number and quality of houses
- It might seem sensible that your relative preferences for steak, sushi and chicken might not be affected by which car you drive or where you live
- We can formalize this
- Write the choice set as a product set, $X = Y \times Z$, where, for example,

  $Y = \mathbb{R}^4_+$ (representing the first four goods) and $Z = \mathbb{R}^5_+$ (representing the rest)
- We say our preferences over Y *do not depend on* z if the following holds:

  for any $y, y' \in Y$ and $z, z' \in Z$, $(y, z) \succsim (y', z)$ if and only if $(y, z') \succsim (y', z')$

  That is, our preferences between y and y' are the same, regardless of what z we're consuming
- Basically, if you prefer sushi to chicken when you're driving an Accord and living in a condo, you still prefer sushi to chicken when you're riding a motorcycle and living in a house

- (Note that we're not talking about money tradeoffs here –

  we haven't yet introduced budgets, this is purely preferences.

  You might choose to drive a BMW and eat chicken cause you spent so much on your car;

  but you would still prefer sushi to chicken, even when you're driving a BMW.)

- (Also note that the definition is not symmetric –

  preferences over z can depend on y even if preferences over y don't depend on z.

  Maybe you always prefer steak to chicken,

  and prefer red wine to white with steak and white wine to red with chicken.

  This just says that even if you're drinking white wine, you'd still rather eat steak with it.)

- Now, separability gives us the following:

- **Proposition.** Suppose preferences on $Y \times Z$ are represented by a utility function $u(y, z)$. Then preferences over y do not depend on z if and only if there exist functions $v : Y \to \mathbb{R}$ and $U : \mathbb{R} \times Z \to \mathbb{R}$ such that

  1. $u(y, z) = U(v(y), z)$, and
  2. U is increasing in its first argument

- So, if preferences over foods don't depend on consumption of transportation or housing,

  we can decompose the utility function into two pieces:

  a food-only utility function v,

  and an aggregator that combines "food utility" with consumption of other goods

- (Note that this isn't just saying our preferences have a utility representation like this;

  it's saying that *every* utility function can be broken down in this way!)

- Let's prove it!
  - One direction is easy: if $u(y,z) = U(v(y),z)$ with U strictly increasing, then

    $$(y,z) \succsim (y',z) \quad \leftrightarrow \quad u(y,z) \geq u(y',z) \quad \leftrightarrow \quad U(v(y),z) \geq U(v(y'),z) \quad \leftrightarrow \quad v(y) \geq v(y')$$

    which doesn't depend on z;
    so if preferences are represented this way, preferences over Y don't depend on z
  - The hard part is the other direction

  - So suppose our preferences over y don't depend on z,
    and we have some utility function $u : Y \times Z \rightarrow \mathbb{R}$ that represents them
  - We'll find functions v and U, show that $U(v(y),z) = u(y,z)$ for every (y,z),
    and show U is increasing in its first argument

  - To do this, fix some element $z_0 \in Z$, and define $v(y) = u(y,z_0)$
  - For any t in the range of v, there exists some y such that $v(y) = t$;
    choose one, and call it $\hat{y}(t)$, so by definition, $v(\hat{y}(t)) = t$
  - So now $\hat{y}$ is a mapping from a subset of the reals to Y,
    with $\hat{y}(t)$ being a a consumption bundle such that $u(y,z_0) = t$

  - Now define $U(t,z) = u(\hat{y}(t),z)$ as a mapping from $\mathbb{R} \times Z \rightarrow \mathbb{R}$

  - Now, we don't need to prove that $U(v(y),z)$ represents preferences –
    all we need to show is that for every (y,z),

    $$U(v(y),z) = u(y,z)$$

    because we already know that u represents preferences

– So now we need to show that $U(v(y), z) = u(y, z)$

– We defined $\hat{y}$ such that $u(\hat{y}(t), z_0) = t$,
and if we pick $t = v(y)$ and plug that in,

$$u(\hat{y}(v(y)), z_0) \quad = \quad v(y) \quad = \quad u(y, z_0)$$

so we know that for any $y \in Y$,

$$(\hat{y}(v(y)), z_0) \quad \sim \quad (y, z_0)$$

– If preferences over y don't depend on z, then this means for any other value of z,

$$(\hat{y}(v(y)), z) \quad \sim \quad (y, z)$$

as well, meaning

$$u(\hat{y}(v(y)), z) \quad = \quad u(y, z)$$

– But we defined our aggregator function U by $U(t, z) = u(\hat{y}(t), z)$, so this becomes

$$U(v(y), z) \quad = \quad u(y, z)$$

which was most of what we needed to prove!

– All that's left is showing that U is strictly increasing in its first argument
– Suppose not – i.e., suppose there was some z, y, and y' such that $v(y) > v(y')$ but $U(v(y), z) \leq U(v(y'), z)$
– Since $U(v(y), z) = u(y, z)$, this would mean $u(y, z) \leq u(y', z)$, or $(y, z) \precsim (y', z)$
– However, $U(v(y), z_0) = u(y, z_0) = v(y)$, so if $v(y) > v(y')$, $(y, z_0) \succ (y', z_0)$
– Together, these would violate our assumption that preferences over y don't depend on z
– That ends the proof.

- so that's the result – if preferences over some subset of the choice space are independent of consumption in the other dimensions,

  then for *any* utility representation of preferences,

  we can decompose it into a "sub-utility" function for the first subspace,

  and a function that combines that sub-utility with the rest of the consumption plan

- Like we said, we can have preferences over y that don't depend on z,

  even if preferences over z do depend on y

- but if both parts are independent of each other, we can apply the theorem a couple of times, and get a stronger characterization

- Suppose $x = (w_1, w_2, w_3, w_4, y_1, y_2, y_3, z_1, z_2)$,

  where w are types of food, y are types of transportation, and z are types of housing

- And suppose that preferences over w don't depend on y or z,

  preferences over y don't depend on w or z, and preferences over z don't depend on w or y

- Then any utility representation of preferences must be of the form

$$u(w, y, z) \quad = \quad U(u_1(w), u_2(y), u_3(z))$$

  where each u_i gives a real number – your "food utility", "car utility", and "housing utility" – and U is strictly increasing in all three of them

- separability is nice if we want to do empirical work on, say, preferences over laundry detergent brands, and don't want to have to ask people about their house and car

- if Y is your consumption of the set of goods we're studying – several different brands of detergent, soap, and fabric softener – and Z is everything outside of our study – housing, hard drugs, etc. – then if preferences over Y don't depend on z, life is a lot easier

- (even with separable preferences, what people *spend* on housing may still affect how much they spend on laundry detergent; but given how much they choose to spend on the Y goods, how they allocate that budget would be independent of their housing choice.)

6 Homothetic preferences

- Preferences are called *homothetic* if for any $\lambda > 0$,

$$x \quad \succsim \quad y \qquad \leftrightarrow \qquad \lambda x \quad \succsim \quad \lambda y$$

- this means your preferences between different goods "scale up and down" – if you prefer x to y, you prefer $2x$ to $2y$ and $\frac{1}{2}x$ to $\frac{1}{2}y$
- you'll show later on that if preferences are homothetic (and continuous), there's a utility representation that's homogeneous of degree 1, and you'll see homothetic preferences lead to some other nice results

7 Quasilinear preferences

- One last property to discuss: quasilinear preferences
- **Proposition.** Suppose $X = \mathbb{R}_+ \times Y$, and preferences $\succsim$ are complete and transitive, and there's a "worst element" $\underline{y} \in Y$ such that $(0, y) \succsim (0, \underline{y})$ for all $y \in Y$. Suppose also that:

  1. the first good is valuable – $(a, \underline{y}) \succsim (a', \underline{y})$ if and only if $a \geq a'$
  2. compensation is possible – for every $y \in Y$, $\exists t \geq 0$ such that $(0, y) \sim (t, \underline{y})$
  3. there are "no wealth effects" – $(a, y) \succsim (a', y')$ if and only if $(a + t, y) \succsim (a' + t, y')$

  Then preferences over X can be represented by a utility function

$$u(a, y) \quad = \quad a + v(y)$$

  (And conversely, if $u(a, y) = a + v(y)$ represents preferences, then these conditions hold.)

- Quasilinear utility is a common assumption – it basically allows you to think of y as some subset of the actual goods, and a as the money you still have left to spend on everything else
- (Again, if you want to analyze preferences over laundry detergents and a few related goods without worrying about someone's car, this is appealing)
- a is generally thought of either as money, or as some other good that's like money, often called a "numeraire" good
- Note that quasilinearity relies on the assumption of no wealth effects – you choose the same way over Y, regardless of how much money you retain to spend on stuff outside of Y
- so this is similar to separability – your preferences over Y don't depend on your spending on a – but stronger, because even the total amount you'll spend on the Y goods doesn't depend on your consumption of a (unless you hit the lower bound)

- The proof of the proposition isn't hard – the result is really baked into the assumptions
  - By the second property, we can define a function $v : Y \to \mathbb{R}^+$ by $(0, y) \sim (v(y), \bar{y})$
  - By the third property, for any (a, y) and (a', y'),

$$(a, y) \sim (a + v(y), \bar{y}) \qquad \text{and} \qquad (a', y') \sim (a' + v(y'), \bar{y})$$

  - So by transitivity,

$$(a, y) \succsim (a', y') \qquad \leftrightarrow \qquad (a + v(y), \bar{y}) \succsim (a' + v(y'), \bar{y})$$

  - By the first condition, this is if and only if $a + v(y) \geq a' + v(y')$, so $u(a, y) = a + v(y)$ represents preferences
  - The other direction is straightforward

- quasilinear utility is a very common assumption when building preferences into a more complicated model, because of how much it can simplify the static part of the problem
- when people want to assume quasilinear utility in a model,

  they don't make these assumptions on primitives, they just assume quasilinear utility,

  but it's nice to know what content that really has