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STRONG LAWS FOR DEPENDENT HETEROGENEOUS PROCESSES

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This paper presents maximal inequalities and strong law of large numbers for weakly dependent heterogeneous random variables. Specifically considered are L^r mixingales for $r > 1$, strong mixing sequences, and near epoch dependent (NED) sequences. We provide the first strong law for L^r-bounded L^r mixingales and NED sequences for $1 < r < 2$. The strong laws presented for α-mixing sequences are less restrictive than the laws of McLeish [8].

1. INTRODUCTION

Strong laws of large numbers (SLLNs) and maximal inequalities are routinely used in theoretical statistics and econometrics, including proofs of consistency of estimators. For weakly dependent data, the standard reference is McLeish [8]. McLeish's results include a maximal inequality for L^2 mixingales, and SLLNs for L^2 mixingales, α-mixing (strong mixing) sequences, and near epoch dependent (NED) sequences. His mixingale and NED results require L^2-bounded random variables. The α-mixing result ([8], Theorem 2.8) requires only that the variables be L^r bounded for some $r > 1$, but requires that the mixing coefficients be of size $-r/(r-1)$. In order to allow for r close to unity, this effectively requires exponential decay for the mixing coefficients. This is unsettling for several reasons. First, moment conditions much greater than unity may be excessive in nonlinear models where complicated functions of the data are being manipulated. Second, not all economic data may possess a finite second moment. Stock returns are an obvious possibility. Third, there has also been substantial recent attention to the possibility of nonexponential decay rates for measures of dependence. The fractionally integrated model of Granger and Joyeux [5], recently investigated in Sowell [11], is one example.

This paper makes the following contributions. First, maximal inequalities and SLLNs are provided for L^r mixingales, $r > 1$. These are the only results available for $1 < r < 2$, but impose stronger conditions than Corollary 1.9 of [8] when $r = 2$. Second, an SLLN is provided for L^r-bounded α-mixing sequences, requiring that $\sum_1^\infty \alpha_m^{1-1/2p} < \infty$ for some $p < r$. This is less restrictive (especially for r close to unity) than $\sum_1^\infty \alpha_m^{1-1/q} < \infty$, for some $q > r$, which is slightly weaker than McLeish's condition that $\{\alpha_m\}$ be of size

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$-r/(r-1)$. Third, an SLLN is provided for L^r-bounded NED sequences, $r > 1$. This is the only result available for $1 < r < 2$. For $r > 2$, our new result complements McLeish's, with weaker (stronger) conditions on the α-mixing (NED) numbers.

The new SLLNs developed in this paper can be used directly to provide consistency proofs for standard linear time series models. Additionally, combined with generic uniform LLNs such as [1] and [10], we can relax the assumptions necessary to prove consistency in nonlinear dynamic models.

Even though the conditions for the SLLNs presented here are the least restrictive known, there still may be some room for improvement. It is known [9] that if we restrict attention to linear processes, strong uniform integrability can replace the L^r bound. The best available WLLN under weak dependence [2] only requires uniform integrability and imposes no decay rate on the mixing coefficients (other than that they converge to zero). This suggests that it may be possible to further relax the requirements imposed here.

Section 2 derives the main results for L^r mixingales, and contrasts these results with those for linear processes. Section 3 applies the results to mixing and near epoch dependent processes. The proofs of the results are self-contained (other than the use of standard inequalities) and are left to the appendix.

2. STRONG LAWS FOR L^r MIXINGALES

Let $(\Omega, \mathcal{F}, \mathcal{P})$ denote a probability space on which there is a sequence of random variables $\{X_i\}_1^\infty$. Let $\{\mathcal{F}_i\}_1^\infty$ be a nondecreasing sequence of sub σ-fields of $\mathcal{F}$, $E_m X_i = E(X_i | \mathcal{F}_m)$ denote the conditional expectation of X_i given $\mathcal{F}_m$, $\|X_i\|_r = (E|X_i|^r)^{1/r}$ denote the L^r norm of a random variable, and $S_j = \sum_{i=1}^j X_i$. We will maintain for the rest of the paper without loss of generality that $EX_i = 0$. Assume that $\mathcal{F}_i$ decreases to $\mathcal{F}_{-\infty}$ which contains only invariant events, and $\mathcal{F}_i$ increases to $\mathcal{F}_\infty$ such that for each $E \in \mathcal{F}_\infty$, there exists $F \in \mathcal{F}$ with $P(E \Delta F) = 0$.

The concept of L^2 mixingales was introduced by McLeish [8], and generalized to L^r mixingales by Andrews [2].

DEFINITION 1. *The sequence $\{X_i, \mathcal{F}_i\}$ is an L^r mixingale if there exist nonnegative constants $\{c_i : i \geq 1\}$ and $\{\psi_m : m \geq 0\}$ such that $\psi_m \downarrow 0$ as $m \uparrow \infty$ and for all $i \geq 0$ and $m \geq 0$ we have*

(a) $\|E_{i-m} X_i\|_r \leq c_i \psi_m$,
(b) $\|X_i - E_{i+m} X_i\|_r \leq c_i \psi_{m+1}$.

Note that condition (b) holds trivially if X_i is adapted to $\mathcal{F}_i$. ∎

Assumption 1. $\{X_i, \mathcal{F}_i\}$ is an L^r mixingale for some $r > 1$.

The approach of this paper follows [8] in using the following representation for X_i which is valid in L^r for L^r mixingales:

$$X_i = \sum_{k=-\infty}^{\infty} X_{ki}, \qquad X_{ki} = E_{i-k}X_i - E_{i-k-1}X_i. \tag{1}$$

The sum in (1) converges since for each i, $\{E_{i-k}X_i, \mathcal{F}_{i-k}\}$ is a reverse martingale which converges a.s. to zero as $k \uparrow \infty$, and for each i, $\{E_{i+k}X_i, \mathcal{F}_{i+k}\}$ is a martingale which converges a.s. to X_i as $k \uparrow \infty$.

The first two results are maximal inequalities.

LEMMA 1. *Under Assumption* 1, *there exists some* $\bar{K} < \infty$ *such that for all* $n \geq 1$,

$$\|\max_{j\leq n} |S_j|\,\|_r \leq \bar{K} \sum_{k=-\infty}^{\infty} \left(\sum_{i=1}^{n} \|X_{ki}\|_r^2 \right)^{1/2}. \qquad ■$$

Assumption 2. $\sum_{k=1}^{\infty} \psi_k < \infty$.

LEMMA 2. *Under Assumptions* 1 *and* 2, *there exists some* $K < \infty$ *such that*

$$\|\max_{j\leq n} |S_j|\,\|_r \leq K \left(\sum_{i=1}^{n} c_i^2 \right)^{1/2}. \qquad ■$$

These maximal inequalities imply the following convergence results.

COROLLARY 1. *Under Assumptions* 1 *and* 2, *and* $\sum_1^{\infty} c_i^2 < \infty$, S_n *converges almost surely.* ■

COROLLARY 2 (SLLN). *Under Assumptions* 1 *and* 2, *and* $\sum_1^{\infty} i^{-2} c_i^2 < \infty$, $n^{-1} S_n \to 0$ *almost surely.* ■

Remark 1. Note that the summability conditions in the corollaries are quite similar to the summability conditions in the classic Kolmogorov SLLN.

Remark 2. McLeish [8] achieved a SLLN for L^2 mixingales when $\{\psi_m\}$ are of size $-1/2$, which is implied by $\sum_1^{\infty} \psi_m^{2-\delta}$ for some $\delta > 0$. (Note that the mixingale coefficients may be taken to be monotone without loss of generality.) This is weaker than Assumption 2 for $r = 2$.

Remark 3. The use of Assumption 2 to prove a strong law is stronger than necessary in more specific cases. Consider the heterogeneous linear process with martingale difference innovations

$$X_i = \sum_{k=-\infty}^{\infty} a_k e_{i-k}, \qquad E_{i-1} e_i = 0$$

where $\mathcal{F}_i = \sigma(e_j : j \leq i)$ is the sigma field generated by $\{e_j : j \leq i\}$. Assume that

$$\sup_{i\geq 1} \|e_i\|_r = B < \infty, \qquad \text{for some } r > 1. \tag{2}$$

The condition

$$\sum_k |a_k| < \infty \tag{3}$$

is sufficient for $\{X_i, \mathcal{F}_i\}$ to be an L^r mixingale. The linear framework allows us to directly compute the random variables $X_{ki} = E_{i-k}X_i - E_{i-k-1}X_i = a_k e_{i-k}$. Thus,

$$\sum_{k=-\infty}^{\infty} \left(\sum_{i=1}^{n} \|X_{ki}/i\|_r^2 \right)^{1/2} \le B \sum_k |a_k| \left(\sum_1^n i^{-2} \right)^{1/2} < CB \sum_k |a_k| < \infty$$

where $C = (\sum_1^\infty i^{-2})^{1/2}$. By Lemma 1, the Cauchy criterion and the Kronecker lemma, (2) and (3) are sufficient for a SLLN:

$$n^{-1} \sum_1^n X_i \to 0 \text{ a.s.}$$

In contrast, an application of Lemma 2 requires the calculation of

$$\|E_{i-m}X_i\|_r = \left\| \sum_{k=m}^{\infty} a_k e_{i-k} \right\|_r \le \sum_m^\infty \|a_k e_{i-k}\|_r \le \sum_m^\infty |a_k| B,$$

and

$$\|X_i - E_{i+m}X_i\|_r = \left\| \sum_{k=-m}^{-\infty} a_k e_{i-k} \right\|_r \le \sum_m^\infty |a_{-k}| B,$$

giving the mixingale coefficients $\psi_m = \sum_m^\infty (|a_k| + |a_{-k}|)$. Assumption 2 requires

$$\sum_{k=1}^{\infty} k(|a_k| + |a_{-k}|) < \infty, \tag{4}$$

which is more restrictive than (3).

Alternative results are possible. For example, [9, Theorem 3.13] provide a SLLN for linear processes when the innovations e_t satisfy strong uniform integrability (rather than L^r boundedness) and (4) (rather than (3)).

3. STRONG LAWS FOR MIXING AND NEAR EPOCH DEPENDENT PROCESSES

We can deduce strong laws for mixing and near epoch dependent (NED) sequences from the lemmas of the previous section. Mixing is frequently useful as a primitive assumption, for functions of mixing processes are mixing, while this is not true (in general) for mixingales. Let $\mathcal{F}_j^k = \sigma(X_i : j \le i \le k)$, and $\mathcal{F}_j = \sigma(X_i : i \le j)$.

DEFINITION 2. *The α-mixing coefficients of $\{X_i\}$ are given by*

$$\alpha_m \equiv \sup_j \sup_{\{F \in \mathcal{F}_{-\infty}^j, G \in \mathcal{F}_{j+m}^\infty\}} |P(G \cap F) - P(G)P(F)|$$

and $\{X_i\}$ is said to be α mixing if $\alpha_m \downarrow 0$ as $m \uparrow \infty$. ■

For the remainder of this section we will assume the existence of numbers r, q, and p which satisfy $1 < q < r$ and $1 \le p < r < 2p$.

Combining McLeish's strong mixing inequality [8, Lemma 2.1] with our Lemma 2 yields the following maximal inequality.

COROLLARY 3. $\sum_{m=1}^\infty \alpha_m^{1/q-1/r} < \infty$ *implies*

$\|\max_{j \le n} |S_j|\,\|_q \le K(\sum_1^n \|X_i\|_r^2)^{1/2}$. ■

Although Corollary 3 could be used to prove a strong law, we can achieve better results by following a similar argument to that in [8].

Assumption 3. $\sum_{m=1}^\infty \alpha_m^{1/q-1/2p} < \infty$.

LEMMA 3. *Assumption* 3 *and* $\sum_{i=1}^\infty \|X_i\|_r^{r/p} < \infty$ *imply that*

$$\sum_1^n (X_i - E[X_i 1(|X_i| \le 1)]) \text{ converges a.s. as } n \uparrow \infty,$$

where $1(\cdot)$ is the indicator function. ■

THEOREM 1 (SLLN). *Assumption* 3 *and* $\sum_1^\infty i^{-r/p} \|X_i\|_r^{r/p} < \infty$ *imply that*

$$n^{-1} \sum_1^n X_i \to 0 \text{ a.s.}$$ ■

COROLLARY 4. $\sum_1^\infty \alpha_m^{1-1/2p} < \infty$ *and* $\sup_i E|X_i|^r < \infty$ *imply that*

$$n^{-1} \sum_1^n X_i \to 0 \text{ a.s.}$$ ■

COROLLARY 5. $\sup_{i \ge 1} E|X_i/i^\alpha|^r < \infty$, *for* $\alpha > 0$, *and either*

(a) $1 < r < 2$, $\sum_1^\infty \alpha_m^{1/2} < \infty$, *and* $\alpha < (r-1)/r$

(b) $r \ge 2$, $\sum_1^\infty \alpha_m^{1-1/r} < \infty$, *and* $\alpha < 1/2 - \delta$ *for some* $\delta > 0$,

imply that

$$n^{-1} \sum_1^n X_i \to 0 \text{ a.s.}$$ ■

Lemma 3 is not particularly useful in itself, but provides an important step in the proof of Theorem 1. Corollaries 4 and 5 give two useful applications of Theorem 1. Corollary 4 probably has widest applicability, imposing a uniform L^r-bound and quite weak conditions on the mixing coefficients. Corollary 5 places more strict requirements on the mixing coefficients, but allows for explosive growth in the rth moment.

Strong laws can also be found for L^r-NED sequences. Near epoch dependence in L^2 (also called functions of mixing processes) was introduced in [7] and used extensively in [3] and [8]. In the econometric literature, [12] and [4] have made extensive use of NED sequences. The concept of NED sequences is useful in cases where a random variable is a function of the infinite history of a process which is assumed to satisfy a mixing condition. It is frequently difficult to verify if the function is mixing, but it may be possible to verify near epoch dependence. A simple example is non-Gaussian AR processes. See [4] for more discussion and examples.

The concept of L^r-NED sequences was introduced in [2] to prove a weak law of large numbers. Here we provide a strong law. Denote for some random sequence $\{X_i\}$ the σ-fields $\mathcal{F}_j^k = \delta(X_i : j \le i \le k)$.

DEFINITION 3. *$\{Y_i\}$ is L^r-near epoch dependent (L^r-NED) with respect to $\{X_i\}$ if $\{X_i\}$ α mixing and there exist nonnegative constants $\{d_i : i \ge 1\}$ and $\{\nu_m : m \ge 0\}$ such that*

$$\| Y_i - E(Y_i \mid \mathcal{F}_{i-m}^{i+m}) \|_r \le d_i \nu_m$$

and $\nu_m \downarrow 0$ as $m \uparrow \infty$. ∎

Corollary 1 and an inequality from [2] allow us to establish the following result.

THEOREM 2. *If for some $q > 1$, $\{Y_i\}$ is L^q-NED on $\{X_i\}$, $EY_i = 0$, for $r > q$, $1 \le p < r < 2p$, $\sum_1^\infty \nu_m < \infty$, $\{X_i\}$ is α mixing with mixing coefficients satisfying Assumption 3, $\sum_1^\infty i^{-r/p} \| X_i \|_r^{r/p} < \infty$, and $\sum_1^\infty i^{-2} d_i^2 < \infty$, then*

$$n^{-1} \sum_1^n Y_i \to 0 \text{ a.s. as } n \uparrow \infty.$$ ∎

Note that the strong LLN for NED sequences proved in [8, Theorem 3.1] required square integrability, while the above theorem only requires r moments finite for some $r > 1$. For $r \ge 2$, the condition on the mixing coefficients in Theorem 2 is weaker than [8], but the latter only requires that the NED numbers ν_m be of size $-1/2$, which is weaker than the condition given in Theorem 2.

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APPENDIX

Proof of Lemma 1. Note that for all k, $\{X_{ki}, \mathcal{F}_{i-k}\}$ is a martingale difference sequence (MDS). Indeed,

$$E_{i-1-k}X_{ki} = E_{i-1-k}(E_{i-k}X_i - E_{i-k-1}X_i) = 0.$$

Now for $r > 1$,

$$\begin{aligned}\left\| \max_{j\le n} \left| \sum_{i=1}^{j} X_i \right| \right\|_r &= \left\| \max_{j\le n} \left| \sum_{i=1}^{j} \sum_{k=-\infty}^{\infty} X_{ki} \right| \right\|_r \le \left\| \sum_k \max_{j\le n} \left| \sum_{i=1}^{j} X_{ki} \right| \right\|_r \\ &\le \sum_k \left\| \max_{j\le n} \left| \sum_{i=1}^{j} X_{ki} \right| \right\|_r \le \sum_k \frac{r}{(r-1)} \left\| \sum_{i=1}^{n} X_{ki} \right\|_r \\ &\le \frac{r}{(r-1)} \sum_k \left(CE\left[\sum_1^n X_{ki}^2 \right]^{r/2} \right)^{1/r}, \qquad \text{for some } C < \infty, \\ &\le C^{1/r} \frac{r}{(r-1)} \sum_k \left(\left(\sum_1^n \|X_{ki}^2\|_{r/2} \right)^{r/2} \right)^{1/r} = \bar{K} \sum_k \left(\sum_1^n \|X_{ki}\|_r^2 \right)^{1/2}.\end{aligned}$$

The equality in the first line follows from (1). The five inequalities are the triangle inequality, Minkowski's, Doob's [6, p. 15], Burkholder's [6, p. 23], and Minkowski's. The final equality sets $\bar{K} = C^{1/r}r/(1 - r) < \infty$. ∎

Proof of Lemma 2. For $k \ge 0$,

$$\|X_{ki}\|_r \le \|E_{i-k}X_i\|_r + \|E_{i-k-1}X_i\|_r \le 2c_i\psi_k,$$

while for $k < 0$,

$$\|X_{ki}\|_r = \|(X_i - E_{i-k-1}X_i) - (X_i - E_{i-k}X_i)\|_r$$
$$\le \|X_i - E_{i-k-1}X_i\|_r + \|X_i - E_{i-k}X_i\|_r \le 2c_i\psi_k.$$

Thus for all k, $\sum_{i=1}^n \|X_{ki}\|_r^2 \le 4\sum_{i=1}^n c_i^2\psi_k^2$, and by Lemma 1

$$\|\max_{j\le n}|S_j|\|_r \le \bar{K}\sum_{k=-\infty}^{\infty}\left(4\sum_1^n c_i^2\psi_k^2\right)^{1/2} = K\left(\sum_1^n c_i^2\right)^{1/2}$$

where $K = 2\bar{K}\sum_{k=-\infty}^{\infty}\psi_k < \infty$ by Assumption 2. ∎

Proof of Corollary 1. For any $\alpha > 0$,

$$P\{\max_{j\le m}|S_{n+j} - S_n| > \alpha\} \le \alpha^{-r}\left\|\max_{j\le m}\left|\sum_{i=n+1}^{n+j} X_i\right|\right\|_r \le \alpha^{-r}K\left(\sum_{i=n+1}^{n+m} c_i^2\right)^{1/2}$$
$$\xrightarrow[m\uparrow\infty]{} \alpha^{-r}K\left(\sum_{n+1}^{\infty} c_i^2\right)^{1/2} \xrightarrow[n\uparrow\infty]{} 0$$

where the first inequality is Markov's and the second is Lemma 2. The convergence results hold since $\sum c_i^2$ converges. The conditions of the Cauchy criterion are satisfied and we conclude that S_n converges almost surely. ∎

Proof of Corollary 2. Since $\|E_{i-m}(X_i/i)\|_r \le (c_i/i)\psi_m$, by Corollary 1, $\sum_1^n(X_i/i)$ converges almost surely as $n \uparrow \infty$. The Kronecker lemma ([6], p. 31) reveals that $n^{-1}\sum_1^n i(X_i/i) = n^{-1}S_n \to 0$ a.s. ∎

Proof of Lemma 3. Define $X_{1i} = X_i 1(|X_i| \le 1)$. Now

$$\sum_{i=1}^{\infty} P\{|X_i| > 1\} \le \sum_1^{\infty}\|X_i\|_{r/p}^{r/p} \le \sum_1^{\infty}\|X_i\|_r^{r/p} < \infty.$$

The first inequality is Markov's and the second uses the fact that $\|X_i\|_{r/p} \le \|X_i\|_r$ when $r/p < r$. This implies that

$$P\{|X_i| > 1 \text{ infinitely often}\} = 0.$$

It follows that $\sum_1^n(X_i - EX_{1i})$ converges a.s. if $\sum_1^n(X_{1i} - EX_{1i})$ converges a.s. The latter will hold if $\{(X_{1i} - EX_{1i}), \mathcal{F}_i\}$ satisfies the conditions of Corollary 1. We show that the sequence is an L^q mixingale. By the α-mixing inequality [8, Lemma 2.1]

$$\|E_{i-m}X_{1i} - EX_{1i}\|_q \le \alpha_m^{1/q-1/2p}\|X_{1i}\|_{2p} = \psi_m c_i$$

where we set $\psi_m = \alpha_m^{1/q-1/2p}$ and $c_i = \|X_{1i}\|_{2p}$. By Assumption 3, $\sum_1^{\infty}\psi_m < \infty$. It remains to show that $\sum_1^{\infty} c_i^2 < \infty$. Now

$$\sum_1^{\infty} c_i^2 = \sum_1^{\infty}(E|X_{1i}|^{2p})^{1/p} \le \sum_1^{\infty}(E|X_{1i}|^r)^{1/p} \le \sum_1^{\infty}\|X_i\|_r^{r/p} < \infty.$$

The first inequality holds since $|X_{1i}| \le 1$ and $r < 2p$ imply that $|X_{1i}|^r \ge |X_{1i}|^{2p}$. ∎

Proof of Theorem 1. Since $\sum_1^\infty i^{-r/p}\|X_i\|_r^{r/p} = \sum_1^\infty \|(i^{-1}X_i)\|_r^{r/p}$, the assumptions and Lemma 3 imply that

$$\sum_{i=1}^{n} ((i^{-1}X_i) - E[(i^{-1}X_i)1(|(i^{-1}X_i)| \le 1)]) \text{ converges a.s.}$$

Hence, by the Kronecker lemma,

$$n^{-1}\sum_1^n (X_i - E[X_i 1(|X_i| \le i)]) \to 0 \text{ a.s.}$$

The proof is completed by showing

$$n^{-1}\sum_1^n E[X_i 1(|X_i| \le i)] = n^{-1}\sum_1^n E[X_i 1(|X_i| > i)] \to 0,$$

where the equality uses $EX_i = 0$. Now

$$\begin{aligned} |E[X_i 1(|X_i| > i)]| &\le E[|X_i| 1(|X_i| > i)] \\ &\le i^{-(r-p)/p} E(|X_i|^{1+(r-p)/p} 1(|X_i| > i)) \\ &= i^{1-r/p} E(|X_i|^{r/p} 1(|X_i| > i)) \\ &\le i^{1-r/p}\|X_i\|_{r/p}^{r/p} \le i^{1-r/p}\|X_i\|_r^{r/p}. \end{aligned}$$

The second inequality uses the fact that the expectation is taken over the region where $|X_i| > i$. We thus have

$$\left| n^{-1}\sum_1^n E[X_i 1(|X_i| > i)] \right| \le n^{-1}\sum_1^n i^{1-r/p}\|X_i\|_r^{r/p} \to 0 \text{ as } n \uparrow \infty$$

by the Kronecker lemma since $\sum_1^\infty i^{-r/p}\|X_i\|_r^{r/p} < \infty$. ■

Proof of Theorem 2. The proof is as above, making use of the following inequality from [2]:

$$\|E_{i-m}Y_{1i} - EY_{1i}\|_q \le c_i \psi_m,$$

where $Y_{1i} = Y_i 1(|Y_i| > 1)$, $c_i = 2d_i + \|Y_{1i}\|_{2p}$, and $\psi_{2m} = \nu_m + 6\alpha_m^{1/q-1/2p}$. ■

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There were some errors made in [2]. The results of Section 2 are stated to hold for L^r-mixingales, $r > 1$. They hold, however, only for $r \geq 2$. The proof of Lemma 1 on page 219 uses Minkowski's inequality in the $r/2$ norm, which requires that $r/2 \geq 1$. The author is grateful to Myoung-jae Lee for pointing out this error.

For $1 < r \leq 2$, $x \geq 0$ and $y \geq 0$, the inequality $(x + y)^{r/2} \leq x^{r/2} + y^{r/2}$ can be used in place of Minkowski's inequality to establish the following results for L^r-mixingales satisfying $\|E_{i-m}X_i\|_r \leq c_i\psi_m$. Set $S_j = \sum_{i=1}^{j} X_i$, $\bar{K} = 18[r/(1-r)]^{3/2}$, and $\Psi = \sum_1^\infty \psi_m$.

LEMMA 1.

$$\|\max_{j\leq n}|S_j|\,\|_r \leq \bar{K}\sum_{m=-\infty}^{\infty}\left(\sum_{i=1}^{n} E|E_{i-m}X_{im} - E_{i-m-1}X_i|^r\right)^{1/r}. \qquad \blacksquare$$

LEMMA 2.

$$\|\max_{j\leq n}|S_j\|_r \leq 2\bar{K}\Psi\left(\sum_{i=1}^{n} c_i^r\right)^{1/r}. \qquad \blacksquare$$

COROLLARY 1. *If $\Psi < \infty$ and $\sum_1^\infty c_i^r < \infty$, then S_n converges almost surely.* ■

COROLLARY 2. *If $\Psi < \infty$ and $\sum_1^\infty (c_i/i)^r < \infty$, then $S_n/n \to 0$ almost surely.* ■

Section 3 concerned zero-mean sequences $\{Y_i\}$ which are L^q near-epoch dependent ($q > 1$) upon some strong-mixing sequence $\{X_i\}$ with mixing coefficients α_m satisfying

$$\|E(Y_i|\mathfrak{F}_m) - Y_i\|_q \leq d_i\nu_m, \quad \text{where } \mathfrak{F}_m = \sigma(X_t : i - m \leq t \leq i + m).$$

For the case $1 < q < 2$ the following theorem follows directly from Corollary 2 and the near-epoch dependent inequality given in [1]:

THEOREM 2. *If for some $p > q$, $\sum_1^\infty \nu_m < \infty$, $\sum_1^\infty i^{-q} d_i^q < \infty$, $\sum_1^\infty \alpha_m^{1/q-1/p} < \infty$ and $\sum_1^\infty i^{-q}\|Y_i\|_q^p < \infty$, then $n^{-1}\sum_1^n Y_i \to 0$ a.s.* ■

It should also be noted that the original proof of Theorem 2 in [2] worked with the truncated sequence $Y_{1i} = Y_i 1(|Y_i| \leq 1)$, implicitly assuming that Y_{1i} is NED with the same coefficients as Y_i. This is not obviously true, and the author is grateful to Don Andrews for pointing out this error.

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2. Hansen, B.E. Strong laws for dependent heterogeneous processes. *Econometric Theory* 7 (1991): 213–221.