Lecture 7: Consumers!

1 On to consumer theory!

- Three weeks ago, I said that microeconomics was basically,
  everyone optimizes – so what?
- Firms maximize profits given their technologies,
  and consumer optimize given their preferences and budgets
- So far, we've talked about the first problem
- Now it's time for us to move on to the second –
  consumers making choices

2 Starting point – preferences and choice

- Pretty much every paper you ever read that has consumers in it
  will start off telling you their utility function
- However, from our point of view,
  a utility function isn't the true primitive;
  what a person actually "has" are preferences,
  and a utility function is just a convenient way to represent them
- So we'll start out today talking more abstractly about preferences and choice,
  and then move on to utility functions
- (Utility functions are an "as-if" model, preferences are a "true primitive";[1]
  even if we want to work with utility functions and make assumptions about them,
  useful to think about what that means in terms of preferences)

[1] For one "counterexample" – an attempt to see utility functions as a "true" model rather than an "as-if" model – check out Antonio Rangel's neuroeconomics lab at Caltech, http://www.rnl.caltech.edu/research/index.html

- Start with the **choice set** X – some abstract set of alternatives one could choose among
  - Could be discrete – $X = \{\text{lager}, \text{porter}, \text{IPA}, \text{stout}\}$
  - Could be continuous and multidimensional –
    $X = \mathbb{R}^k_+$, like we used for producer theory,
    giving the quantity of k different commodities you get to consume
- We'll mostly be working with $X = \mathbb{R}^k_+$, but for now,
  let's think about the more abstract version where X could be anything
- We'll see two different ways of thinking about a consumer knowing what they like
- We can define *preferences* over X as a binary relation –
  given two elements a and b, which do I prefer?
- Or we can define a *choice rule* over X –
  given any set of elements of X, which subset of those are my favorites?
- We'll define what it means for a preference relation to be rational,
  and what it means for a choice rule to be rational,
  and we'll see that the two are equivalent –
  if we start with rational preferences, we can derive a rational choice rule from them,
  and if we start with a rational choice rule, we can derive rational preferences from them

- Once we have that, we'll move on to think about when preferences can be represented by a utility function,
  and what properties of preferences imply about properties of utility functions

3 Preferences as a Primitive

- Define **preferences** over the elements of X:[2]
  - $x \succsim y$ ("x is weakly preferred to y") means the consumer likes x at least as much as y
  - $\succsim$ is a binary relation on X – that is, for some pairs $(x, y) \in X^2$, $x \succsim y$, and for some pairs, $x \not\succsim y$, and the subset of pairs such that $x \succsim y$ defines the preference relation
- From weak preferences $\succsim$, we can define indifference and strict preference:
  - $x \sim y$ if $x \succsim y$ and $y \succsim x$
  - $x \succ y$ if $x \succsim y$ but $y \not\succsim x$
- For preferences to be well-defined and coherent, we need them to be...
  - **Complete**: for any x and y, either $x \succsim y$ or $y \succsim x$ (or both)
  - **Transitive**: for any x, y and z, $x \succsim y$ and $y \succsim z$ impliy $x \succsim z$

  This is what we will mean when we say preferences are **rational**.
- (Completeness seems innocuous, but which do you prefer, beer or umbrellas?)
- (Transitivity basically means, no scissors-paper-rock cycles.)
- (Note that completeness also implies *reflexivity*: $x \succsim x$.)

[2] If you use LaTeX: $\succsim$, $\succ$, $\not\succsim$, $\not\succ$, $\precsim$, $\prec$, $\not\precsim$, $\not\prec$, and $\sim$ are \succsim, \succ, \not\succsim, \not\succ, \precsim, \prec, \not\precsim, \not\prec, and \sim

4 Choice Rule as a Primitive

- Instead of defining preferences via a binary relation,

  we could alternatively define preferences as your rule for how you choose from sets

- Let $\mathcal{P}(X)$ denote the power set of X, that is, the set of subsets of X

- (Also sometimes written as 2^X.)

- Without starting with pairwise preferences, we could simply define a choice rule as a function

$$C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

  such that for every $A \in \mathcal{P}(X)$, $C(A) \subseteq A$.

- That is, for every subset A of X, C chooses a subset of A as the "chosen set"

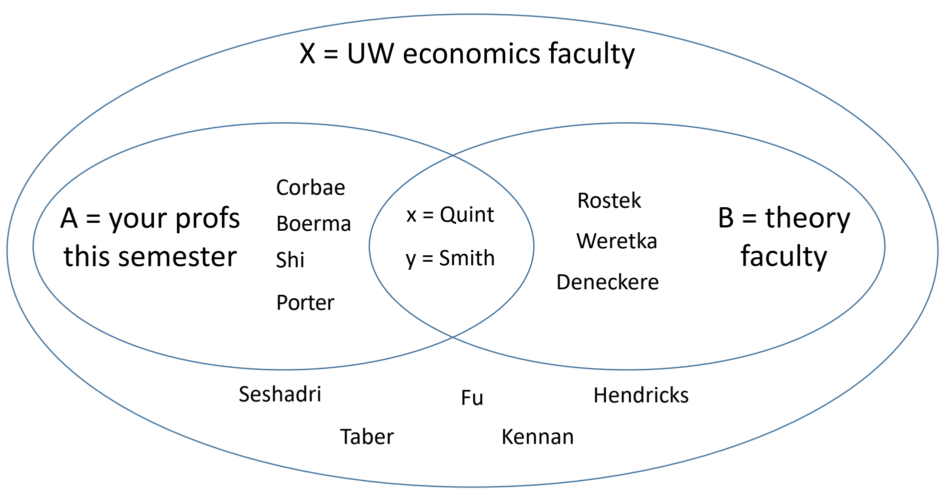
- We'll say a choice rule C is **nonempty** if $A \neq \emptyset$ implies $C(A) \neq \emptyset$

- A key property we'll want a choice rule to satisfy is the Weak Axiom of Revealed Preference:

  $C(\cdot)$ satisfies WARP if for any sets $A, B \subset X$ and any $x, y \in A \cap B$,

  if $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$.

- To visualize what this means:



- WARP says if I'm one of your favorite professors teaching you this fall,

  and Lones is one of your favorite theory profs in the department,

  then I'm also one of your favorite theory profs, and Lones is also one of your favorite fall teachers

5 Equivalence between choice rules and binary preferences

- Turns out, these two formulations of preferences are pretty much equivalent – a rational preference relation defines a rational choice rule, and vice versa

- For any preference relation $\succsim$, we can define the choice rule it induces,

$$C(A, \succsim) = \{x \in A : x \succsim y \ \ \forall y \in A\}$$

- We can show:

Proposition 1. *If $\succsim$ is complete and transitive, then the choice rule it induces satisfies WARP. If X is finite and $\succsim$ is complete and transitive, then the choice rule it induces is non-empty.*

Proposition 2. *Suppose $C : \mathcal{P}(X) \to \mathcal{P}(X)$ is nonempty. Then there exists a complete, transitive preference relation $\succsim$ on X such that $C(\cdot) = C(\cdot, \succsim)$ if and only if C satisfies WARP.*

- So, rational preferences induce a choice rule satisfying WARP;
  and if a nonempty choice rule satisfies WARP,
  it corresponds to rational choice for some rational preference relation

- Why is it useful to think about choice rules as a primitive?
- If you have empirical work in mind, it's often nice to think of the choice rule as the primitive, rather than preferences,

  since you can hope to observe peoples' choices, not their underlying preferences;

  this says that if you fully understand how they choose, you can fully infer their preferences

- But there are a couple important limitations to this

- First of all, in choice data,

  you typically observe one thing someone chose, not the set of all their favorite elements

- That is, we're imagining we know the whole choice rule C,

  meaning we know the whole set $C(A)$ when it has multiple elements,

  not just one element from it

- If you only observed one element of $C(A)$ for each A, what could you say?
- Well, not that much
- For example, if you couldn't observe all of $C(A)$, you could never learn what's *not* in $C(A)$,

  and you could never rule out the trivial preferences where the consumer is indifferent among every element in X

- So we typically imagine, even though it's unrealistic, that you know all of $C(A)$
- (If X is finite, we could also assume there are no indifferences,

  although this is usually impossible if X is continuous and multi-dimensional)

- The other limitation is, empirically,

  you're unlikely to get to see the result of every possible choice problem, only some of them

- So, what would happen if you knew $C(A)$ for *some* sets A but not for others?

  - (Not to get ahead of ourselves, but once we state the consumer problem, we'll think about observing choice only over *budget sets*, which are only some of the subsets of X)

- Is satifying WARP on limited observations enough to tell whether the choice rule is compatible with rational choice?

- It turns out, not always

- For a simple example, let $X = \{scissors, paper, rock\}$, and suppose that we observe a choice rule C on each subset of two elements, with

$$C(\{s,p\}) = \{s\}, \qquad C(\{p,r\}) = \{p\}, \qquad \text{and} \qquad C(\{s,r\}) = \{r\}$$

- Note that on the sets for which we observe C, it satisfies WARP vacuously – there are no two sets A and B whose intersection contains more than one item, so WARP is not violated

- However, it should also be clear that these choices are not compatible with rational preferences

  We could never find a complete, transitive preference relation $\succsim$ such that $C(\cdot, \succsim)$ coincided with this C on the sets we know

  Or to put it another way, we could never extend this choice rule to a choice rule on all of $\mathcal{P}(X)$ which satisfied WARP

- So if we only know C on a sample of sets,

  when do we know it's compatible with rational choice?

  That is, how much data do we need to recover someone's preferences,

  or to know whether those preferences are rational?[3]

- It turns out, if we know the choice rule C on *every set with three or fewer elements*,

  and it satisfies the Weak Axiom on those sets,

  that's enough to be sure.

[3] De Clippel and Rozen, "Bounded Rationality and Limited Datasets" (forthcoming, *Theoretical Economics*) asks this same question for models of "bounded rationality" (i.e., models in behavioral economics).

6 Utility

- Next, we move into a more concrete, numerical representation of choice
- Why? Two reasons:
  - Once we rephrase individual choice problem as mathematical optimization,
    there are a bunch of tools we can use to understand the problem and its solution
  - Also comes from a desire to have a way to compare societal outcomes,
    or outcomes for more than one person –
    although for this second purpose,
    our theory of utility will have a lot of important shortcomings

- Define a *utility function* as a real-valued function on X
- **Definition.** A preference relation $\succsim$ on X is *represented by* a utility function $u : X \to \mathbb{R}$ if

$$x \succsim y \qquad \text{if and only if} \qquad u(x) \geq u(y)$$

  for every $x, y \in X$

- So, a utility function is just a real-valued function on X
  whose *ordinal* ranking coincides with the ordinal preference relation
- Why is this good? Well, we defined the choice rule induced by preferences as

$$C(A; \succsim) \quad = \quad \{x \in A \quad : \quad x \succsim y \quad \forall y \in A\}$$

  If u represents $\succsim$, then we can rewrite this as

$$\begin{aligned} C(A; \succsim) \quad &= \quad \{x \in A \quad : \quad u(x) \geq u(y) \quad \forall y \in A\} \\ &= \quad \left\{x \in A \quad : \quad x \quad \text{solves} \quad \max_{y \in A} u(y)\right\} \end{aligned}$$

- That is, a utility representation lets us write the consumer's choice problem as an optimization problem, which is something we're good at understanding

- Now, we've said what a utility function is, but we haven't yet established that they exist
- Under what conditions on preferences does a utility representation exist?

- **Proposition.** If X is **finite**, then any complete and transitive preference relation $\succsim$ on X can be represented by a utility function u.
  - I'll skip the formal proof, but it's by induction on the number of elements of X
  - First, we show by induction (you'll do this on the homework) that for any complete, transitive preferences on any finite set,
    there's a nonempty set of "best" elements –
    those weakly preferred to everything else in the set
  - Then if there are $k+1$ elements in X,
    we assign the best elements utility of $k+1$,
    and are left with a set of at most k other elements,
    which we can assign utility of k or less to

- So if X is finite, then any rational preferences admit a utility representation
- If X is infinite, this isn't always true
- A nice example of preferences that **don't** admit a utility representation is the *lexicographic* preferences
- Let $X = \mathbb{R}^2$, and define $\succsim$ on X such that

$$(x_1, x_2) \succ (y_1, y_2) \quad \text{if either} \quad (x_1 > y_1) \quad \text{or} \quad (x_1 = y_1 \quad \text{and} \quad x_2 > y_2)$$

- These are preferences where you care infinitely more about the first good than the second –
  you'd give up any amount of good 2 to get a little more of good 1 –
  but you still prefer more of good 2 to less of it
- (Also note that there are no indifferences – $x \sim y$ only if $x = y$,
  otherwise one is strictly better than the other.)
- (These preferences are called "lexicographic" because this is how we alphabetize things –
  the first letter is all that matters, unless the first letter is the same,
  in which case the second letter is all that matters, and so on –
  if you're sorting names alphabetically, Aziz comes before Barbara)
- These preferences are rational – they're complete and transitive –
  but there's a cute proof that they can't be represented by a real-valued utility function

- Proof the lexicographic preferences can't be represented by a utility function
  - let's start with two things we know:
    * the rational numbers are countable
    * the real numbers are not
  - This means we can make a list that includes every rational number,
    and we *can't* make a list that includes every real number
  - Which also means that we can't assign a unique rational number to every real –
    there's no mapping $f : \mathbb{R} \to \mathbb{Q}$ that's one-to-one
  - (If you could assign a different rational number to each real number,
    that would prove there are at least as many rationals as reals,
    which we know there aren't)

  - So now consider lexicographic preferences on $\mathbb{R}^2$,
    and suppose there was a utility representation u
  - Pick an arbitrary number $x \in \mathbb{R}$, and consider the two points $(x, 1)$ and $(x, 2)$
  - Since $(x, 2) \succ (x, 1)$, $u(x, 2) > u(x, 1)$
  - the rationals are dense in the reals – so for any two real numbers, there's a rational number between them
  - define $f(x)$ as some arbitrarily-chosen rational number in the range $[u(x, 1), u(x, 2)]$
  - for $x' > x$, $(x', y') \succ (x, y)$ regardless of y' and y, so $u(x', y') > u(x, y)$, so $f(x') > f(x)$
  - so f is a mapping that assigns a rational to every real, with no repetition
  - since that can't exist, we have a contradiction,
    which proves there's no utility representation for lexicographic preferences

- So, not *all* preferences can be represented by a utility function
- Next, we'll introduce a restriction on preferences which *will* guarantee a utility representation exists
- We already know a utility representation exists when X is finite,

  so we're only worried about the infinite case now
- We could do this in full generality, but going forward, we'll be focusing on the case where X is $\mathbb{R}^k$ or some subset of it, so that's what we'll use here

- **Definition.** A preference relation $\succsim$ on X is **continuous** if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$, if $x_n \to \bar{x}$, $y_n \to \bar{y}$, and $x_n \succsim y_n$ for each n, then $\bar{x} \succsim \bar{y}$.
- Now, the value of continuity is this:
- **Proposition.** If $X \subseteq \mathbb{R}^k$, then any complete, transitive, **and continuous** preference relation on X can be represented by a continuous utility function $u : X \to \mathbb{R}$.

- Before we prove this, note that we just showed the lexicographic preferences cannot be represented by any utility function, continuous or not
- In order to not be contradicting ourselves, it better be that lexicographic preferences aren't continuous
- And indeed they're not.

  To see this, let $x_n = (2 + \frac{1}{n}, 3)$ and $y_n = (2, 4)$.

  For every n, $x_n \succ y_n$, because $2 + \frac{1}{n} > 2$;

  but $x_n \to (2, 3)$ and $y_n \to (2, 4)$, so $\bar{x} \prec \bar{y}$.

- We'll prove a slightly simpler case of the proposition:
  we'll prove it for $X = \mathbb{R}^k_+$, and the case where $\succsim$ is *monotone*,
  meaning if $x \gg y$ (meaning $x_j > y_j$ for every dimension j), $x \succ y$

- Note that if preferences are monotone and continuous,
  then if $x \geq y$ (meaning $x_j \geq y_j$ for every dimension j), $x \succsim y$:
  - Let $x_n = x + \frac{1}{n}e \rightarrow x$, and $y_n = y$
  - Since $x_n \gg x \geq y$, $x_n \gg y$, so $x_n \succ y$, so $x_n \succsim y$
  - Since $x_n \rightarrow x$, continuity means $x \succsim y$

- Let $e = (1, 1, \ldots, 1)$ be the bundle consisting of one unit of each good, and $\alpha e = (\alpha, \alpha, \ldots, \alpha)$.
- We first show that for every $x \in \mathbb{R}^k_+$, there exists a unique α such that $x \sim \alpha e$
- We'll then define $u(x)$ as that α, show this utility function represents $\succsim$, and prove it's continuous.

- first, α exists:
  - Fix x; define $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$
  - Both are nonempty: by monotonicity, $x \succsim 0e$, so $0 \in A^-$;
    and if we let $\bar{\alpha} = \max_k x_k$, then $\bar{\alpha}e \geq x$ and therefore $\bar{\alpha}e \succsim x$, so $\bar{\alpha} \in A^+$
  - By completeness of $\succsim$, for any α, either $\alpha e \succsim x$ or $x \succsim \alpha e$,
    so every $\alpha \in \mathbb{R}_+$ is in either A^+ or A^- (or both), so $A^+ \cup A^- = \Re_+$
  - By continuity, both A^+ and A^- contain all their limit points, so they are *closed*
    We want to show that if $\{y_n\}_{n=1}^{\infty}$ are all in A^+ and $y_n \rightarrow \bar{y}$, then $\bar{y} \in A^+$ as well
    By continuity, if $y_n e \succsim x$ for every y_n, then $\bar{y}e \succsim x$ as well; so $\bar{y} \in A^+$
    Same for A^-
  - So now we have two sets, A^+ and A^-, which are both nonempty and closed and whose union is $\mathbb{R}^+$, which is connected; this means the two sets must have nonempty intersection
  - Now for any $\alpha \in A^+ \cap A^-$, $\alpha e \succsim x$ and $x \succsim \alpha e$, or $\alpha e \sim x$, so we're done

- second, α is unique:
  - Suppose there were two values α and α' with $\alpha e \sim x$ and $\alpha' e \sim x$
  - If $\alpha > \alpha'$, then $\alpha e \gg \alpha' e$, so if $\succsim$ is monotone, $\alpha e \succ \alpha' e$
  - But that's impossible if $\alpha e \sim x$ and $\alpha' e \sim x$, since $\sim$ is transitive

- So now that we know α exists and is unique, let $\alpha(x)$ be the value of α such that $\alpha e \sim x$; we claim $u(x) = \alpha(x)$ is a utility representation of $\succsim$

  - We need to show that if $x \succsim y$ if and only if $\alpha(x) \geq \alpha(y)$
  - If $x \succsim y$, then $\alpha(x)e \succsim x \succsim y \succsim \alpha(y)e$, so $\alpha(x)e \succsim \alpha(y)e$;
    by monotonicity, this requires $\alpha(x) \geq \alpha(y)$ (because if $\alpha(x) < \alpha(y)$, then $\alpha(y)e \gg \alpha(x)e$ and therefore $\alpha(y)e \succ \alpha(x)e$)
  - Similarly, if $\alpha(x) \geq \alpha(y)$, then $\alpha(x)e \gg \alpha(y)e$, so by monotonicity, $\alpha(x)e \succsim \alpha(y)e$
    By transitivity, then, $x \succsim \alpha(x)e \succsim \alpha(y)e \succsim y$ implies $x \succsim y$, and we're done

- Finally, the last bit of theorem is that this utility function is continuous (SKIPPED IN CLASS)

  - Continuity means that for any sequence of points $\{x_n\}$, if $\lim_n x_n = \bar{x}$,

    $$\lim_{n\to\infty} u(x_n) \quad = \quad u(\bar{x})$$

  - It's actually not that hard to prove that *if* $u(x_n)$ converges to any limit,
    that limit must be $u(\bar{x})$
  - But showing that it converges at all ends up being some extra steps
    (things like extra notation and invoking Bolzano-Weierstrass –
    MWG 3.C covers it if you're curious)
  - If we skip that and suppose we already know $u(x_n)$ converges to some limit,
    let's prove that limit must be $u(\bar{x})$
  - Suppose it wasn't
  - That is, suppose $\bar{x} = \lim x_n$, but $\alpha(\bar{x}) \neq \lim \alpha(x_n) \equiv \alpha'$
  - Suppose $\alpha(\bar{x}) < \alpha'$, and let $\alpha^* = \frac{1}{2}\alpha' + \frac{1}{2}\alpha(\bar{x})$
  - By assumption, $\alpha' > \alpha^* > \alpha(\bar{x})$, so by monotonicity, $\alpha' e \succ \alpha^* e \succ \alpha(\bar{x})e$
  - Now, $\alpha(x_n) \to \alpha'$, so as n gets big, every $\alpha(x_n)$ must be close to α', so every $\alpha(x_n)$ must be above α^*, and therefore $\alpha(x_n)e \succ \alpha^* e$, and therefore $x_n \succsim \alpha^* e$
  - On the other hand, $\alpha(\bar{x}) < \alpha^*$, so $\alpha(\bar{x})e \prec \alpha^* e$, or $\bar{x} \not\succsim \alpha^* e$
  - But this violates continuity – we have $x_n \to \bar{x}$, with $x_n \succsim \alpha^* e$ but $\bar{x} \not\succsim \alpha^* e$, which means the preferences couldn't be continuous

- So, if $X \subseteq \mathbb{R}^k$, then any complete, transitive, monotone, continuous preferences on x can be represented by a continuous utility function

- (Same holds if we replace "monotone" with a weaker condition)