Lecture 5: Monotone Compar. Statics

1 Where we are

- Last time, we saw the setup for the one-dimensional version of Monotone Comparative Statics
- We start with a parameterized optimization problem over a one-dimensional choice set,

$$\max_{x \in X} g(x,t) \qquad X \subseteq \mathbb{R}$$

  and let $x^*(t) = \arg\max_{x \in X} g(x,t)$ be the set of maximizers
- We define a partial order of subsets of $\mathbb{R}$, the Strong Set Order:

  $A \geq_{SSO} B$ if for any $a \in A$ and $b \in B$, $\max\{a,b\} \in A$ and $\min\{a,b\} \in B$,

  which reduces to $a \in A \geq b \in B$ in the case of singleton sets
- We define the function g as having *increasing differences* if for any $x' > x$,

  the difference $g(x',t) - g(x,t)$ is weakly increasing in t,

  which is equivalent to $\frac{\partial g}{\partial x}$ increasing in t, or $\frac{\partial g}{\partial t}$ increasing in x, or $\frac{\partial^2 g}{\partial x \partial t} \geq 0$,

  whenever these derivatives exist
- And we gave the "punchline", which I called "Baby Topkis":

Theorem. *Let $x^*(t) = \arg\max_{x \in X} g(x,t)$, where $X \subseteq \mathbb{R}$. If g has increasing differences, then $x^*(t)$ is increasing in t via the Strong Set Order.*

- Today we'll build more intuition with the single-dimensional case,

  and then move on to the case where X is multi-dimensional
- **But first: any questions?**

1.1 How do we use the result?

- Consider a single-output firm with cost function c
- The firm's problem is

$$\max_{q \geq 0} \{pq - c(q)\}$$

  so $g(q, p) = pq - c(q)$, where p is the parameter and q the choice variable
- Note that $\frac{\partial g}{\partial p} = q$ is increasing in q, so the objective function has increasing differences
- So now we're guaranteed that $q^*(p)$ is increasing in p (via the strong set order)
- If the firm has a unique optimal production level q, then q must be weakly increasing in p
- And we didn't have to make *any* assumptions about c –

  it doesn't have to be convex, or differentiable, or anything!

- (To be fair, we already knew this –

  q^* increasing in output price is a consequence of the Law of Supply –

  but it's still nice to see that this works!)

- Another example: a single-input, single-output firm with production function f, solving

$$\max_{z \geq 0} \{pf(z) - wz\}$$

where f is increasing

- Since f is increasing, $\frac{\partial g}{\partial p} = f(z)$ is increasing in z,

  so g has increasing differences in p and z –

  so when p goes up, input used (and output produced) goes up

- What about when w changes?
- Well, $\frac{\partial g}{\partial w} = -z$, so $\frac{\partial^2 g}{\partial w \partial z} = -1$, so the problem does *not* have increasing differences in z and w
- So what can we do?

- Well, basically, we can flip the sign of w
- We can introduce a new variable $\hat{w} = -w$, and think of the problem in terms of $\hat{w}$
- That is, consider

$$z^*(\hat{w}) \quad = \quad \arg\max_{z \geq 0} \{pf(z) + \hat{w}z\}$$

- This is obviously the same problem; but the new objective function has increasing differences in z and $\hat{w}$
- So z^* is increasing in $\hat{w}$
- But since $\hat{w}$ is just $-w$, we learn z^* is **decreasing** in w
  - In practice, we don't need to formally define a new variable,

    we can just think of $-w$ instead of w as the parameter of interest,

    and note that g has increasing differences in z and $-w$

- (Note that we can apply this parameter by parameter to any parameter in the problem –

  we don't need to worry about the relationship between the different parameters,

  we can just think of holding the others fixed while we change one,

  so we just worry about the relationship between the choice variable and one parameter at a time)

- Let's do one more little complication,

  so at least we're proving something that we didn't already know from the Law of Supply

- Suppose the firm isn't a price taker in the output market,

  but faces a downward-sloping demand curve giving inverse demand $P(q)$ at each q

- The firm is now solving

$$\max_z \{P(f(z))f(z) - wz\}$$

- Here's the fun part – the objective function still has increasing differences in z and $-w$,

  so without knowing anything about the shape of demand $P(\cdot)$ or the production function $f(\cdot)$,

  we know that when w goes up, z must go down (at least via the SSO)

2 Minor Extensions

- First of all, recall that the Strong Set Order is just regular weak inequality when the sets are singletons, so:
- **Corollary.** If g has increasing differences and $x^*(t)$ is single-valued, then x^* is weakly increasing in t in the "usual" sense.
- We can also show a stronger result if we have **strictly** increasing differences:
- **Theorem.** Suppose g has **strictly** increasing differences, that is, $g(x', t) - g(x, t)$ is **strictly** increasing in t for any $x' > x$. Then for any $x \in x^*(t)$ and $x' \in x^*(t')$, $x' \geq x$.
- That is, $x^*(t)$ is increasing in t in the more intuitive sense –

  every solution at t' is at least as big as every solution at $t < t'$

  - To prove it, suppose $t' > t$, $x \in x^*(t)$, and $x' \in x^*(t')$
  - $x \in x^*(t)$ requires $g(x, t) - g(x', t) \geq 0$
  - and $x' \in x^*(t')$ requires $g(x, t') - g(x', t') \leq 0$
  - So together, $g(x, t') - g(x', t') \leq g(x, t) - g(x', t)$
  - If g has strictly increasing differences, this is impossible for $x > x'$,

    which proves $x' \geq x$
- This is called the **Monotone Selection Theorem**:

  for any selection x and x' from $x^*(t)$ and $x^*(t')$,

  $x' \geq x$
- Our examples so far have had strictly increasing differences, so we get the stronger result
- (which we already knew from the Law of Supply)

- Note, though, that even with strictly increasing differences,

  we're **not** claiming x^* is *strictly* increasing in t –

  that would require some differentiability assumptions –

  although with those, we could get a strictly-increasing result.

  - To see why strictly increasing differences does not imply $x^*(t)$ *strictly* increasing, consider the example

    $$g(x,t) \quad = \quad \begin{cases} tx & \text{if} \quad x \leq 0 \\ (t-3)x & \text{if} \quad x > 0 \end{cases}$$

    on $X = \mathbb{R}$ and $T = \{1, 2\}$

  - g has strictly increasing differences –

    for any $x' > x$, $g(x', t) - g(x)$ is strictly increasing in t –

  - but $x = 0$ is optimal for both $t = 1$ and $t = 2$

  - What goes wrong here is the "kink" –

    since the objective function is kinked at the optimum (not differentiable in x),

    even a strictly positive interaction between x and t does not ensure that $x^*(t)$ moves strictly as t moves

- One other technical note
- We made the assumption that

$$g(x', t) - g(x, t)$$

is increasing in t,

but all we actually need is that when this is positive for one value of t,

it's also positive for higher values of t

- That is, we don't really care whether this difference is 5 or 7, we only care about when it's positive and when it's negative, and whether *that* is increasing in t
- So we can weaken the "increasing differences" condition to what's called **single crossing differences** – that

$$g(x', t) - g(x, t) \quad \geq \quad 0 \qquad \longrightarrow \qquad g(x', t') - g(x, t') \quad \geq \quad 0$$

for any $x' > x$ and $t' > t$

- It's easy to show that the proof we gave of Topkis' Theorem only relies on this, not increasing differences
- We go with increasing differences because it's typically easier to check

(Whether g has increasing differences depends only on "interaction effects" between x and t;

single-crossing differences depends on levels,

so adding a function of x that isn't a function of t to g could change whether single crossing differences holds, but doesn't change whether increasing differences holds

so I think it's easier to check increasing differences;

but it's good to know the weaker condition still gives the same result)

3 Motivating the bigger problem

- So we get a nice clean result – when the choice variable is one-dimensional,

  if the objective function has increasing differences in the choice variable and the parameter,

  the optimal choice is increasing in the parameter

- But what if there's more than one choice variable?

- What about a firm with two inputs, capital and labor, say, and a production function f

- The firm's problem is

$$\max_{k,\ell \geq 0} \{pf(k,\ell) - wk - r\ell\}$$

- We already know that if p goes up, output will go up –

  because we can restate this as a one-dimensional quantity-setting problem with some cost function $c(q)$,

  or from the Law of Supply

- But what about inputs?

- If p goes up, will the firm use more capital and more labor?

  Or more capital and less labor? Or more labor and less capital?

- If w goes up, the Law of Supply says k will go down,

  but what about ℓ, and output?

- What do we need to answer this question?

- That's where we're headed next – generalizing Topkis to cover the case of a multi-dimensional choice problem like the firm's choice of inputs

4 The Multi-Dimensional Case

4.1 Setup

- So let's consider a general **multi-dimensional** optimization problem,

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x, t)$$

  where $g : X \times T \to \mathbb{R}$ and now $X \subseteq \mathbb{R}^m$

  (We can still let $T \subseteq \mathbb{R}$, since we only need to consider one parameter at a time,

  although for notational convenience I'll often think of $T \subset \mathbb{R}^n$ as well)

- Our goal is the same as last time:

  to say when the solution x^* changes in a predictable direction when a parameter changes

- To do this, we'll need to do the following:

  1. Extend the Strong Set Order to a way to rank sets that are subsets of $\mathbb{R}^m$ rather than $\mathbb{R}$,

     so we'll know what it means to say x^* is increasing in t
  2. Put a condition on the choice set X to make our approach work
  3. Generalize increasing differences to a condition on the objective function g in a multi-dimensional problem
  4. Show the analogous result: given the conditions on X and g, $x^*(t)$ is increasing in t

4.2 First: when is a subset of $\mathbb{R}^m$ above another one?

- To be able to say that x^* is increasing in t,

  we need to know what it means for $x^*(t')$ to be greater than $x^*(t)$,

  when they are both sets of points in $\mathbb{R}^m$

- To do that, we introduce a generalization of the Strong Set Order

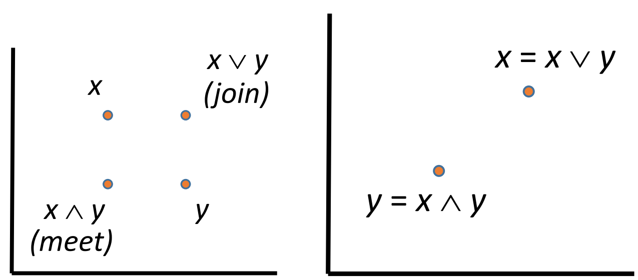
- For two points $a, b \in X \subset \mathbb{R}^m$, we'll define their componentwise maximum

$$a \vee b \quad = \quad \text{"}a \text{ join } b\text{"} \quad = \quad (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_m, b_m\})$$

  and their componentwise minimum

$$a \wedge b \quad = \quad \text{"}a \text{ meet } b\text{"} \quad = \quad (\min\{a_1, b_1\}, \min\{a_2, b_2\}, \ldots, \min\{a_m, b_m\})$$

- In two dimensions:



  (Note that if $x \geq y$, then the join is just x and the meet is just y)

- If we consider the partial order on individual points where $x \geq y$ if it's weakly higher in every dimension,

  then the join of two points is their least upper bound – the "lowest" point bigger than both;

  and the meet is the greatest lower bound – the highest point lower than both

- With the meet and the join defined,

  we'll say that a set A is bigger than a set B, $A \geq B$, if

$$a \in A \quad \text{and} \quad b \in B \qquad \longrightarrow \qquad a \vee b \in A \quad \text{and} \quad a \wedge b \in B$$

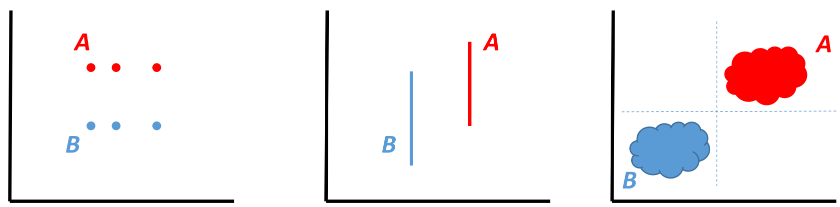
- If X is one-dimensional, this is identical to the Strong Set Order,

  because if $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$

- If X is multi-dimensional but A and B are singleton sets $\{a\}$ and $\{b\}$,

  then this requires that $a \geq b$, that is, the point in A is weakly bigger than the point in B in every dimension

- But this also allows a bunch of other configurations:



- This is what we'll mean when we say $A \geq B$ when they're both subsets of $\mathbb{R}^m$;

  so our goal will be to show $x^*(t') \geq x^*(t)$ via this ranking when $t' > t$

4.3 Second: what conditions do we need for X?

- So, we're defining a set $x^*(t')$ to be above another set $x^*(t)$ if for any points in the two sets, $x^*(t')$ also contains the join, and $x^*(t)$ also contains the meet
- Since x^* is a subset of X,

  this will only make sense if the meet and the join are also in the choice set X
- For this reason, we can only apply Monotone Comparative Statics to choice sets X that have a certain shape
- Specifically, for any x and y in X, we need X to contain $x \vee y$ and $x \wedge y$ as well

- This actually rules out a lot of the problems we consider this semester
- When we solved the firm's profit maximization over a production set Y,

  we were optimizing over some weird shape that would not satisfy this condition
- When we thought about cost minimization,

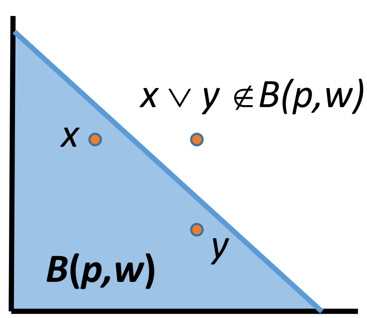
  we were minimizing over the set of input vectors generating enough output,

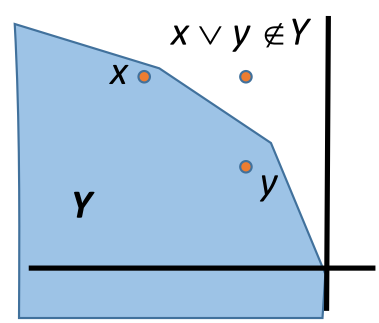
  or the upper contour set of a production function –

  this would also not satisfy this condition
- When we get to consumer theory,

  we'll generally be assuming that consumers choose from budget sets,

  which are triangles, and don't satisfy this condition





- So, what kind of choice sets *do* work?
- It suffices for X to be a product set

$$X \quad = \quad X_1 \times X_2 \times \ldots \times X_m$$

  where each $X_i \subseteq \mathbb{R}$

- (X doesn't need to be a product set –
  formally, it needs to be a *sublattice* of $\mathbb{R}^m$,
  which just means for any two points in X, the meet and join are also in X –
  but this is a natural assumption, and a sufficient one)
- So basically, X is a grid or a rectangle, not a triangle or some other funny-shaped thing

- We're also assume that X is fixed while the parameter changes;
  this can also be relaxed some, but not completely, and it's safest to just leave it fixed

- (For the firm, we can't analyze the general maximize-profits-over-Y problem this way,
  because Y is almost certainly not a product set.
  This is another reason the single-output, production-function formulation is useful:
  if we think about just choosing input combinations,
  we're choosing over $\mathbb{R}^m_+$, so we can do it this way.)

4.4 Third: what do we need for g?

- So now we know when a set A is greater than a set B,
  so we'll know how to say that x^* is increasing in t,
  and we know what type of choice set X we're able to consider
- What we need now is conditions on the objective function g,
  which will allow us to say that $x^*(t)$ is increasing in t
- Basically, we need to extend Increasing Differences to a multi-dimensional environment

- For now, fix t, so we can think of g as a function from X to $\mathbb{R}$
- **Definition.** For X a product set in $\mathbb{R}^m$, a function $g : X \to \mathbb{R}$ is **supermodular** if

$$g(x \vee y) + g(x \wedge y) \quad \geq \quad g(x) + g(y)$$

  for any $x, y \in X$.

- This sounds like a tough condition to check, but it turns out to be equivalent to a simpler one:
- **Equivalent Definition.** A function $g : X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in x_i and x_j for every pair (i, j), holding the other variables fixed.
- This is awesome, because we already know that if g is twice differentiable,
  this just means all its mixed partials $\frac{\partial^2 g}{\partial x_i \partial x_j} \geq 0$, which is easy to check if we know g
- (We'll get intuition for why pairwise increasing differences is good enough,
  when we talk about the intuition for the upcoming result)

- We'll also say g has **increasing differences in** (X, T) if it has increasing differences in (x_i, t_j) for each i and j.

- So basically, the conditions we want come down to pairwise increasing differences –
  increasing differences between any two of the choice variables,
  and increasing differences between any choice variable and any parameter we're considering
- This will ensure that all "feedback loops" and indirect effects reinforce the primary effects,
  which will give us strong results

- So now, we know what it means to say $x^*(t') \geq x^*(t)$;
  we know what type of choice set X we want to allow;
  and we have a condition on g that we can impose
- And that gives us the result:

Theorem (Topkis)**.** *Let X be a product set in $\mathbb{R}^m$, $T \subseteq \mathbb{R}^n$, $g : X \times T \to \mathbb{R}$, and*

$$x^*(t) \quad = \quad \arg\max_{x \in X} g(x, t)$$

If...

1. *g is supermodular in X, and*
2. *g has increasing differences in X and T,*

then $x^*(t)$ is increasing in t.

- That is, if $x \in x^*(t)$ and $x' \in x^*(t')$, with $t' > t$, then $x \vee x' \in x^*(t')$, and $x \wedge x' \in x^*(t)$

- **Corollary.** If x^* is single-valued, this means $x^*(t)$ is weakly increasing in every dimension
  (That is, if $t' > t$, then $x^*(t') \geq x^*(t)$, meaning $x_i^*(t') \geq x_i^*(t)$ for every dimension i)

- Before getting into the proof, an example will help clarify exactly what's going on
- Let's consider the two-input firm I mentioned last time,

  which uses capital k and labor ℓ as inputs, and solves

$$\max_{k,\ell \geq 0} \{pf(k,\ell) - w\ell - rk\}$$

- For simplicity, let's suppose that f is twice differentiable, and that $\frac{\partial^2 f}{\partial k \partial \ell} \geq 0$
- Then g is differentiable, and $\frac{\partial^2 g}{\partial k \partial \ell} = p\frac{\partial^2 f}{\partial k \partial \ell} \geq 0$,

  so g is supermodular in the choice variables $X = (k, \ell)$
- What about increasing differences in (X, T)?
- In differentiable cases, I find the easiest way to check is to take first derivatives of g with respect to each choice variable, and check whether they're monotonic in each parameter
- In this case, we're best off thinking of the parameters as $T = (p, -w, -r)$
- $\frac{\partial g}{\partial k} = p\frac{\partial f}{\partial k} - r$ is increasing in p and $-r$,

  and since it doesn't depend on w, we're free to say it's (weakly) increasing in $-w$
- And $\frac{\partial g}{\partial \ell} = p\frac{\partial f}{\partial \ell} - w$ is increasing in p and $-w$,

  and (weakly) increasing in $-r$ as well
- So g has increasing differences in (X, T), where $X = (k, \ell)$ and $T = (p, -w, -r)$

- Since g is supermodular in X and has increasing differences in (X, T), we can apply Topkis' Theorem
- In this case, if we assume that the firm's problem has a unique solution (so we don't have to worry about stating things in terms of sets above other sets),

  we simply get that (k^*, ℓ^*) is increasing in p and decreasing in w and ℓ
- So if the price of output goes up,

  the firm demands more labor and more capital,

  and therefore produces more output (as we already knew);
- and if either w or r goes up,

  the firm demands less capital and less labor,

  and therefore produces less output

- Why does this make sense?
- Suppose the price of labor, w, goes up
- The obvious first response is that the firm reduces the labor input ℓ
- But since $\frac{\partial f}{\partial k}$ is increasing in ℓ,

  when ℓ goes down, that reduces the marginal product of capital;

  so the firm reduces its use of capital k
- But since $\frac{\partial f}{\partial \ell}$ is increasing in k,

  when the firm reduces its use of capital,

  that reduces the marginal product of labor,

  so the firm reduces its labor demand again
- And so on, and so on
- Supermodularity basically ensures that all the feedback loops go in the same direction – every change the firm wants to make, reinforces the other changes

- Here, we assumed f was differentiable and the solution was single-valued,

  but we could easily drop these assumptions;

  the only really substantive assumption we needed was that f is supermodular,

  i.e., more capital makes labor more productive and vice versa,

  i.e., capital and labor are complements in production!