Lecture 2: Rationalizability and Recoverability

1 Last time...

- Last week, we laid out the basic model of firms and production
- We're working in $\mathbb{R}^k$, where each dimension represents a different commodity good that could be used as an input or an output
- A production plan y is a vector in $\mathbb{R}^k$,

  with each entry representing the quantity of one good –

  positive numbers being the amount produced by the firm,

  negative numbers being the amount used by the firm in production
- A production set is a set $Y \subset \mathbb{R}^k$ containing all the feasible production plans for a firm
- Firms are price-takers – they see market prices p for all the goods,

  and can buy or sell any quantity at those prices
- And firms maximize profits given their production set Y and the prices, solving

$$\max_{y \in Y} p \cdot y$$

  with $\pi(p)$ the value of that problem and $Y^*(p)$ the set of solutions

- We wrapped up last week proving one result: the Law of Supply
- If we think of the change from one set of prices p to another set of prices p',

  Δp, as a vector in $\mathbb{R}^k$;

  and we look at the change from any solution to the firm's problem at the first prices, y,

  to any solution to the firm's problem at the second set of prices, y',

  as a vector $\Delta y \in \mathbb{R}^k$;

  the Law of Supply says these two vectors have a non-negative dot product,

$$(\Delta p) \cdot (\Delta y) \quad \geq \quad 0$$

- And as one corollary of that, if Δp is just an increase in a single price,

  then the level of y corresponding to that good must weakly increase –

  if the price of an output goes up, the firm produces weakly more,

  and if the price of an input goes up, the firm uses weakly less

2 Today

- Thursday, we won't meet, so I can atone for my sins of the past year;
  instead, I'll post a video of a lecture from last year,
  where I'll talk more about the profit function π, and optimal production correspondence Y^*,
  and derive some more of their properties
- Today, we consider the question of what we can learn about Y from observations
- That is, suppose we don't know Y, but we have several observations of market prices p and the production plan a firm chose, $y(p)$;
  or several observations of market prices p and the firm's resulting profit level $\pi(p)$
- We'll ask the following questions: **(write 'em on the board)**
  if we observe production and/or profit level at some set of price vectors...
  1. When is the data consistent with profit maximization for *some* production set –
     that is, when are the firm's actions *rationalizable*?
     (Or to put it in the negative,
     is there data we see that would tell us our model must be wrong –
     what observations could *not* be explained by a profit-maximizing firm?)
  2. Assuming the data is consistent with our model,
     what can we infer from the data about the firm's production set?
     (That is, how much do our observations restrict what Y could be?)
  3. With enough data – i.e., if we observed firm behavior at *every* price vector $p > 0$ –
     could we infer Y exactly, or is Y *recoverable*?
- In some sense, we're working backwards here
- Last week, we started with the production set Y as the primitive,
  which pins down a profit function π and optimal firm choices Y^*,
  and we can ask about their properties
- Today, we're going the other way –
  if we start out knowing something about π and Y^*,
  is that information consistent with *any* Y,
  what can we say about the range of Y it's consistent with,
  and can we learn Y completely with enough data?
- **Before we start: any questions?**

3 Rationalizability

- We start with the first question, rationalizability
- Rationalizability is basically the flip side of Karl Popper's idea of falsifiability
- Popper wrote that a theory isn't scientific if it could explain anything –
  a scientific theory must make predictions that could in principle be observed to be false
- (So "all swans are white" is a scientific theory,
  because one could in principle see a black swan – they do, in fact, exist –
  and you'd know that the theory was wrong;
  a theory that could explain any observations,
  such as "the world was created this morning but made to seem like it had existed for longer,"
  is not scientific, since there's nothing you could possibly observe that would prove it false)
- So in some sense, we're wondering whether profit maximization is a scientific theory –
  does it make non-vacuous predictions that might or might not hold in empirical data,
  or could it explain everything?
- (If it does make predictions that might or might not hold empirically,
  that starts us down the road toward developing *tests* of whether profit maximization seems to hold...)
- So... when is, or isn't, a collection of observations consistent with profit maximization?

3.1 Definitions

- Let $P \subset \mathbb{R}^k_+$ be the set of price vectors for which we have observations
- That is, suppose we observe

$$\pi \quad : \quad P \quad \rightarrow \quad \mathbb{R}$$

(the profit level $\pi(p)$ for each $p \in P$),

or suppose we observe

$$y \quad : \quad P \quad \rightrightarrows \quad \mathbb{R}^k$$

(one or more production plans $y(p)$ for each $p \in P$),

or suppose we observe both.

(**Notation:** $Y^*(p)$ is still the full set of the firm's optimal choices; $y(p)$ is our observations – which might be a single point or might be multiple, because maybe we observed the same price vector multiple times.

We do not assume $y(p) = Y^*(p)$, only $y(p) \subseteq Y^*(p)$ – at each price vector $p \in P$, we observe some subset of the firm's optimal choices, but not necessarily all of them.)

- Informally, the question is, whatever our data is, when is there *some* production set Y which would "explain" it as the result of profit maximization?
- More formally, a profit function π is **rationalized by** a production set Y if for all $p \in P$,

$$\pi(p) \quad = \quad \max_{y \in Y} p \cdot y$$

and π is **rationalizable** if there exists some production set Y that rationalizes it.

- Similarly, a production correspondence $y : P \rightrightarrows \mathbb{R}^k$ is rationalized by Y if for all $p \in P$,

$$y(p) \quad \subseteq \quad Y^*(p) \quad = \quad \arg\max_{y \in Y} p \cdot y$$

and y is rationalizable if there exists such a set Y.

- And finally, π and y are **jointly rationalizable** if there's a single production set Y that rationalizes both of them.

3.2 What's Rationalizable?

Adding-Up

- If we observe prices p and the firm's production plan $y(p)$,

  it doesn't really matter whether we also observe $\pi(p)$, since we can infer it;

  but if we do observe both, our data are only internally consistent if

  $$p \cdot y(p) \quad = \quad \pi(p)$$

  or really, $p \cdot y = \pi(p)$ for every $y \in y(p)$ and every $p \in P$

- This is called the **adding-up condition**

- If that doesn't hold, our data is nonsense, so we'll need that to hold

- (Just to fix ideas, we'll focus on the case where we observe both production and profit level, and they're consistent;

  but everything is the same if we just observe $y(p)$ and infer $\pi(p)$ as $p \cdot y(p)$

- If we only observe the firm's profit level,

  we can't necessarily infer what production plan was chosen,

  although we'll see that just observing π does still tell us a lot)

Inner Bound

- So suppose we observe both y and π, and they satisfy the adding-up condition;

  what do we know about Y if it exists?

- Clearly, everything the firm actually did must be feasible,

  so Y must contain all the observed choices $y(p)$

- So if we want to think about the smallest that Y could possibly be,

  we can define the "inner bound"

  $$Y^I \quad = \quad \bigcup_{p \in P} y(p)$$

  and note that if $y(\cdot)$ is rationalized by Y, then $Y^I \subseteq Y$

Outer Bound

- We can also ask the question, "what's the biggest that Y could be?"
- Another way to ask this is, "what points *can't* be in Y"?
- To answer this, consider a single observation $(p, y(p))$
- If $y(p)$ was chosen at price p, then the firm's production set must not contain any of the points which would have been more profitable than $y(p)$ at prices p
- That is, if a profit maximizing firm chose $y(p)$ at prices p,

  it must not have had any other feasible choices that would have been more profitable

- If we draw $y(p)$ and p, and draw the line[1] normal to it,

  this divides $\mathbb{R}^k$ into three regions:

  the line itself, which contains points exactly as profitable as $y(p)$;

  the region above, which is the points which are more profitable than $y(p)$ at prices p;

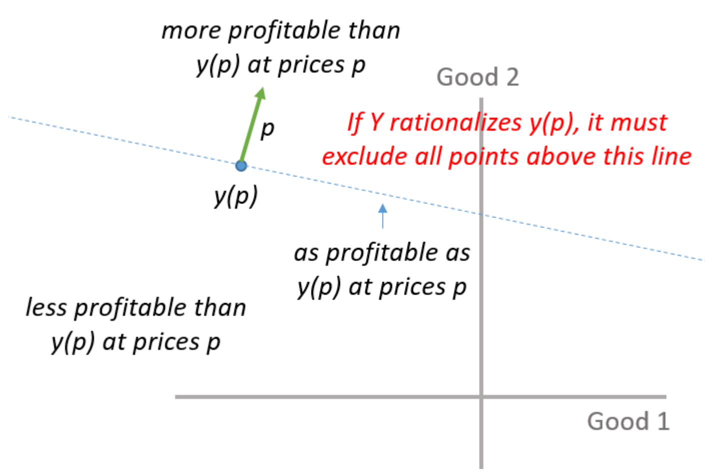
  and the region below, which is points less profitable than $y(p)$ at prices p

- If a production set Y exists which can rationalize the data,

  it must exclude all the points above the line –

  since if any of them were in Y,

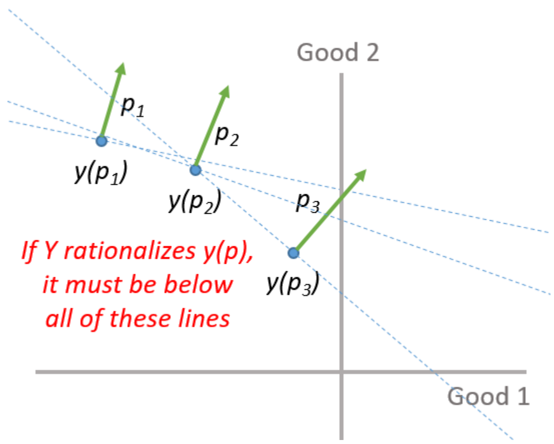
  the firm wouldn't have been maximizing profits by choosing $y(p)$



[1]This would be a $k-1$-dimensional hyperplane if $k > 2$

- Of course, if we have more than one observation,

  we get more than one half-space that has to be excluded from Y



- We'll take what's left – all the points that can't be ruled out in this way – and call it the "outer bound"

$$Y^O \quad = \quad \left\{ y \in \mathbb{R}^k \quad : \quad p \cdot y \leq \pi(p) \quad \forall p \in P \right\}$$

- If π is rationalized by Y, then $Y \subseteq Y^O$

- (Pick a point z that's not in Y^O. Then there's some price vector $p \in P$ such that $p \cdot z > \pi(p)$. This means the day we observed those prices, if z were in Y, the firm would have preferred z to whatever it did. Since z wasn't chosen, it can't have been feasible, so $z \notin Y$.)

Results

- So that's two restrictions on Y that must hold if Y rationalizes the data –

$$Y \supseteq Y^I \qquad \text{and} \qquad Y \subseteq Y^O$$

- It turns out, if we don't make any additional assumptions about Y, those two bounds are all we know!

- **Proposition.** Start with a nonempty-valued supply correspondence y and profit function π defined on a price set $P \subset \mathbb{R}^k_+$, and assume the adding-up condition holds $(p \cdot y(p) = \pi(p))$. A production set $Y \subset \mathbb{R}^k$ jointly rationalizes y and π if and only if $Y^I \subseteq Y \subseteq Y^O$.

- The proof that $Y^I \subseteq Y \subseteq Y^O$ is necessary has already been done –
  if $Y^I \not\subseteq Y$, then Y doesn't contain all of our observations,
  and if $Y \not\subseteq Y^O$, the firm wouldn't have been maximizing profits.
- The proof that it's sufficient: pick a price vector $p \in P$,
  and pick a production plan $y \in y(p)$.
  We know that $y \in Y^I \subseteq Y$, and we know that $p \cdot y = \pi(p)$ (adding up).
  For any y' with $p \cdot y' > \pi(p)$, y' would be excluded from Y^O,
  so since $Y \subseteq Y^O$, y' is also excluded from y.
  So Y contains y, and doesn't contain any points such that $p \cdot y' > p \cdot y$;
  so y is an optimal choice from Y at prices p.

- **Corollary.** If y and π satisfy the adding-up condition,
  they're jointly rationalizable if and only if $Y^I \subseteq Y^O$.

The Weak Axiom of Profit Maximization

- So, assuming our data is internally consistent, it's are rationalizable if and only if $Y^I \subseteq Y^O$
- What does this condition mean?
- It means each point in Y^I must also be in Y^O
- Or, pick a point $y' \in Y^I$; we need $y' \in Y^O$ as well
- $y' \in Y^O$ means $p \cdot y' \leq \pi(p)$ for every $p \in P$, or $p \cdot y' \leq p \cdot y$, for every $y \in y(p)$ and $y' \in Y^I$
- Or equivalently, for any $y \in y(p)$ and $y' \in y(p')$,

$$p \cdot y \quad \geq \quad p \cdot y'$$

- This is called the **Weak Axiom of Profit Maximization.**
- (This is also the condition we assumed last week, to prove the Law of Supply.)
- And that's the empirical implication of profit maximization: a profit-maximizing firm must generate data satisfying the Weak Axiom, *and nothing more* – any data satisfying WAPM can be rationalized

Empirical Tests of WAPM

- So, to see whether a dataset can be explained by profit maximization, all we need to do is check whether the observations satisfy WAPM
- And in fact, there are a number of empirical papers, (seemingly all on agriculture, and half of them written by this one guy at Cornell) that do exactly this[2]
- A few minor complications:
  - Some of these focus on cost minimization rather than profit maximization – given its output, is the firm producing it as cheaply as possible – a problem we'll consider next week (which gives a similar set of restrictions to WAPM)
  - In our model, we assume a static production set Y; if the data is successive observations of the same firm, it might be reasonable to assume Y is expanding (technology improves and never regresses), which leads to a weaker condition than WAPM
  - And of course, if you have a large number of observations, you always expect to see some small number of violations; so the question is less whether WAPM holds exactly, and more whether the violations are significant (either statistically or economically) – people have proposed different ways to measure this

[2]For example:

- Chris Fawson and Richard Shumway (1998), "A Nonparametric Investigation of Agricultural Production Behavior for U.S. Subregions," *American Journal of Agricultural Economics* 70.2;
- Subhash Ray and Dipasis Bhadra (1993), "Nonparametric Tests of Cost Minimizing Behavior: A Study of Indian Farms," *American Journal of Agricultural Economics* 75.4;
- Allen Featherstone, Ghassan Moghnieh and Barry Goodwin (1995), "Farm-level Nonparametric Analysis of Cost-Minimization and Profit-Maximization Behavior," *Agricultural Economics* 13.2;
- Loren Tauer (1995), "Do New York Dairy Farmers Maximize Profits or Minimize Costs?" *American Journal of Agricultural Economics* 77
- Loren Tauer and Zdenko Stefanides (1998), "Success in Maximizing Profits and Reasons for Profit Deviation on Dairy Farms," *Applied Economics* 30.2
- Masato Nakane and Loren Tauer (2009), "Empirical Dairy Profits under Fluctuating Prices," *Applied Economics* 41.1
- Anne Dooley, Nicola Shadbolt, Koohyar Khatami, and Loren Tauer (2017), "Application of the Adjusted Weak Axiom of Profit Maximization to New Zealand Dairy Farming," *Journal of Applied Farm Economics* 1.2

For more on the theory side, see Varian (1984), "The Nonparametric Approach to Production Analysis," *Econometrica* 52.3, and Christopher Chambers and Federico Echenique (2016), *Revealed Preference Theory*, Cambridge University Press Econometric Society Monographs, ch. 6.; and for an application of similar ideas to a different model of production, see Andrés Carvajal, Rahul Deb, James Fenske and John Quah (2013), "Revealed Preference Tests of the Cournot Model," *Econometrica* 81.6.

4 Recoverability

- So, now we've answered our first two questions:

  1. When is data consistent with a profit-maximizing firm?
     When the inner bound we calculate from the data
     is contained in the outer bound we calculate from the data,
     or $Y^I \subseteq Y^O$, meaning $p \cdot y(p) \geq p \cdot y(p')$
  2. What can we infer about Y?
     Y must contain Y^I, and be contained inside Y^O

- Our third question: what if we had complete data, that is, what if P were all of $\mathbb{R}^k_+ - \{0\}$?
  Could we recover Y exactly?

- (This is the same as asking, is our model of production *identified* from firm production data?

- In general, we can think of a *model* – such as profit maximization –
  as being a mapping from the unknown primitives of the model (in this case, Y)
  to the observations they would lead to (values of $\pi(p)$ and $y(p)$ for each p)

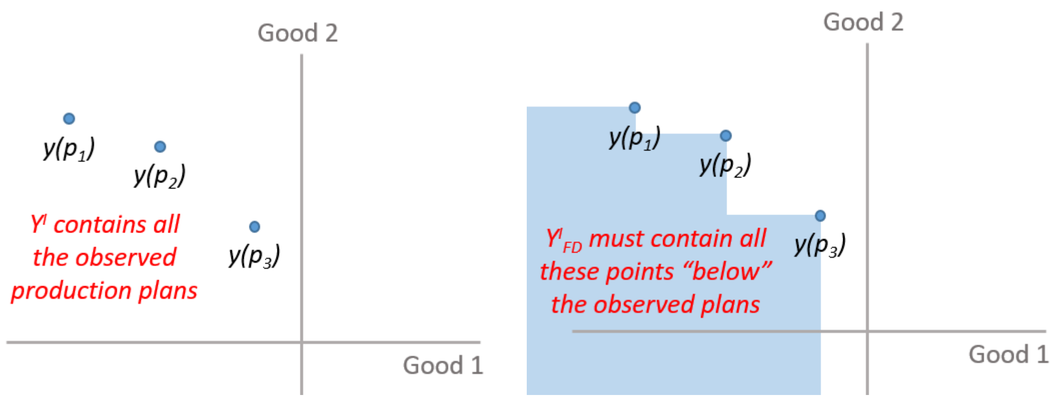
- *Identification* is the question of whether that mapping is invertible –
  whether a given set of observations could only be explained by one "version" of the model,
  or whether there are multiple production sets Y which could have led to the same observables)

- In this case, the answer turns out to be yes – **if** Y is convex

- For simplicity, we'll consider production sets with free disposal, and expand our inner bound accordingly to

$$Y^I_{FD} \quad = \quad \left\{ y \in \mathbb{R}^k \quad : \quad y \leq x \quad \text{for some} \quad x \in Y^I \right\}$$

- So Y^I_{FD} is Y^I, plus all the points that would have to be included by free disposal



- So if Y has free disposal and contains Y^I, then it contains Y^I_{FD}
- Since prices are positive,
  the points that get added this way are never more profitable than the points in Y^I,
  so this never changes whether the data are rationalizable –
  $Y^I \subset Y^O$ if and only if $Y^I_{FD} \subset Y^O$

Proposition 1. *Suppose Y is closed and has free disposal, and $P = \mathbb{R}^k_+ - \{0\}$.*

1. *If Y is convex, then $Y = Y^O$.*
2. *If Y is convex and we observe* **all** *of a firm's optimal choices ($y(p) = Y^*(p)$) at each price, then $Y = Y^I_{FD}$.*

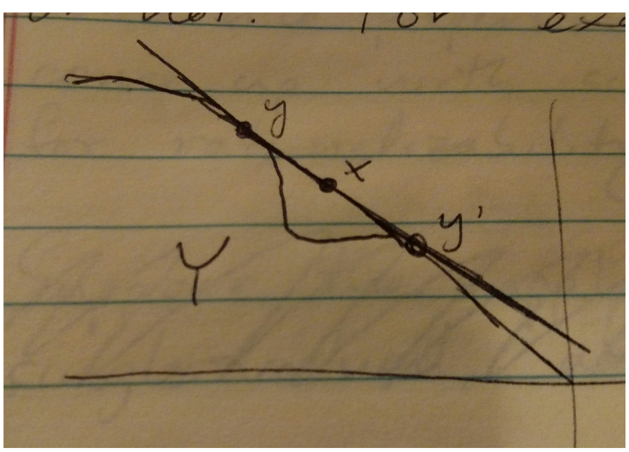
- Among other things, what's cool here is that Y^O is defined only from π, not y; and Y^I is defined just from y, not π
- So if we know the production set is convex, we can recover it from *either* production choices *or* profit levels

- This proposition also has a nice corollary
- Y^O is the intersection of a bunch of convex sets
  (the half-spaces below the isoprofit hyperplane at each p)
  and is therefore always convex
- Which means that if Y is not convex, then $Y \subsetneq Y^O$
- And since $Y^I_{FD} \subseteq Y$, this means $Y^I_{FD} \subsetneq Y^O$ if Y is not convex
- On the other hand, if Y is convex, then $Y^I_{FD} = Y = Y^O$
- So if we observe all of $Y^*(p)$ at every p, we can *learn* whether Y is convex or not
- I'll write that as a corollary:

  Corollary. Suppose $P = \mathbb{R}^k_+ - \{0\}$ and $y(p) = Y^*(p)$ (you observe all prices, and all optimal production plans at each price).

  - if $Y^I_{FD} = Y^O$, then Y is convex (and equal to both);
  - if $Y^I_{FD} \subsetneq Y^O$, then Y is not convex
  - (and of course, if $Y^I_{FD} \supsetneq Y^O$, then the data is not rationalizable)
- (Whether Y is convex may be important for other reasons –
  for example, it's an assumption we make to guarantee existence of general equilibrium...)

- I should emphasize that $Y = Y^I_{FD}$ requires that we "see" **all** of $Y^*(p)$ for each p
- If we only see a subset, even if we observe every p,
  that's not enough to learn whether Y is convex
- Example – if you only see y or y' chosen at this price vector, don't know whether x isn't chosen because it's not in Y, or because y or y' was just as good, so can't tell if Y is convex



One other thing

- One other thing I'll mention before we get into the proof
- If Y is not convex, then we can never learn exactly what it is
- The reason for this is that the points that are "missing" are never optimal
- Going back to the pictures from before,

  for any point in that hole below x,

  if those points were in the production set Y,

  for any price vector, either y or y' would be more profitable,

  so we would never observe the firm choosing those points
- So there's no way for us to learn whether those points are in Y
- All we can learn – assuming $y(p) = Y^*(p)$ – is that Y is not convex,

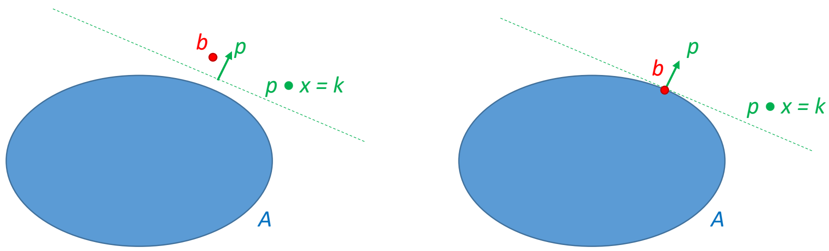
  and somewhere between Y^I_{FD} and Y^O

4.1 Proof of Proposition

The proofs of this proposition use a pair of results known as the Separating Hyperplane Theorem:

Theorem. *Let A be a closed, convex subset of $\mathbb{R}^k$.*

1. *If $b \notin A$, then there exists a vector $p \neq 0$ with $p \cdot b > p \cdot a$ for every $a \in A$.*
2. *If $b \in Bd(A)$, then there exists a vector $p \neq 0$ with $p \cdot b \geq p \cdot a$ for every $a \in A$.*



(For the first, to prove it, you first show that there's a unique point $a^* \in A$ that's closest to b, and then set $p = b - a^*$ and show it works. For the second, there's a very clever proof by Marty Weitzman that I've recorded as "online appendix" material that I'll post next week.)

For now we'll take these as given, and use them to prove the two results on recoverability

Proof of the first part

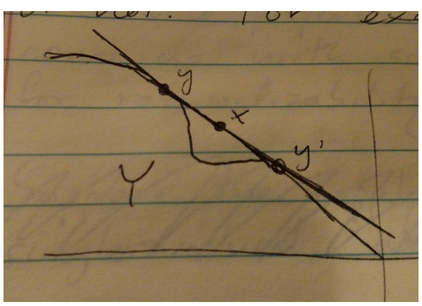
- We want to prove that if Y is closed, **convex**, and has free disposal, and $P = \mathbb{R}^k_+ - \{0\}$, then $Y = Y^O$.
- We know that $Y \subseteq Y^O$, by construction, so we just need to show $Y^O \subseteq Y$
- Or, if we pick a point in Y^O, it has to also be in Y
- Or, if we pick a point $x \notin Y$, we need to show $x \notin Y^O$
- So pick $x \notin Y$
- From the separating hyperplane theorem, since Y is closed and convex and $x \notin Y$, we know there's a vector q with $q \cdot x > \sup_{y \in Y} q \cdot y$
- If q exists, it must have all nonnegative elements, because of free disposal – if there were an element of q that was negative, then the right-hand side would be infinite
- So now we know $q > 0$, which means $q \in P$; and we know $q \cdot x > \sup_{y \in Y} q \cdot y = \pi(q)$ – so $x \notin Y^O$

Proof of the second part (DIDN'T GET TO IN CLASS)

- Now let's prove the second part – if $y(p) = Y^*(p)$ for each p, then $Y^I_{FD} = Y$
- We know $Y^I_{FD} \subseteq Y$, so we need to show $Y \subseteq Y^I_{FD}$
- So pick $x \in Y$, we'll show it's in Y^I_{FD}

- First, suppose x is on the boundary
- We use the second separating hyperplane result:
  since Y is convex and x is on its boundary, there's a vector q with $q \cdot x = \max_{y \in Y} q \cdot y$
- Like before, $q \geq 0$, or else the RHS would be infinite; so $q \in P$ (the set of observed prices)
- So $x \in Y^*(q)$, which was completely observed, so $x \in Y^I$ and therefore Y^I_{FD}.
- If x is in the interior of Y, find a point "above" it that's on the boundary;
  that point must be in Y^I, which means x must be in Y^I_{FD}.

- (Again recall that we do need to observe all of $Y^*(p)$, not just part of it;
  in the same picture from before, if Y is convex,
  we might still never observe x just because y and y' are always just as good)



- And agin, with complete data – if we observe the firm's choice for every p,
  and observe every element of $Y^*(p)$ – we can check whether Y is convex or not:
  we can just calculate Y^O and Y^I_{FD}, and see whether they're equal
  (If $Y^O = Y^I_{FD}$, then Y is convex, and equal to both of them;
  (If $Y^O \supsetneq Y^I_{FD}$, then Y must not be convex)