Lecture 11: Expenditure minimization and Slutsky

1 Where are we?

- Tuesday, we continued to explore the relationship between the consumer problem and the Lagrangian
- We did a quick example of solving the consumer problem

  and saw how the Lagrangian gave an elegant Envelope Theorem proof of Roy's Identity
- And we introduced the Expenditure Minimization Problem and Hicksian demand
- Today, we'll explore some properties of $h(p, u)$ and $e(p, u)$,

  the link between Marshallian and Hicksian demand,

  and how we can use Hicksian demand to test our model
- First... any questions?

2 Expenditure Minimization

- We defined the Expenditure Minimization Problem as

$$\min_{x \geq 0} p \cdot x \quad \text{subject to} \quad u(x) \geq u$$

  with

$$e(p,u) \quad = \quad \min_{x \geq 0} p \cdot x \quad \text{subject to} \quad u(x) \geq u$$

  the value function, which we called the expenditure function, and

$$h(p,u) \quad = \quad \arg\min_{x \geq 0} p \cdot x \quad \text{subject to} \quad u(x) \geq u$$

  the solution, which we called Hicksian demand (or compensated demand)

- It's called "compensated" demand because when we think about Hicksian demand changing in response to a price change,

  we're choosing to hold the utility level u constant;

  so it's as if we're, say, increasing a price,

  but we're also compensating the consumer for it by giving them enough additional money to afford the same level of utility as they were getting before

- (In contrast, Marshallian demand is called "uncompensated" demand)

- (Also note that our main interest in Hicksian demand is in how it responds to price changes,

  we'll see why a little later today)

2.1 Does the Expenditure Minimization Problem Have a Solution?

- With Marshallian demand, when $p \gg 0$, we noted that $B(p, w)$ is compact,
  so we knew the consumer problem has a solution
- With expenditure minimization, we're minimizing over the upper contour set of u –
  the part of $\mathbb{R}^k_+$ that gives sufficiently high utility –
  which is likely to be unbounded
- So it's not immediately obvious a solution exists
- Turns out, though, it does:
- **Theorem.** If $p \gg 0$, u is continuous, and there exists an x with $u(x) \geq u$, then the expenditure minimization problem has a solution.
  - Proof. Fix $\hat{x}$ with $u(\hat{x}) \geq u$. Define

    $$S = \{x | u(x) \geq u\} \cap \{x | p \cdot x \leq p \cdot \hat{x}\}$$

    the intersection of the set of points better than u and the set of points cheaper than $\hat{x}$
  - The first set is closed, because preferences are continuous;
    the second set is closed and bounded, since it's a budget set;
    so the intersection is closed and bounded, therefore compact
  - So a minimum exists over S;
    and the solution over S must solve the original problem (over all $\mathbb{R}^k_+$),
    because the points we've excluded were all strictly more expensive than those in S.
- So, we let $h(p, u)$ be the set of minimizers – the Hicksian, or Compensated, Demand
  And $e(p, u)$ the min value – the Expenditure Function

2.2 Properties of Hicksian Demand and the Expenditure Function

- Expenditure minimization is the problem of minimizing a linear function $(p \cdot x)$ over an arbitrary set $(\{x : u(x) \geq x\})$
- Which means it has the exact same structure as a firm's cost minimization problem; and the same structure as the firm's profit maximization problem, just with the sign flipped
- So Hicksian demand, and the expenditure function, will have a whole bunch of familiar-looking properties
- **Proposition.** Suppose u is a continuous utility function, representing a preference relation $\succsim$ on $\mathbb{R}^k_+$.
  1. $h(p,u)$ is homogenous of degree 0 in p – $h(\lambda p, u) = h(p,u)$ for any $\lambda > 0$
  2. if preferences are convex, $h(p,u)$ is a convex set;
     if preferences are strictly convex and $p \gg 0$, $h(p,u)$ is a singleton
  3. "no excess utility" – if $u \geq u(0)$ and $p \gg 0$, then for any $x \in h(p,u)$, $u(x) = u$
- Proofs.
  1. $h(\lambda p, u)$ solves
     $$\min_{u(x)\geq u} (\lambda p) \cdot x \quad = \quad \lambda \min_{u(x) \geq u} p \cdot x$$
     the same minimization problem as $h(p,u)$, so it has the same solution
  2. if x and x' are both in $h(p,u)$, then $u(x) \geq u$, $u(x') \geq u$, and $p \cdot x = p \cdot x' = e(p,u)$
     if preferences are convex, $u(x^t) \geq u$ and $p \cdot x^t = p \cdot x$, so $x^t \in h(p,u)$ as well
     if preferences are strictly convex, then if $x' \neq x$, $u(x^t) > u(x)$, which would contradict "no excess utility" proven below
  3. If $u(x) > u$, then since u is continuous, there's some $t < 1$ with $u(tx) \geq u$
     - If this weren't the case, then consider the sequence $\{u(\frac{1}{2}x), u(\frac{3}{4}x), u(\frac{7}{8}x), \ldots\}$
     - Since $\{\frac{1}{2}x, \frac{3}{4}x, \ldots\} \to x$, if u is continuous, the sequence converges to $u(x)$, which by assumption was strictly greater than u
     - But then the later elements of this sequence must be close to $u(x) > u$, and therefore also above u

     But if for some $t < 1$, $u(tx) \geq u$, then the bundle tx, which is strictly cheaper than x, gives a cheaper way to achieve the target utility, so x couldn't solve the EMP
     (Also note this result doesn't require preferences to be LNS, just that u is continuous)

- **Proposition.** Suppose u is a continuous utility function representing **locally non-satiated** preferences on $\mathbb{R}^k_+$. Then $e(p,u)$ is...

  1. homogeneous of degree 1 in p: $e(\lambda p, u) = \lambda e(p,u)$
  2. continuous in p and u
  3. nondecreasing in p, strictly increasing in u if $p \gg 0$
  4. concave in p

- Proofs.

  1. Well,
  $$e(\lambda p, u) \quad = \quad \min_{u(x)\geq u} (\lambda p)\cdot x \quad = \quad \lambda \min_{u(x)\geq u} p \cdot x \quad = \quad \lambda e(p,u)$$
  or, we already showed $h(\lambda p, u) = h(p,u)$, and
  $$e(\lambda p, u) \quad = \quad (\lambda p)\cdot h(\lambda p, u) \quad = \quad (\lambda p)\cdot h(p,u) \quad = \quad \lambda e(p,u)$$
  2. We'll skip the proof, but the intuition is similar to continuity of $v(p,w)$ – if p changes a little, the min value changes a little; if u changes a little, the set $\{x : u(x) \geq u\}$ only changes a little since $u(\cdot)$ is continuous
  (This part does depend on preferences being LNS, though.)
  3. Suppose $p' > p$, and pick $x \in h(p', u)$. So $u(x) \geq u$, and
  $$e(p', u) \quad = \quad p' \cdot x \quad \geq \quad p \cdot x$$
  so $e(p,u) \leq p \cdot x \leq e(p', u)$.
  For u, let $u' > u$, and pick $x \in h(p, u')$, so $u(x) = u'$. If $e(p,u) \geq e(p,u') = p \cdot x$, then x would be a minimizer at (p,u), so $x \in h(p,u)$, violating no-excess-utility since $u(x) > u$.
  4. For concavity, we need to show
  $$e(tp + (1-t)p', u) \quad \geq \quad te(p,u) + (1-t)e(p',u)$$
  Pick $x \in h(tp + (1-t)p', u)$, so that $u(x) \geq u$.
  Then $e(p,u) \leq p \cdot x$, and $e(p', u) \leq p' \cdot x$
  So
  $$e(tp+(1-t)p', u) \quad = \quad (tp+(1-t)p')\cdot x \quad = \quad t(p\cdot x)+(1-t)(p'\cdot x) \quad \geq \quad te(p,u)+(1-t)e(p',u)$$
  and we're done.
  (More intuition: e is the min of a bunch of linear functions...)

- **Proposition (Shepard's Lemma).** Let u be a continuous utility function representing LNS preferences, and suppose $h(p,u)$ is single-valued. Then $e(p,u)$ is differentiable in p, and

$$\frac{\partial e}{\partial p_i}(p,u) \quad = \quad h_i(p,u)$$

- Proof is via the Envelope Theorem:

$$e(p,u) \quad = \quad \min_{\{x:u(x)\geq u\}} \{p_1 x_1 + \ldots + p_k x_k\}$$

so

$$\frac{\partial}{\partial p_i} e(p,u) \quad = \quad \frac{\partial}{\partial p_i}\left(p_1 x_1 + \ldots + p_k x_k\right)|_{x=h(p,u)} \quad = \quad x_i|_{x=h(p,u)} \quad = \quad h_i(p,u)$$

- **Proposition (Law of Compensated Demand).** Suppose $p, p' \geq 0$, $x \in h(p,u)$, and $x' \in h(p',u)$. Then

$$(p'-p)\cdot(x'-x) \quad \leq \quad 0$$

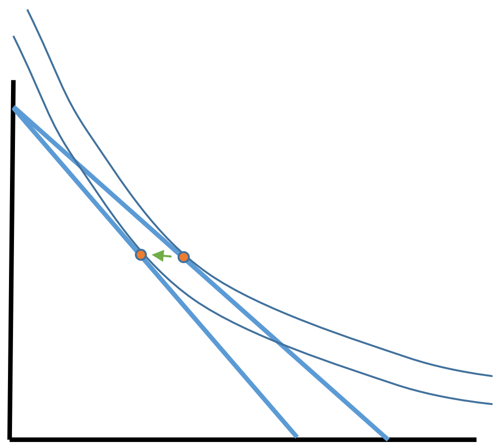
- Like with the Law of Supply, $p'-p$ is a vector pointing in the direction of a price change; $x'-x$ is a vector pointing in the direction of the change in Hicksian demand; and the negative dot product means these go in opposite directions
- We'll skip the proof, since it's analogous to the Law of Supply
- And of course, if p' and p differ in just one price, we get

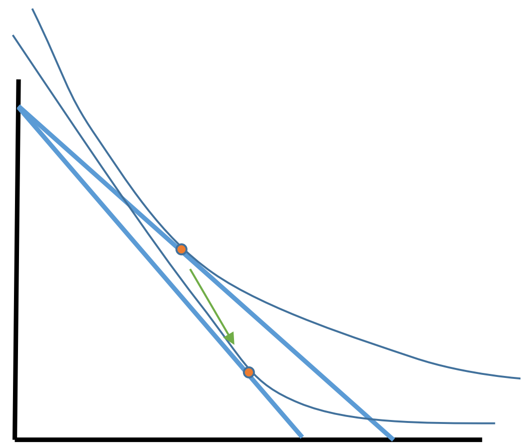
$$(p_i'-p_i)(h_i(p',u)-h_i(p,u)) \leq 0$$

or, if $p_i' > p_i$, $h_i(p',u) \leq h_i(p,u)$

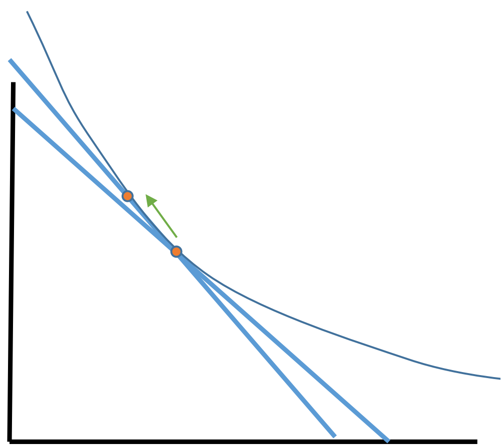
- Or, the Hicksian demand for a good is decreasing in that good's price
- While this seems obvious, this is *not* always true with Marshallian demand

- With Marshallian demand, when p_1 goes up, you shift to a different (lower) indifference curve; since the utility function could have any shape, the effect on x_1 is ambiguous





- But with Hicksian demand, when p_1 goes up, since utility stays constant, you move along the same indifference curve – so the effect on h_1 is clear



- (In fact, there's a name for goods where the Marshallian demand is increasing in its own price, we'll get to that soon)

3 More results when h is single-valued and differentiable

- If Hicksian demand is single-valued and differentiable,

  consider the matrix of its partial derivatives with respect to price:

$$D_p h(p,u) \quad = \quad \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} & \cdots & \frac{\partial h_1}{\partial p_k} \\ \frac{\partial h_2}{\partial p_1} & \frac{\partial h_2}{\partial p_2} & \cdots & \frac{\partial h_2}{\partial p_k} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_k}{\partial p_1} & \frac{\partial h_k}{\partial p_2} & \cdots & \frac{\partial h_k}{\partial p_k} \end{bmatrix}$$

- Here are some properties that should look pretty familiar from producer theory:

Proposition. Suppose u represents a preference relation $\succsim$,
and $h(p,u)$ is single-valued and differentiable at (p,u), with $p \gg 0$.
Then the matrix $D_p h(p,u)$ is symmetric and negative semi-definite, and $[D_p h(p,u)]p = 0$.

- The first two come straight from Shepard's Lemma –

  since $h_i = \frac{\partial e}{\partial p_i}$, this is the matrix of second derivatives of e,

  which is symmetric (since cross-partials are symmetric)

  and negative semidefinite (since e is concave)

- But stepping back from the math, the symmetry result is pretty surprising –

  the change in compensated demand for butter due to a change in price of beer,

  is the same as the change in demand for beer with a change in the price of butter

- In fact, it's a strong prediction we could potentially test with data,

  if we could observe Hicksian demand

  (foreshadowing)

- Negative semidefinite also implies the diagonal elements are negative – $\frac{\partial h_i}{\partial p_i} \leq 0$ –

  which we already knew from the Law of Demand

- For the last result, $[D_p h]p = 0$, the textbook offers a couple of mechanical ways to prove this, but there's a more (I think) intuitive explanation

- Knowing $D_p h$ is symmetric, we can use its transpose;
  matrix math gives

$$(D_p h(p,u))^T p \quad = \quad \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_2}{\partial p_1} & \cdots & \frac{\partial h_k}{\partial p_1} \\ \\ \frac{\partial h_1}{\partial p_2} & \frac{\partial h_2}{\partial p_2} & \cdots & \frac{\partial h_k}{\partial p_2} \\ \\ \vdots & \vdots & & \vdots \\ \\ \frac{\partial h_1}{\partial p_k} & \frac{\partial h_2}{\partial p_k} & \cdots & \frac{\partial h_k}{\partial p_k} \end{bmatrix} \begin{bmatrix} p_1 \\ \\ p_2 \\ \\ \vdots \\ \\ p_k \end{bmatrix} \quad = \quad \begin{bmatrix} \frac{dh}{dp_1} \cdot p \\ \\ \frac{dh}{dp_2} \cdot p \\ \\ \vdots \\ \\ \frac{dh}{dp_k} \cdot p \end{bmatrix}$$

  so saying this is 0 is the same as saying that

$$\frac{dh}{dp_i} \cdot p \quad = \quad 0$$

  for each p_i

- We know that Hicksian demand always achieves the target utility level u,
  so as p_i changes, the change in Hicksian demand is movement along an indifference curve

- $\frac{dh}{dp_i} \cdot p = 0$ is saying the indifference curves are perpendicular to the price vector –
  which is the same as saying the indifference curve is tangent to the budget line –
  which we know must be true at the optimum!

4 Relating Marshallian and Hicksian Demand

- So, Marshallian Demand is the solution to utility maximization,

$$\max u(x) \quad \text{subject to} \quad p \cdot x \leq w$$

  and Hicksian demand is the solution to expenditure minimization,

$$\min p \cdot x \quad \text{subject to} \quad u(x) \geq u$$

- As I've hinted at before,

  I think of Marshallian demand as more or less the problem people face in the real world;

  but Hicksian demand is something we have a lot of strong theoretical results about

- Next, we'll see how the two relate to each other,

  and in particular, how we can infer Hicksian demand from Marshallian demand;

  so if we observe someone's Marshallian demand,

  we can test whether their Hicksian demand is consistent with the properties we just saw,

  to test whether their choices are consistent with utility maximization

Proposition. Suppose u is a continuous utility function representing LNS preferences on $\mathbb{R}^k_+$. Suppose $p \gg 0$, $w \geq 0$, and $u \geq u(0)$.

1. Fixing p, "e and v are inverses":

$$e(p, v(p, w)) \quad = \quad w \qquad \text{and} \qquad v(p, e(p, u)) \quad = \quad u$$

2. Hicksian and Marshallian demand match at corresponding wealth/utility levels:

$$h(p, u) \quad = \quad x(p, e(p, u)) \qquad \text{and} \qquad x(p, w) \quad = \quad h(p, v(p, w))$$

- Proofs.
  - Fix p and w, and pick $\hat{x} \in x(p,w)$
  - Since preferences are LNS, $p \cdot \hat{x} = w$; and define $u = v(p,w) = u(\hat{x})$
  - Since $\hat{x}$ gives us one way to achieve utility u at cost w, $e(p,u) \leq w$
  - But if there were a way to achieve u at cost less than w, this would give another bundle $x' \in x(p,w)$ with $p \cdot x' < w$, which would violate Walras' Law, so $e(p,u) \geq w$
  - Thus, $e(p,u) = w$ when $u = v(p,w)$, or $e(p, v(p,w)) = w$

  - Next, let's do $v(p, e(p,u)) = u$
  - Pick $\hat{x} \in h(p,u)$, so that $e(p,u) = p \cdot \hat{x}$
  - By no-excess-utility, $u(\hat{x}) = u$
  - Of course, this means if we let $w = e(p,u)$, we must have $v(p,w) \geq u$, since one option is to consume $\hat{x}$
  - On the other hand, if $v(p,w)$ were strictly greater than u, this would mean there was some other bundle x' with $p \cdot x' \leq w = p \cdot x$ giving $u(x') > u$
  - Since $\hat{x} \in h(p,u)$ and x' is the same cost, $x' \in h(p,u)$ as well; but this would violate the no-excess-utility condition

  - Finally, to show equivalence of Marshallian and Hicksian demand,
    we can write Marshallian demand as

$$x(p,w) \quad = \quad \{x : u(x) \geq v(p,w)\} \cap \{x : p \cdot x \leq w\}$$

    and Hicksian demand as

$$h(p,u) \quad = \quad \{x : u(x) \geq u\} \cap \{x : p \cdot x \leq e(p,u)\}$$

  - If we plug in $u = v(p,w)$ and use the fact that $e(p, v(p,w)) = w$, these are the same, so $x(p,w) = h(p, v(p,w))$
  - Similarly, if we plug in $w = e(p,u)$ and use the fact that $v(p, e(p,u)) = u$, they're the same, so $x(p, e(p,u)) = h(p,u)$

- So we've shown "equivalence" between the solutions to the two problems
- We can think of $e(p, v(p, w)) = w$ as basically a restatement of Walras' Law:

  whatever maximal utility $v(p, w)$ you can get for wealth w,

  the cheapest way to get it costs the full w
- And similarly, $v(p, e(p, u)) = u$ is basically No Excess Utility:

  however much it costs you to buy utility u, you can't get higher utility for the same cost
- And equivalence of the solutions to the two problems is bascially the fact that in either case, the solution can be thought of as the intersection of two sets –

  the set of bundles weakly better than x, and the set of bundles weakly cheaper than x

- However, saying that when $u = v(p, w)$, Hicksian and Marshallian demand coincide does not mean they are always identical
- The whole point is that they respond differently to price changes
- That is, when a price changes, Hicksian and Marshallian demand respond differently –

  because Hicksian demand must leave utility unchanged,

  while Marshallian demand must leave budget unchanged
- How the two relate in terms of changes brings us to the Slutsky Equation

5 The Slutsky Equation

- Begin with

$$x(p, e(p, u)) = h(p, u)$$

which we just proved

- Focus on good i,

$$x_i(p, e(p, u)) = h_i(p, u)$$

- Differentiate with respect to the price of good j,

$$\frac{\partial x_i}{\partial p_j}(p, e(p, u)) + \frac{\partial x_i}{\partial w}(p, e(p, u))\frac{\partial e}{\partial p_j}(p, u) = \frac{\partial h_i}{\partial p_j}(p, u)$$

- Recall that $\frac{\partial e}{\partial p_j} = h_j$ (Shepard's Lemma), and rearrange, giving

$$\frac{\partial x_i}{\partial p_j}(p, e(p, u)) = \frac{\partial h_i}{\partial p_j}(p, u) - \frac{\partial x_i}{\partial w}(p, e(p, u))h_j(p, u)$$

- Letting $u = v(p, w)$, so $e(p, u) = w$, and noting $h_j(p, u) = x_j(p, w)$, this is

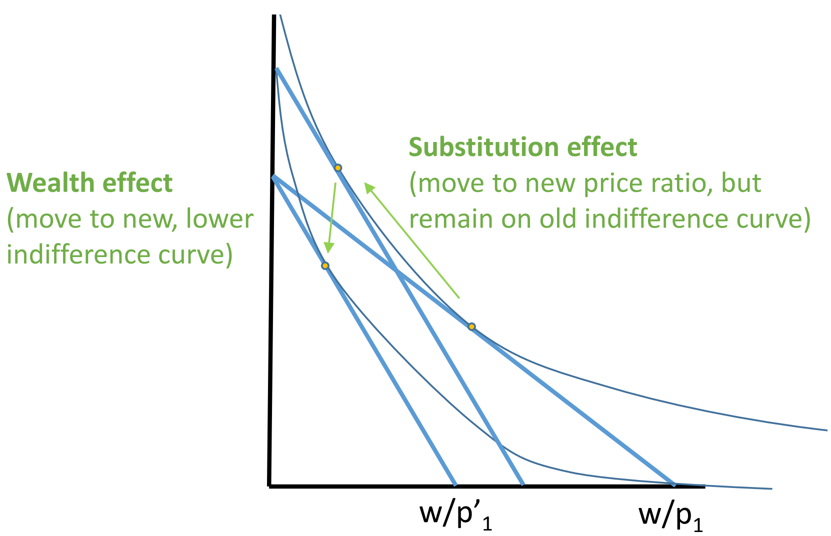
$$\frac{\partial x_i}{\partial p_j}(p, w) = \frac{\partial h_i}{\partial p_j}(p, u) - \frac{\partial x_i}{\partial w}(p, w)x_j(p, w)$$

which is called the Slutsky Equation

- **Theorem. (Slutsky Equation.)** Let u be a continuous utility function representing LNS preferences on $\mathbb{R}^k_+$; pick $p \gg 0$, and let $w = e(p, u)$. If $h(p, u)$ and $x(p, w)$ are single-valued and differentiable at (p, u, w), then

$$\frac{\partial x_i}{\partial p_j}(p, w) = \frac{\partial h_i}{\partial p_j}(p, u) - \frac{\partial x_i}{\partial w}(p, w)x_j(p, w)$$

- This lets us decompose $\frac{\partial x_i}{\partial p_j}$ into two effects – a **substitution effect** and a **wealth effect**
  - From a Marshallian perspective, an increase in p_j does two things –
    it makes good j *relatively* more expensive, and it makes you effectively poorer
  - That is, it changes the cheapest way to achieve a particular level of utility,
    and it reduces the level of utility you expect to achieve
  - The first effect is how Hicksian demand responds
  - The second effect is how Marshallian demand responds to wealth, times how much your wealth goes down
  - How much does your effective wealth go down?
    Well, if p_j goes up by ϵ, then for every unit of good j that you were planning to consume, you're now ϵ over your budget constraint –
    so you're effectively made poorer by ϵx_j for every ϵ the price rises
  - So the effect is whatever the effect of losing some wealth is on your Marshallian demand, multiplied by x_j



- Note that both the substitution and wealth effects can have either sign,
  depending on the nature of the goods
- The Slutsky equation still holds when $i = j$, giving

$$\frac{\partial x_i}{\partial p_i}(p,w) \quad = \quad \frac{\partial h_i}{\partial p_i}(p,u) \quad - \quad \frac{\partial x_i}{\partial w}(p,w)x_i(p,w)$$

  where here, we know the substitution effect is negative ($\partial h_i/\partial p_i \leq 0$),
  but the wealth effect can still have either sign

6 Inferior Goods, Giffen Goods, Substitutes, Complements

- In fact, we have names for the different signs these effects can have...

- Good i is a **normal good** if x_i is increasing in w,

  and an **inferior good** if x_i is decreasing in w

  Luxury goods tend to be normal – when you get richer, you consume more BMWs and caviar than you used to.

  Inferior goods might be things like canned tuna or ramen – you consume them a lot when you're poor, but then when you get richer, you start swtiching to other, "better" goods, so your demand decreases

- Good i is a **regular good** if x_i is decreasing in p_i,

  and a **Giffen good** if x_i is increasing in p_i

  Going back to the Slutsky equation, a Giffen good is an inferior good where the wealth effect overwhelms the substitution effect

  The "classic" example of a Giffen good is potatoes during the Irish potato famine. The poor subsisted largely on potatoes, but had some money left over for other things (like other foods). When the price of potatoes went up, they still needed to eat enough to survive, and potatoes were still the cheapest option, so people ate *more* potatoes because it's all they could afford.

  There's a 2008 experimental paper that finds this effect for rice and wheat in two provinces in China.[1]

  Another weird recent sort-of-example was US government debt. In August 2011, the U.S. government was in a political fight over whether to increase the debt ceiling, and due to the uncertainty, Standard and Poors downgraded US debt from AAA (their highest rating) to AA (their second-highest). This led to an increase in demand for US debt, driving prices up rather than down. Why? Even if it was being judged to be riskier than before, US debt is still perceived as the safest investment available; the added riskiness led to a "flight to quality" and attracted more investment in the still-pretty-safe asset![2]

[1] Jensen and Miller (2008), "Giffen Behavior and Subsistence Consumption," *American Economic Review* 98.4

[2] G. I., "Treasury Potatoes – Bonds as Giffen Goods," *The Economist* Nov 22 2011; also see Komal Sri-Kumar, "The Bond Market Is Now a Giffen Good," *Bloomberg*, Jul 12 2019, for a more recent example of the same phenomenon.

- Good i is a **substitute** for good j if h_i is increasing in p_j,
  and a **complement** if h_i is decreasing in p_j

- Note that this is symmetric:
  if i is a substitute for j, then j is a substitute for i, since $\partial h_i/\partial p_j = \partial h_j/\partial p_i$

- But two goods can be substitutes at some prices/utility levels and complements at others
  (If you're incredibly poor, maybe peanut butter and jelly are substitutes,
  because you're just trying to get enough calories to survive and you'll eat whichever is cheaper;
  but at higher income levels, they're complements,
  because you choose to eat them together as opposed to other alternatives like ham and cheese.)

- Also note that if there are only two goods, they have to be substitutes –
  since h_i decreases in p_i, you need to consume more of j to get the same level of utility.
  We need 3 or more goods to have complements in the Hicksian sense.

- Finally, good i is a **gross substitute** for j if x_i is increasing in p_j,
  and a **gross complement** if x_i is decreasing in p_j

- This one is *not* always symmetric, because the wealth effects may be different
  - For a couple examples of this asymmetry:[3]
    suppose there are two goods, rice and meat, and rice is a Giffen good
    If the price of meat goes up, you eat less meat and more rice,
    so rice is a gross substitute for meat;
    but if the price of rice goes up, you eat more rice, which means you eat less meat,
    so meat is a gross complement for rice.
  - For another example, consider "one-sided complements": cake mix and cake frosting
    If the price of cake mix goes up, you bake fewer cakes, so you demand less frosting,
    so frosting is a gross complement to cake mix
    But if the price of frosting goes up, you use less frosting, but maybe you substitute with a bigger unfrosted cake,
    so you use more cake mix, so cake mix is a gross substitute for frosting

- If we say two goods are gross substitutes, we mean each is a gross substitute for the other.

- And again, two goods could be substitutes in one price range but complements in another.

[3] from De Jaegher (2009), "Asymmetric Substitutability: Theory and Some Applications," *Economic Inquiry* 47.4