Ben-Porath Model

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As we discussed before, the Mincer model is not structural in the classic sense

The Ben-Porath model is a structural model of investment on the job

The model is

- Finite lived to time T
- continuous time
- interest rate r
- earnings $E(t)$

People make human capital investment decisions to maximize the present value of income

$$\int_0^T e^{-rt} E(t) dt$$

We assume that earnings take the form

$$E(t) = H(t) [1 - I(t)] - D(t)$$

Where

- $I(t)$: time spent investing in human capital
- $H(t)$: Human capital itself
- $D(t)$: Direct costs of human capital investment

Thus the present value of earnings can be written as

$$\int_0^T e^{-rt} \left(H(t) [1 - I(t)] - D(t) \right) dt$$

The human capital production function is defined as

$$\dot{H} = A\,(IH)^{\alpha}\,D^{\beta} - \sigma H$$

where σ is the rate of depreciation in human capital

The one other thing we need to solve this model is the initial level of human capital $H(0)$

Now we can write down the Hamiltonian as

$$\mathcal{H} = e^{-rt}\left(H(t)\left[1 - I(t)\right] - D(t)\right) + \mu\left(t\right)\left[A\left(IH\right)^{\alpha} D^{\beta} - \sigma H\right]$$

This gives first order conditions:

$$\begin{aligned} I &: \quad e^{-rt}H = \mu\alpha A I^{\alpha-1} H^{\alpha} D^{\beta} \\ D &: \quad e^{-rt} = \mu\beta A\left(IH\right)^{\alpha} D^{\beta-1} \end{aligned}$$

and

$$\begin{aligned} \dot{\mu} &= \frac{-\partial\mathcal{H}}{\partial H} \\ &= -e^{-rt}\left(1 - I\right) - \mu\left[\alpha A I^{\alpha} H^{\alpha-1} D^{\beta} - \sigma\right] \end{aligned}$$

Take the ratio of the first two first order conditions:

$$\begin{aligned} H &= \frac{\mu\alpha A I^{\alpha-1} H^{\alpha} D^{\beta}}{\mu\beta A (IH)^{\alpha} D^{\beta-1}} \\ &= \frac{\alpha D}{\beta I} \end{aligned}$$

or

$$D = \frac{\beta}{\alpha} IH$$

Since direct costs of investment D are just a multiple of time costs IH, the distinction between the two is not interesting (of course with borrowing constraints this would no longer be true)

That is we can redefine the model so that

$$\begin{aligned} I^* &= \left(1+\frac{\beta}{\alpha}\right) I \\ \alpha^* &= \alpha+\beta \\ A^* &= A\left(\frac{\beta}{\alpha}\right)^{\beta}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha+\beta} \end{aligned}$$

With this notation, you can see that

$$\begin{aligned} A^*\left(I^* H\right)^{a^*} &= A\left(\frac{\beta}{\alpha}\right)^{\beta}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha+\beta}\left(\left(1+\frac{\beta}{\alpha}\right) IH\right)^{\alpha+\beta} \\ &= A\left(\frac{\beta}{\alpha}\right)^{\beta}(IH)^{\alpha+\beta} \\ &= A\,(IH)^{\alpha}\left(\frac{\beta}{\alpha} IH\right)^{\beta} \\ &= A\,(IH)^{\alpha}(D)^{\beta} \end{aligned}$$

It is easy to show that everything else goes through as well.

Thus there is no need to worry about D

Lets abstract from it by using the redefined model (without the * notation)

Then we have first order conditions:

$$
\begin{aligned}
e^{-rt} &= \mu\alpha A I^{\alpha-1} H^{\alpha-1} \\
\dot{\mu} &= -e^{-rt}(1-I) - \mu\left[\alpha A I^{\alpha} H^{\alpha-1} D^{\beta} - \sigma\right] \\
&= -e^{-rt} + \sigma\mu + I\left[e^{-rt} - \mu\alpha A I^{\alpha-1} H^{\alpha-1}\right] \\
&= -e^{-rt} + \sigma\mu
\end{aligned}
$$

Define

$$g(t) = e^{rt}\mu$$

Then

$$\begin{aligned}\frac{\partial g}{\partial t} &= re^{rt}\mu + e^{rt}\dot{\mu} \\ &= re^{rt}\mu - 1 + e^{rt}\sigma\mu \\ &= (r+\sigma)g - 1\end{aligned}$$

We want to solve for this differential equation, but we don't know $g(0)$.

However, we do know that $\mu(T) = 0$ which implies that $g(T) = 0$.

This is straight forward to solve, it yields

$$g(t) = \frac{1 - e^{(\sigma+r)(t-T)}}{\sigma + r}$$

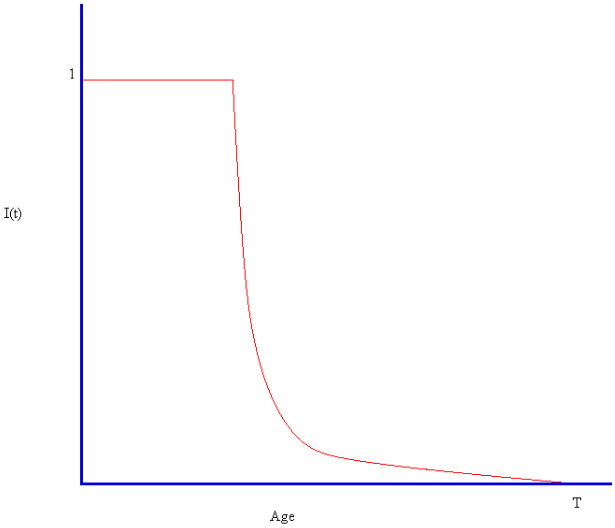
You can see that

- $g(T) = 0$
- $g(t)$ is strictly decreasing with t
- From the first order condition for investment

$$\begin{aligned} I(t)H(t) &= (\alpha A g(t))^{\frac{1}{1-\alpha}} \\ I(t) &= \frac{(\alpha A g(t))^{\frac{1}{1-\alpha}}}{H(t)} \end{aligned}$$

- investment IH, is decreasing with t
- IH doesn't depend on $H(0)$ (Ben-Porath neutrality)
- Investment is decreasing with H
- What happens to H(t) depends on investment versus depreciation
- It makes sense to impose that Investment time is bounded from above by 1

This gives a pattern of investment



And a pattern of earnings