Some Beautiful Theorems
with Beautiful Proofs

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Introduction – Why Are We Here?

The basic goal of this semester is to prove five or so elegant results in micro theory. The list:

1. *The First Welfare Theorem* – any Walrasian equilibrium gives a Pareto-efficient allocation
2. *Arrow's Impossibility Theorem* – individual preferences don't "aggregate up" to societal preferences well
3. *A "No Trade" Theorem* – access to different information can't be the sole basis for trade
4. *Revenue Equivalence and the Optimal Auction* – under certain conditions, many standard auctions are equally good for the seller, and the revenue-maximizing auction (out of all possible formats) can be determined
5. *Impossibility of Efficient Bilateral Trade* – when both buyer and seller have private information, no mechanism can realize all possible gains from trade

Of course, doing this will require learning a bunch of different setups/environments that are standard in micro theory. We'll see an exchange economy with production; a simple voting world with ordinal preferences over policies; a world with an unknown true state of the world and differential private information about that state; and the benchmark environment for analyzing auctions and other trading mechanisms, the independent private values setup. And we'll deal with Bayesian Nash equilibrium – the standard extension of Nash equilibrium to settings with incomplete information.

The lectures are generally organized around the result to be proven. We'll typically start by defining the environment we're considering, and the goal – what problem we're interested in. Then once we've got everything defined, we'll state the theorem, and prove it. And hopefully briefly discuss its relevance.

Lecture 1

The First Welfare Theorem

Any Walrasian equilibrium allocation is Pareto-efficient.

- Basic exchange economy – lots of consumption goods, lots of individuals endowed with some of each good
- Money is a means of exchange, but has no consumption value and nobody's endowed with it, it just facilitates trade
- Firms are technologies for turning some goods into some other goods
- Each good has a "market price" – nobody knows where prices come from, but everyone's free to trade as much as they want at the market price
  - So you can sell some of your endowment and use that money to buy other stuff you want
- The key assumption is *price-taking behavior* – individuals, and firms, assume that market prices are fixed and outside their own control
  - All you can do is trade, or not trade, at the prices you see
- We're thinking about *general equilibrium*, or *competitive equilibrium*, or *Walrasian equilibrium*, which is when market prices equate supply and demand for each good
  - Given market prices, individuals demand the best consumption bundle they can afford
  - Given market prices, firms choose production to maximize profit (and pay that profit out to their shareholders)
  - And markets clear
- Today's result: any such equilibrium is a Pareto-efficient allocation.

1 What Environment Are We In?

We begin by defining a very general exchange economy with production.

- There are M different goods
  - We'll let p_m denote the price per unit of good m, and $p \in (\Re^+)^M$ the vector of all prices
- There are N consumers
  - Each consumer $i \in \{1, 2, \ldots, N\}$ has some endowment $e_i \in (\Re^+)^M$ of goods...
  - ...and a utility function $u_i : (\Re^+)^M \to \Re$ over how much he/she consumes of each good
  - We'll let $x_i \in (\Re^+)^M$ denote how much i consumes of each good, giving utility $u_i(x_i)$
- There are F firms
  - Each firm $j \in \{1, 2, \ldots, F\}$ is represented by a *production set* $Y_j \subset \Re^M$, which consists of all feasible production plans for that firm
  - For example, if $M = 5$ and the vector $(-1, -1, 1, 0, 0) \in Y_j$, then firm j has the ability to turn one unit each of goods 1 and 2 into one unit of good 3
  - (Technology need not scale – if $y \in Y_j$, it does not imply that $2y \in Y_j$)
  - We'll let $y_j \in Y_j$ denote the production plan that firm j chooses
- The firms are owned by the consumers
  - Consumer i owns a share θ_{ij} of firm j, with $\sum_{i=1}^N \theta_{ij} = 1$ for each j
  - If firm j produces y_j at prices p, it earns profits $\pi_j = p \cdot y_j$, and pays $\theta_{ij}\pi_j$ to consumer i
- Finally, one assumption on preferences:

  each consumer's preferences are **locally non-satiated**:

  for any i, any x_i, and any $\epsilon > 0$, there's some x_i' such that $\|x_i' - x_i\| \leq \epsilon$ and $u_i(x_i') > u_i(x_i)$

2 What Are We Trying To Do?

We're interested in two things: Pareto efficient consumption plans, and general equilibrium.

2.1 Pareto efficiency

- Let $x = (x_1, x_2, \ldots, x_N) \in (\Re^+)^{MN}$ be a consumption plan for each consumer
- x is *feasible* if it's technologically possible to produce that much of each good: that is, if there is some production plan $(y_1, y_2, \ldots, y_F) \in Y_1 \times Y_2 \times \ldots \times Y_F$ such that

$$\sum_{i=1}^{N} x_i \quad \leq \quad \sum_{i=1}^{N} e_i + \sum_{j=1}^{F} y_j$$

- A feasible consumption plan x is *Pareto efficient* if it's not Pareto-dominated by any other feasible consumption plan: that is, if there does not exist a feasible plan $x' = (x'_1, x'_2, \ldots, x'_M)$ with

$$u_i(x'_i) \quad \geq \quad u_i(x_i)$$

  for every i, with strict inequality holding for at least one i.

2.2 Walrasian Equilibrium

Walrasian equilibrium is defined, basically, as two things happening:

1. Consumers and firms are "price takers" – they take market prices as being outside their control – and behave optimally given the prices they see
2. Market prices are such that the demand for each good equals the supply, i.e., markets clear

Formally, a Walrasian equilibrium is a vector of prices p^*, a consumption plan for each consumer $x^* = (x_1^*, \ldots, x_N^*)$, and a production plan for each firm $y^* = (y_1^*, \ldots, y_F^*)$ such that:

- Given prices p^*, each firm is maximizing profits: for each $j \in \{1, 2, \ldots, F\}$,

$$y_j^* \quad \in \quad \arg\max_{y_j \in Y_j} p^* \cdot y_j$$

- Given prices p^* and their wealth (from both their endowment of goods and their income from the firms they own), each consumer is maximizing utility: for each i,

$$x_i^* \quad \in \quad \arg\max_{x_i} \left\{ u_i(x_i) \quad : \quad p^* \cdot x_i \leq p^* \cdot e_i + \sum_j \theta_{ij}(p^* \cdot y_j^*) \right\}$$

- Each market clears: element by element,

$$\sum_i x_i^* \quad = \quad \sum_i e_i + \sum_j y_j^*$$

3 Today's Big Result

Theorem (The First Fundamental Theorem of Welfare Economics)**.** *If* (p^*, x^*, y^*) *is a Walrasian equilibrium, then* x^* *is Pareto efficient.*

4 But before we prove this...

- Before we prove this, one more preliminary – **proof by contradiction**
- Many of our results will be proven this way
- We want to show that some claim B is true – or really, that it follows from some set of assumptions A
- So we want to show $A \longrightarrow B$
- Very often, it's easier to show that contrapositive: $\neg B \longrightarrow \neg A$
- Which can be summarized: "Suppose B was false. Show this leads to a contradiction. That proves B must be true."
- Classic example of this: Euclid's proof there must be an infinite number of prime numbers.
  - Suppose there was a finite set of prime numbers
  - Label them $\{p_1, p_2, \ldots, p_r\}$
  - Let $P = p_1 p_2 \cdots p_r + 1$
  - If P is prime, we have a contradiction, because it's bigger than any of the numbers on our list
  - If P is not prime, then some prime number divides it, call that p
  - p can't be on our list, because $P = ap + 1$; but then p is a prime that wasn't on our list, another contradiction
  - So either way, we've found a prime that wasn't on our list
  - Thus, the number of primes is infinite
- Now we're ready to prove the First Welfare Theorem

5 Proof of the First Welfare Theorem

- The proof is surprisingly straightforward. We will prove this by contradiction.
- Suppose the theorem is false, which means there is some Walrasian equilibrium (p^*, x^*, y^*) which is not Pareto-efficient
- Then x^* must be Pareto-dominated by some other feasible consumption plan

  Let $x' = (x'_1, x'_2, \ldots)$ be that consumption plan,

  and $y' = (y'_1, y'_2, \ldots)$ a feasible production plan that generates it ($\sum_i x'_i = \sum_i e_i + \sum_j y'_j$)
- First thing to show: **at prices p^*, x' must be more expensive than x^***
  - Since x' Pareto-dominates x^*, at least one guy *strictly* prefers x'_i to x^*_i; call him Bob
    * Since Bob chose x^*_{Bob} at prices p^*, he can't afford anything better
    * Since x'_{Bob} is better, that means he can't afford it – or $p^* \cdot x'_{Bob} > p^* \cdot x^*_{Bob}$
  - For everyone else, x'_i is *at least as good* as x^*_i – and this means it must also be *at least as expensive* at prices p^*
    * **I did this in a kind of hand-wavy way in class**, so let me do it properly here
    * Suppose this was false, i.e., for some i, $u_i(x'_i) \geq u_i(x^*_i)$ but $p^* \cdot x'_i < p^* \cdot x^*_i$
    * Let $\delta = p^* \cdot x^*_i - p^* \cdot x'_i$, and let $\epsilon = \frac{\delta}{\|p^*\|}$
    * Local non-satiation means there exists an x''_i with $\|x''_i - x'_i\| \leq \epsilon$ and $u_i(x''_i) > u_i(x'_i)$
    * Now, $p^* \cdot x''_i = p^* \cdot (x''_i - x'_i) + p^* \cdot x'_i = p^* \cdot (x''_i - x'_i) + (p^* \cdot x^*_i - \delta)$
    * Recall from years ago that $a \cdot b = \|a\|\|b\| \cos\theta$, where θ is the angle between vectors a and b; or more significantly, $a \cdot b \leq \|a\|\|b\|$
    * This means $p^* \cdot (x''_i - x'_i) \leq \|p^*\|\epsilon = \|p^*\| \frac{\delta}{\|p^*\|} = \delta$
    * So $p^* \cdot x''_i \leq \delta + (p^* \cdot x^*_i - \delta) = p^* \cdot x^*_i$
    * But $u_i(x''_i) > u_i(x'_i) \geq u_i(x^*_i)$, which contradicts x^*_i being chosen in equilibrium
    * So the contradiction proves that $p^* \cdot x'_i \geq p^* \cdot x^*_i$
    * (Note that this is the only step of the proof where we need local non-satiatedness.)
  - So if we sum up over all consumers,

$$\textstyle\sum_{i=1}^N (p^* \cdot x'_i) \quad > \quad \sum_{i=1}^N (p^* \cdot x^*_i)$$

- Now, we assumed the production plan y' would generate x', or $\sum_i x'_i = \sum_i e_i + \sum_j y'_j$
- And since markets cleared in equilibrium, $\sum_i x^*_i = \sum_i e_i + \sum_j y^*_j$

- Plugging these into our last equation,

$$p^* \cdot \left(\sum_i e_i + \sum_j y'_j\right) \quad > \quad p^* \cdot \left(\sum_i e_i + \sum_j y^*_j\right)$$

or, subtracting $p^* \cdot \sum_i e_i$ from both sides and pulling the dot product inside the summation,

$$\sum_j p^* \cdot y'_j \quad > \quad \sum_j p^* \cdot y^*_j$$

- But if the left-side sum is bigger than the right, then at least one of the summands must be bigger – there's some firm k such that

$$p^* \cdot y'_k \quad > \quad p^* \cdot y^*_k$$

- But by definition, firm k was supposed to be maximizing profits at p^* by choosing y^*_k, and there's our contradiction!

6 Where do we go from here?

- We just proved: if we "get to" an equilibrium, it's efficient
- But we don't yet even know for sure an equilibrium exists
- Turns out, we can prove it does, but only with some additional assumptions – it's pretty easy to find examples of environments that do not have a Walrasian equilibrium.
- If there are no firms, just consumers, existence of an equilibrium is guaranteed if...
  - each u_i is continuous
  - each u_i is increasing
  - each u_i is concave
  - and every consumer has a strictly positive endowment of every good ($e^i \gg 0$)

  But if any of these conditions is violated, you can generate examples with no equilibrium.

- For one simple example, take two goods, and let $u_1(x, y) = \min\{x, y\}$ and $u_2(x, y) = \max\{x, y\}$, which is convex

  Let $e^1 = e^2 = (1, 1)$

  The problem is that at any positive prices, the first guy demands equal amounts of the two goods, which means he refuses to trade; but the second guy demands to trade his endowment of one good for as much as he can afford of the other good; so markets can't clear at positive prices

  But if either price is 0, the second guy demands an infinite amount of that good, and again, markets can't clear

- For another example, suppose there are two goods and two consumers, and endowments are not strictly positive: $e_1 = (10, 0)$ and $e_2 = (0, 10)$. Utility functions are $u_1(x, y) = x$ and $u_2(x, y) = \sqrt{x} + y$. It turns out, no prices exist that clear markets:
  - If either price is zero, player 2 will demand an infinite amount of that good, so markets won't clear
  - If both prices are strictly positive, player 1 will demand exactly his endowment; but player 2 will demand some of good 1, so markets won't clear
- Even if we know a Walrasian equilibrium exists, we haven't said why we would expect it to be reached...
- But if we do believe that Walrasian equilibrium will naturally occur, then the First Welfare Theorem is sort of the competitive-markets analog of the Coase Theorem – let people trade, and an efficient outcome will inevitably be reached
- (The "price-takers" assumption can be thought of as the assumption that each person, and each firm, is small relative to the "market," so that everyone believes his own impact on market prices is negligible.)
- There's also the Second Welfare Theorem – that given an economy, *any* Pareto-efficient consumption plan is a Walrasian equilibrium for some set of prices and endowments – but this also requires some additional assumptions
  - Basically, we need continuous, concave utility functions and convex production sets (if we allow firms)
  - Under those assumptions, any Pareto-efficient allocation is a Walrasian equilibrium for some set of initial endowments

References

- Mas-Colell, Whinston and Green (1995), *Microeconomic Theory*, Oxford University Press (the standard first-year PhD micro textbook), ch. 16.