Econometrica, Vol. 72, No. 2 (March, 2004), 641–646

ASYMPTOTIC NORMALITY OF THE QMLE ESTIMATOR OF ARCH IN THE NONSTATIONARY CASE

BY SØREN TOLVER JENSEN AND ANDERS RAHBEK[1]

We establish consistency and asymptotic normality of the quasi-maximum likelihood estimator in the linear ARCH model. Contrary to the existing literature, we allow the parameters to be in the region where no stationary version of the process exists. This implies that the estimator is always asymptotically normal.

KEYWORDS: ARCH, asymptotic normality, asymptotic theory, consistency, GARCH, nonstationarity, quasi-maximum likelihood estimation.

1. INTRODUCTION

CONSIDER THE FIRST ORDER ARCH (autoregressive conditional heteroscedastic) model introduced by Engle (1982), as given by

$$
\begin{aligned}
&y_t = \sigma_t z_t, \\
&\sigma_t^2 = \omega + \alpha y_{t-1}^2,
\end{aligned} \tag{1}
$$

for $t = 1, \dots, T$, $\alpha > 0$, $\omega > 0$, and with z_t an i.i.d.(0, 1) process with

$$V(z_t^2) = E(z_t^4 - 1) = \zeta < \infty.$$

Asymptotic inference for the ARCH(1) and more general ARCH models, including GARCH models, has been studied in, e.g., Weiss (1986), Lee and Hansen (1994), Lumsdaine (1996), and Kristensen and Rahbek (2002). These papers, as well as others in the econometric literature, all assume as a minimal requirement that the ARCH process y_t is suitably ergodic or stationary such that laws of large numbers apply. We relax the condition on the stability of the y_t process and allow it to be nonstationary and in particular not to have any moments. Our only condition is that $V(z_t^2) < \infty$.

In the proofs we have aimed at a detailed presentation of the analysis of the score, information, and third derivatives of the likelihood function. We note that the arguments for the score and information in Lemma 3 and Lemma 4 are easily carried over to the stationary case by using the ergodic theorem similar to Lee and Hansen (1994) and Lumsdaine (1996). Moreover, we note that our result in Lemma 5 regarding the uniform boundedness of the third derivatives of the likelihood function does not use the nonstationary condition, and hence can be applied to the stationary case as well. As can be deduced from Remark 2, the established uniform boundedness corrects and significantly extends existing proofs of asymptotic normality for the ARCH(1) model.

[1]Anders Rahbek is grateful for support from the Danish Social Sciences Research Council, Centre for Analytical Finance (CAF) as well as the EU network DYNSTOCH. We thank David Lando for comments and discussions on our paper. We would also like to thank the two anonymous referees as well as the coeditor for constructive comments on our paper.

2. INFERENCE

By Nelson (1990) and Bougerol and Picard (1992), y_t is stationary (a stationary version exists) and ergodic if and only if $E\log(\alpha z_t^2) < 0$. In particular, if z_t is Gaussian, then the if and only if condition is that $\alpha < \frac{1}{2}\exp(-\Psi(\frac{1}{2})) \simeq 3.56$, where $\Psi(\cdot)$ is the Euler psi function; see Nelson (1990).

As mentioned our analysis is under the assumption that y_t does not have a stationary version or equivalently,

$$E\log(\alpha z_t^2) \geq 0. \tag{2}$$

Consider the likelihood estimator based on maximizing the quasi likelihood

$$\ell_T(\alpha) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log\sigma_t^2 + \frac{y_t^2}{\sigma_t^2}\right) \tag{3}$$

from which the QMLE (quasi-maximum likelihood estimator) $\hat{\alpha}$ is found. Note that this is the true likelihood if z_t is Gaussian and that we are conditioning on the initial value y_0. Our main result is the following:

THEOREM 1: *Assume that the ARCH process y_t in* (1) *does not allow a stationary version or equivalently* (2) *holds. Assume further that the* iid(0, 1) *process z_t is such that $V(z_t^2) = \zeta$ is finite, and the scale parameter ω is known. Then as $T \to \infty$ the sequence of QMLE $\hat{\alpha}$ is consistent, and asymptotically normal,*

$$\sqrt{T}(\hat{\alpha} - \alpha) \xrightarrow{D} N(0, \sigma^2),$$

where

$$\sigma^2 = \zeta\alpha^2 > 0.$$

REMARK 1: Note that if z_t is Gaussian, then $\sigma^2 = 2\alpha^2$ in Theorem 1.

PROOF: Together Lemmas 3, 4, and 5 in the next section establish the classical Cramér type conditions; see, e.g., Lehmann (1999). *Q.E.D.*

3. DERIVATION

For exposition and without loss of generality we henceforth set $\omega = 1$.

With the likelihood function given by (3), the score, information, and the third derivative of the log-likelihood with respect to α are found to be given by

$$\frac{\partial}{\partial\alpha}\ell_T(\alpha) = -\frac{1}{2}\sum_{t=1}^{T}\left(1 - \frac{y_t^2}{\sigma_t^2}\right)\frac{y_{t-1}^2}{\sigma_t^2}, \tag{4}$$

$$\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha) = \frac{1}{2}\sum_{t=1}^{T}\left(1 - 2\frac{y_t^2}{\sigma_t^2}\right)\frac{y_{t-1}^4}{\sigma_t^4}, \tag{5}$$

$$\frac{\partial^3}{\partial\alpha^3}\ell_T(\alpha) = -\sum_{t=1}^{T}\left(1 - 3\frac{y_t^2}{\sigma_t^2}\right)\frac{y_{t-1}^6}{\sigma_t^6}. \tag{6}$$

In the following we study the asymptotic behavior of these in order to establish consistency and asymptotic normality of the QMLE. First, consider the asymptotic behavior of y_t:

LEMMA 1: *Assume that* (2) *holds*; *then*

$$y_t^2 \overset{a.s.}{\to} \infty$$

as $t \to \infty$.

PROOF: This follows by Theorem 2 of Nelson (1990). *Q.E.D.*

Next, consider the asymptotic behavior of the following type of averages:

LEMMA 2: *Assume that* (2) *holds*; *then with* $m \leq k$ *positive integers*, *as* $t \to \infty$,

$$\frac{y_{t-1}^{2m}}{(1+\alpha y_{t-1}^2)^k} \overset{a.s.}{\to} \begin{cases} \dfrac{1}{\alpha^m} & \textit{if } m = k, \\ 0 & \textit{if } m < k, \end{cases} \tag{7}$$

and likewise, *as* $T \to \infty$,

$$\frac{1}{T}\sum_{t=1}^{T} \frac{y_{t-1}^{2m}}{(1+\alpha y_{t-1}^2)^k} \overset{a.s.}{\to} \begin{cases} \dfrac{1}{\alpha^m} & \textit{if } m = k, \\ 0 & \textit{if } m < k. \end{cases} \tag{8}$$

PROOF: The results follow by Lemma 1. *Q.E.D.*

Next turn to the score and the information:

LEMMA 3: *Under the assumptions of Theorem* 1, *then with* $\partial \ell_T(\alpha)/\partial\alpha$ *given by* (4),

$$\left(1/\sqrt{T}\right)\frac{\partial}{\partial\alpha}\ell_T(\alpha) \overset{D}{\to} N\left(0, \frac{\zeta}{4\alpha^2}\right)$$

as $T \to \infty$.

PROOF: By definition $(1/\sqrt{T})\partial\ell_T(\alpha)/\partial\alpha = (1/\sqrt{T})\sum_{t=1}^{T} s_t$ where

$$s_t = -\frac{1}{2}\left(1 - \frac{y_t^2}{\sigma_t^2}\right)\frac{y_{t-1}^2}{\sigma_t^2}.$$

The process s_t is a Martingale difference sequence with respect to $\mathcal{F}_t = \sigma\{y_t, y_{t-1}, \ldots, y_0\}$ as $E|s_t| \leq E|1 - z_t^2|/2\alpha < \infty$ and

$$E(s_t|\mathcal{F}_{t-1}) = -\frac{1}{2}E(1 - z_t^2)\frac{y_{t-1}^2}{\sigma_t^2} = 0.$$

Next, using (8),

$$\frac{1}{T}\sum_{t=1}^{T} E(s_t^2|\mathcal{F}_{t-1}) = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{y_{t-1}^2}{1+\alpha y_{t-1}^2}\right)^2 \frac{\zeta}{4} \overset{\text{a.s.}}{\to} \frac{\zeta}{4\alpha^2} > 0,$$

where $\zeta = E(1-z_t^2)^2 = V(z_t^2)$. Furthermore, as s_t^2 is bounded by $\mu_t^2 = (1-z_t^2)^2/4\alpha^2$ we derive the Lindeberg type condition,

$$\frac{1}{T}\sum_{t=1}^{T} E\big(s_t^2 \mathbb{1}\{|s_t| > \sqrt{T}\delta\}\big) \leq E\big(\mu_t^2 \mathbb{1}\{|\mu_t| > \sqrt{T}\delta\}\big) \to 0,$$

for some $\delta > 0$ and as T tends to ∞ using $V(z_t^2) = \zeta < \infty$. By the central limit theorem in Brown (1971) the desired result follows. *Q.E.D.*

LEMMA 4: *Under the assumptions of Theorem* 1, *then with the observed information* $\partial^2 \ell_T(\alpha)/\partial\alpha^2$ *given by* (5),

$$\frac{1}{T}\left(-\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha)\right) \overset{a.s.}{\to} \frac{1}{2\alpha^2} > 0$$

as $T \to \infty$.

PROOF: Rewrite minus the observed information as

$$\frac{1}{T}\left(-\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha)\right) = \frac{1}{2T}\sum_{t=1}^{T}\kappa_t\gamma_t$$

with

$$\kappa_t = 2z_t^2 - 1 \quad \text{and} \quad \gamma_t = \frac{y_{t-1}^4}{\sigma_t^4} = \frac{y_{t-1}^4}{(1+\alpha y_{t-1}^2)^2}.$$

The strong law of large numbers implies

$$\frac{1}{T}\sum_{t=1}^{T}\kappa_t \overset{\text{a.s.}}{\to} 1,$$

$$\frac{1}{T}\sum_{t=1}^{T}|\kappa_t| \overset{\text{a.s.}}{\to} \kappa < \infty,$$

while (7) implies $\gamma_t \overset{\text{a.s.}}{\to} (1/\alpha^2)$ and hence the desired result follows. *Q.E.D.*

Finally, we turn to the uniform boundedness of the third derivative of the likelihood function. Note that the proof does not require nonstationarity of the process.

LEMMA 5: *Denote by* $I(\alpha,\delta)$ *the interval* $[\alpha-\delta, \alpha+\delta]$, $0 < \delta < \alpha$. *Then with* $\partial^3\ell_T(\alpha)/\partial\alpha^3$ *given by* (6), *it holds that*

$$\sup_{\tilde{\alpha}\in I(\alpha,\delta)}\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| \leq g(\alpha,\delta,T) \overset{a.s.}{\to} \beta < \infty$$

as $T \to \infty$.

PROOF: With $\alpha_l = \alpha - \delta$,

$$\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| = \left|\frac{1}{T}\sum_{t=1}^{T}\left(3\frac{y_t^2}{\sigma_t^2}-1\right)\frac{y_{t-1}^6}{\sigma_t^6}\right| \leq \left|\frac{1}{T}\sum_{t=1}^{T}\left(3\frac{y_t^2}{\sigma_t^2}-1\right)\right|\frac{1}{\alpha_l^3}$$

$$= \left|\frac{1}{T}\sum_{t=1}^{T}\left(3\frac{\{1+\alpha y_{t-1}^2\}}{\{1+\tilde{\alpha} y_{t-1}^2\}}z_t^2-1\right)\right|\frac{1}{\alpha_l^3}$$

$$\leq \frac{1}{T}\sum_{t=1}^{T}\left(3\left\{1+\frac{\alpha}{\alpha_l}\right\}z_t^2+1\right)\frac{1}{\alpha_l^3} := g(\alpha, \delta, T)$$

and the results follows by the law of large numbers. *Q.E.D.*

REMARK 2: When deriving consistency and asymptotic normality, the classical sufficient condition regarding bounds of the third derivatives of the likelihood function is that

$$E \sup_{\tilde{\alpha}\in I(\alpha,\delta)}\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| < \infty.$$

In Basawa, Feigin, and Heyde (1976, condition (B.7)) this is incorrectly stated as

$$\sup_{\tilde{\alpha}\in I(\alpha,\delta)} E\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| < \infty.$$

The mistake is reproduced in Weiss (1986) and next in Lumsdaine (1996), and their proofs may therefore not be complete.

4. CONCLUSION

In this paper we have shown that for the ARCH(1) model the QMLE is always asymptotically Gaussian so long as the fourth order moment of the innovations z_t is finite. This is somewhat surprising as most researchers have assumed one needs strict stationarity.

For financial applications most often it is the GARCH(1, 1) as opposed to the ARCH(1) model that is applied and hence of much interest. In a forthcoming paper, which follows on from the developments given in this paper, Jensen and Rahbek (2003) show that indeed the results hold for the GARCH(1, 1) model as well. That is, whether or not the process is stationary, asymptotic normality holds and hence there are no "knife edge results like [in] the unit root case" as conjectured by Lumsdaine (1996, p. 580). The derivations for the GARCH(1, 1) case are more involved and lengthy due to the added complexity of the lagged variance in the σ_t^2 specification.

Department of Applied Mathematics and Statistics, University of Copenhagen, Denmark; soerent@math.ku.dk

and

Department of Applied Mathematics and Statistics, University of Copenhagen, Denmark; rahbek@math.ku.dk.

Manuscript received October, 2002; final revision received July, 2003.

REFERENCES

BASAWA, I. V., P. D. FEIGIN, AND C. C. HEYDE (1976): "Asymptotic Properties of Maximum Likelihood Estimators for Stochastic Processes," *Sankyā*, Series A, 38, 259–270.

BOUGEROL, P., AND N. PICARD (1992): "Stationarity of GARCH Processes and of Some Nonnegative Time Series," *Journal of Econometrics*, 52, 115–127.

BROWN, B. M. (1971): "Martingale Central Limit Theorems," *Annals of Mathematical Statistics*, 42, 59–66.

ENGLE, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of United Kingdom Inflation," *Econometrica*, 50, 987–1007.

JENSEN, S. T., AND A. RAHBEK (2003): "Asymptotic Normality for Non-Stationary, Explosive GARCH," Preprint No. 4, Department of Applied Mathematics and Statistics, University of Copenhagen.

KRISTENSEN, D., AND A. RAHBEK (2002): "Asymptotics of the QMLE for a Class of ARCH(q) Models," Preprint No. 10, Department of Applied Mathematics and Statistics, University of Copenhagen.

LEE, S.-W., AND B. HANSEN (1994): "Asymptotic Theory for the GARCH(1, 1) Quasi-Maximum Likelihood Estimator," *Econometric Theory*, 10, 29–53.

LEHMANN, E. L. (1999): *Elements of Large-Sample Theory*. New York: Springer-Verlag.

LUMSDAINE, R. (1996): "Consistency and Asymptotic Normality of the Quasi-Maximum Likelihood Estimator in IGARCH(1, 1) and Covariance Stationary GARCH(1, 1) Models," *Econometrica*, 64, 575–596.

NELSON, D. B. (1990): "Stationarity and Persistence in the GARCH(1, 1) Model," *Econometric Theory*, 6, 318–334.

WEISS, A. (1986): "Asymptotic Theory for ARCH Models," *Econometric Theory*, 2, 107–131.