Econometrica, Vol. 72, No. 3 (May, 2004), 885–925

ECONOMETRIC ANALYSIS OF REALIZED COVARIATION: HIGH FREQUENCY BASED COVARIANCE, REGRESSION, AND CORRELATION IN FINANCIAL ECONOMICS

BY OLE E. BARNDORFF-NIELSEN AND NEIL SHEPHARD[1]

This paper analyses multivariate high frequency financial data using realized covariation. We provide a new asymptotic distribution theory for standard methods such as regression, correlation analysis, and covariance. It will be based on a fixed interval of time (e.g., a day or week), allowing the number of high frequency returns during this period to go to infinity. Our analysis allows us to study how high frequency correlations, regressions, and covariances change through time. In particular we provide confidence intervals for each of these quantities.

KEYWORDS: Power variation, realized correlation, realized covolatility, realized regression, realized variance, semimartingales, covolatility.

1. INTRODUCTION

1.1. *Motivation and Definitions*

THIS PAPER PROVIDES a set of probabilistic laws for the analysis of the covariation between asset returns. Based on a fixed interval of time (e.g., a trading day or a calendar month), the number of high frequency returns during this period is assumed to go to infinity. We are able to derive an asymptotic distributional analysis of realized covariation—the sum of outer products of high frequency vectors of returns. The new theory allows us to study how covariances, correlations, and regression coefficients change through time by carrying out inference on these quantities over sequences of nonoverlapping intervals of time.

Our theoretical development is motivated by the advent of complete records of quotes or transaction prices for many financial assets. Although market microstructure effects (e.g., discreteness of prices, bid/ask bounce, irregular trading, etc.) mean that there is a mismatch between asset pricing theory based on semimartingales and the data at very fine time intervals, it does suggest the desirability of establishing an asymptotic distribution theory for estimators as we use more and more highly frequent observations.

[1]Ole E. Barndorff-Nielsen's work is supported by CAF (www.caf.dk), which is funded by the Danish Social Science Research Council, and by MaPhySto (www.maphysto.dk), which is funded by the Danish National Research Foundation. Neil Shephard's research is supported by the UK's ESRC through the grant "High Frequency Financial Econometrics Based upon Power Variation." All the calculations made in this paper are based on software written by the authors using the Ox language of Doornik (2001). We thank Torben Andersen, Tim Bollerslev, Clive Bowsher, Frank Gerhard, and Nour Meddahi for very helpful conversations on this topic. Finally, we thank the co-editor and referees for their detailed and constructive comments on earlier revisions of this paper.

Let the log-price of a q dimensional vector of assets be written as $y^*(t)$ for $t \geq 0$. Here t represents continuous time. The starred superscript is used to denote the fact that later we will assume that y^* is an integrated process in continuous time. Next, consider a fixed interval of time of length $\hbar > 0$. For concreteness we typically refer to $\hbar$ as representing a day. Traditional daily returns are computed as

$$y_i = y^*(i\hbar) - y^*((i-1)\hbar) \qquad (i = 1, 2, \ldots),$$

where i indexes the day. However, our focus will be on the case where we additionally have M equally spaced intra-$\hbar$ high frequency observations during each $\hbar$ time period. The jth intra-$\hbar$ return for the ith period (e.g., if $\hbar$ is a day and $M = 1440$, then this is the return for the jth minute on the ith day) will be calculated as

$$(1) \qquad y_{j,i} = y^*\left((i-1)\hbar + \frac{\hbar j}{M}\right) - y^*\left((i-1)\hbar + \frac{\hbar(j-1)}{M}\right) \qquad (j = 1, \ldots, M).$$

High frequency returns allow us to compute

$$(2) \qquad [y_M^*]_i = \sum_{j=1}^{M} y_{j,i} y_{j,i}',$$

the *realized covariation matrix* for the ith day. In this introductory section we will see that $[y_M^*]_i$ is, in a sense that will be made precise in (8) and (9), an ex-post estimator of the covariability of the unpredictable part of y^* over the interval of time from $(i-1)\hbar$ to $i\hbar$. The notation $[y_M^*]_i$ is designed to reflect that this matrix is based on the y^* process using M intra-$\hbar$ observations and computed on the ith day. The reason for the use of the square brackets will become clearer in a moment when we recall the idea of quadratic variation.

The realized covariation matrix is clearly different from the empirical covariance matrix of high frequency returns

$$\frac{1}{M}\sum_{j=1}^{M} y_{j,i} y_{j,i}' - \left(\frac{1}{M}\sum_{j=1}^{M} y_{j,i}\right)\left(\frac{1}{M}\sum_{j=1}^{M} y_{j,i}\right)' = \frac{1}{M}\sum_{j=1}^{M} y_{j,i} y_{j,i}' - \frac{1}{M^2} y_i y_i'.$$

In high frequency finance this quantity does not make sense for it will converge in probability to a matrix of zeros as $M \to \infty$. Realized covariation is roughly M times the empirical covariance of returns, the difference being that realized covariation ignores the $(1/M) y_i y_i'$ term as it is stochastically of smaller order than $\sum_{j=1}^{M} y_{j,i} y_{j,i}'$.

The properties of the realized covariation matrix are the main topic of our paper. We will establish the asymptotic distribution of $[y_M^*]_i$ as $M \to \infty$, thus providing a guide to its behavior for a finite value of M. The result is important in its own right, but it also implies a distribution theory for quantities derived from $[y_M^*]_i$. Examples of this include realized regression and realized correlation.

1.2. *Quadratic Variation and Semimartingales*

The probability limit of $[y_M^*]_i$ is well known when we assume y^* is a semimartingale ($\mathcal{SM}$) by using the theory of quadratic variation. Here we briefly review this material before going beyond it to develop the corresponding asymptotic distribution theory.

Recall if $y^* \in \mathcal{SM}$, then it can be decomposed as

$$y^*(t) = \alpha^*(t) + m^*(t), \tag{3}$$

where α^*, a drift term, is a process with *finite variation* ($\mathcal{FV}$) paths and m^* is a *local martingale* ($\mathcal{M}_{loc}$). For a very accessible discussion of probabilistic aspects of this, see Protter (1990), while its attraction from an economic viewpoint is discussed by Back (1991). We will sometimes restrict various classes of processes to those with continuous sample paths. We generically denote this with superscripts c; e.g., $\mathcal{M}_{loc}^c$ stands for the class of continuous local martingales.

For all $y^* \in \mathcal{SM}$ the *quadratic variation* (QV) or *covariation* process can be defined as

$$[y^*](t) = \mathrm{p}\text{-}\lim_{M\to\infty} \sum_{j=0}^{M-1} \{y^*(t_{j+1}) - y^*(t_j)\}\{y^*(t_{j+1}) - y^*(t_j)\}', \tag{4}$$

for any sequence of partitions $t_0 = 0 < t_1 < \cdots < t_M = t$ with $\sup_j\{t_{j+1} - t_j\} \to 0$ for $M \to \infty$. Here p$-$lim denotes the probability limit of the sum. Thus QV can be thought of as the sum of outer products of return vectors computed over infinitesimal time intervals calculated during the period from time 0 up to time t. In general (e.g., Jacod and Shiryaev (1987, p. 55)) if $y^* \in \mathcal{SM}$ and $\alpha^* \in \mathcal{FV}^c$, then it can be shown that $[y^*](t) = [m^*](t)$, which holds irrespective of the presence of jumps in the local martingale component.

The definition of QV immediately implies that for all $y^* \in \mathcal{SM}$ and as $M \to \infty$,

$$[y_M^*]_i \xrightarrow{p} [y^*](\hbar i) - [y^*](\hbar(i-1)) = [y^*]_i,$$

meaning realized covariation, $[y_M^*]_i$, consistently estimates increments of QV, $[y^*]_i$.

In the univariate case the connection between realized covariation and quadratic variation is discussed in the econometric literature by independent and concurrent work by Comte and Renault (1998), Barndorff-Nielsen and Shephard (2001), and Andersen and Bollerslev (1998). It was later developed and applied in some empirical work by Andersen, Bollerslev, Diebold, and Labys (2001). See also Barndorff-Nielsen and Shephard (2001) and Andersen, Bollerslev, Diebold, and Labys (2003) for a discussion of the multivariate case and Andersen, Bollerslev, and Diebold (2004) for an incisive survey of this area. Andersen, Bollerslev, Diebold, and Ebens (2001) discuss the use of the multivariate theory in the context of equity prices.

1.3. *Continuous Stochastic Volatility Semimartingales*

The above theoretical framework is too general for us to be able to derive a distribution theory for $[y_M^*]_i - [y^*]_i$, the difference between the realized covariation and its probability limit. As a result we have had to specialize. We do this by imposing more structure on y^*.

DEFINITION 1: A *continuous stochastic volatility semimartingale* (the class of such semimartingales will be denoted $\mathcal{SVSM}^c$) is a vector semimartingale $y^* = \alpha^* + m^*$ satisfying the following two additional conditions:

(a) that $\alpha^* \in \mathcal{FV}^c$ and $\alpha^*(0) = 0$;

(b) that m^* is a multivariate stochastic volatility (SV) process

$$m^*(t) = \int_0^t \Theta(u)\, dw(u), \tag{5}$$

where Θ, the instantaneous or spot covolatility process, has elements that are all càdlàg and w is a vector standard Brownian motion. We also define the *spot covariance* as

$$\Sigma(t) = \Theta(t)\Theta(t)',$$

and assume that (for all $t < \infty$)

$$\int_0^t \Sigma_{kk}(u)\, du < \infty \qquad (k = 1, \ldots, q), \tag{6}$$

where $\Sigma_{kl}(t)$ is the notation for the (k, l)th element of the $\Sigma(t)$ process.

REMARK 1: (i) Assumption (b) means that $m^* \in \mathcal{M}_{loc}^c$. All continuous local martingales with absolutely continuous QV can be written in the form of assumption (b). This result, which is due to Doob (1953), is discussed in, for example, Karatzas and Shreve (1991, pp. 170–172). In the univariate case, using the Dambis–Dubins–Schwartz theorem, we know that the difference between the entire $\mathcal{M}_{loc}^c$ class and those given in Assumption (b) are the local

martingales that have only continuous, not absolutely continuous, QV.[2] Hence $\mathcal{SVSM}^c$ is only a slightly smaller class than $\mathcal{SM}^c$.

(ii) If $y^* = \alpha^* + m^*$ is a semimartingale with $m^* \in \mathcal{M}^c_{loc}$ and $\alpha^* \in \mathcal{FV}$ and if α^* is additionally predictable, then if we impose a lack of arbitrage opportunities, $\alpha^* \in \mathcal{FV}^c$. This result is discussed in, for example, Back (1991, p. 380). Hence, in the context of financial economics, Assumption (a) follows from (b) when we assume α^* is predictable. In fact a stronger result holds. Under these assumptions α^* is absolutely continuous (Karatzas and Shreve (1998, p. 3) and Andersen, Bollerslev, Diebold, and Labys (2003, p. 583)).

(iii) Although Σ can exhibit jumps, the implied $y^* \in \mathcal{SM}^c$. The unusual technical càdlàg assumption (rather than the standard càglàd condition) on Θ will be discussed in some detail in the next section where we will note that it is of no consequence from a modelling viewpoint.

(iv) Overall, for $y^* \in \mathcal{SVSM}^c$,

$$y^*(t) = \alpha^*(t) + \int_0^t \Theta(u)\, dw(u), \quad y^*(0) = 0. \tag{7}$$

A key feature of this model class is

$$\Sigma^*(t) = \int_0^t \Sigma(u)\, du,$$

the integrated covariance matrix. It plays a central role in the analysis of SV models (e.g., Ghysels, Harvey, and Renault (1996)). An example of α^* is $\alpha^*(t) = \mu t + \Sigma^*(t)\beta$, linking the drift process to the covariance (see Bollerslev, Engle, and Wooldridge (1988) for discussions of the related ARCH-M models).

Importantly, if $y^* \in \mathcal{SVSM}^c$, then writing $\mathcal{F}$ as the natural filtration of y^* we have that

$$\alpha^*(t) = \int_0^t \mathrm{E}(dy^*(u)|\mathcal{F}(u)), \tag{8}$$

the integral of the expected instantaneous returns and

$$[y^*](t) = \Sigma^*(t) = \int_0^t \mathrm{Cov}(dy^*(u)|\mathcal{F}(u)) = \int_0^t \mathrm{Cov}(dm^*(u)|\mathcal{F}(u)).$$

[2]An example of a continuous local martingale that has no SV representation is a time change Brownian motion where the time change takes the form of the so-called "devil's staircase," which is continuous and nondecreasing but not absolutely continuous (see, e.g., Munroe (1953, Section 27)). This relates to the work of, for example, Calvet and Fisher (2002) on multifractals.

This implies that

$$[y_M^*]_i \xrightarrow{p} \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du = \int_{\hbar(i-1)}^{\hbar i} \mathrm{Cov}(dm^*(u)|\mathcal{F}(u)). \tag{9}$$

Thus $[y_M^*]_i$ consistently estimates the integral of the conditional covariance of the increments of the local martingale component of y^* over the interval from time $\hbar(i-1)$ to $\hbar i$. Much economic theory is postulated in terms of increments of these types of quantities, e.g., the work of Chamberlain (1988) and Back (1991). We call $\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$ the *actual covariance matrix* of the local martingale component. Under stronger conditions Andersen, Bollerslev, Diebold, and Labys (2003) relate $[y_M^*]_i$ to the variability of y_i, conditioning on a variety of information sets, while Foster and Nelson (1996) develop estimators for $\Sigma(t)$, the spot covariability. We will discuss the latter paper in more detail in a moment.

1.4. *The Literature*

In the special case of y^* being univariate ($q = 1$), we call $[y_M^*]_i$ the *realized variance* (reserving $\sqrt{[y_M^*]_i}$ for the *realized volatility*). In that case Barndorff-Nielsen and Shephard (2002b) have considerably strengthened the univariate consistency result implied by (4). They showed that if $y^* \in \mathcal{SVSM}^c$, and obeys some regularity assumptions discussed in the next section, the following three results hold. The first is that

$$\frac{\sqrt{\frac{M}{\hbar}}\left\{[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du\right\}}{\sqrt{2\int_{\hbar(i-1)}^{\hbar i} \Sigma^2(u)\,du}} \xrightarrow{L} N(0,1). \tag{10}$$

The second result is that

$$\frac{[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du}{\sqrt{\frac{2}{3}\sum_{j=1}^{M} y_{j,i}^4}} \xrightarrow{L} N(0,1). \tag{11}$$

These two limit theorems are linked together by the third result, which is that

$$\frac{M}{3\hbar}\sum_{j=1}^{M} y_{j,i}^4 \xrightarrow{p} \int_{\hbar(i-1)}^{\hbar i} \Sigma^2(u)\,du. \tag{12}$$

The result (11) is statistically feasible (i.e., except for the unknown integral $\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$, it can be computed directly from the data and so can be used to, for example, construct confidence intervals for $\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$), while (10) is perhaps more informative from a theoretical viewpoint.

The following points can be made about this analysis.

REMARK 2: (i) $[y_M^*]_i$ converges to $\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$ at rate $\sqrt{M}$.

(ii) Knowledge of the volatility dynamics is not required in order to use (11).

(iii) $\Sigma(t)$ can be nonstationary, have long-memory, jumps, no moments, and include intra-day effects.

(iv) $[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$ has a mixed Gaussian limit and so will generally have heavier tails than a normal.

We should note that Bai, Russell, and Tiao (2000), Meddahi (2002), Meddahi (2003), and Andreou and Ghysels (2002) have some interesting additional insights into the accuracy of $[y_M^*]_i$. Barndorff-Nielsen and Shephard (2004a) have studied the finite sample behavior of (11). Following Barndorff-Nielsen and Shephard (2002b), Meddahi (2001) has studied the first two moments of the difference between elements of the multivariate $[y_M^*]_i$ and $\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$. This is the only other paper of which we know that has discussed the properties of $[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du$.

As we have already briefly mentioned, these results are also related to the work of Foster and Nelson (1996) (note also the contributions of Genon-Catalot, Laredo, and Picard (1992) and Hansen (1995)). They provided an asymptotic distribution theory for an estimator of $\Sigma(t)$, the spot (not integrated) covariance. Their idea was to compute a local covariance from the lagged data, e.g.,

$$\widehat{\Sigma}(t) = \hbar^{-1} \sum_{j=1}^{M} \left\{ y^*\left(t - \frac{\hbar j}{M}\right) - y^*\left(t - \frac{\hbar(j-1)}{M}\right) \right\} \times \left\{ y^*\left(t - \frac{\hbar j}{M}\right) - y^*\left(t - \frac{\hbar(j-1)}{M}\right) \right\}'.$$

They then studied its behavior as $M \to \infty$ and $\hbar \downarrow 0$ under various assumptions. This "double asymptotics" yields a Gaussian limit theory so long as $\hbar \downarrow 0$ and $M \to \infty$ at the correct, related rates. They extended their analysis to provide a distributional theory for estimators of the spot regression coefficients. In the univariate case, Andreou and Ghysels (2002) have used the distribution theory for the estimator of the spot covariance to assess the distribution of the difference between the realized variance and quadratic variation. It would be informative to try to establish links between our results and those of Foster and Nelson (1996) in the multivariate case.

1.5. *Contribution of This Paper*

In this paper we extend the univariate results in (10), (11), and (12) to cover the multivariate case. The contributions of the paper will be as follows:

(i) Our main result is that as $M \to \infty$, assuming that (α^*, Σ) is independent of w,[3] and conditioning on the path of α^* and Σ,

$$\sqrt{\frac{M}{\hbar}}\left\{\text{vech}([y_M^*]_i) - \text{vech}\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du\right)\right\} \xrightarrow{L} N(0, \Pi_i), \tag{13}$$

where Π_i will be given explicitly in Section 2. Recall that the vech notation stacks the (unique) lower triangular elements of the columns of a matrix into a vector (e.g., Lutkepohl (1996, Chapter 7)). The matrix Π_i is a function of the path of Σ and so, as we argue conditionally on Σ, Π_i is deterministic. Moreover, defining $x_{j,i} = \text{vech}(y_{j,i} y_{j,i}')$ and

$$G_i = \sum_{j=1}^{M} x_{j,i} x_{j,i}' - \frac{1}{2}\sum_{j=1}^{M-1}(x_{j,i} x_{j+1,i}' + x_{j+1,i} x_{j,i}'), \tag{14}$$

we have that $M\hbar^{-1}G_i$, which is positive semi-definite, converges in probability to Π_i. Unconditionally (that is averaging over paths of α^* and Σ) this yields a feasible, mixed Gaussian limit theory. In the univariate version of (13) our assumptions are substantially weaker than those used in Barndorff-Nielsen and Shephard (2002b). Further, of course, our result covers the multivariate extension. These issues will be explicitly spelled out in Section 2. Further, (14) is a different feasible estimator than that used in Barndorff-Nielsen and Shephard (2002b), even in the univariate case (given in (12)). This is because it uses lags of returns. We will see that the direct application of the feasible strategy developed by Barndorff-Nielsen and Shephard (2002b) for the univariate case does not work in the multivariate situation.

(ii) In the bivariate case we derive the asymptotic distribution of the *realized regression estimator*. When we take one of the assets as a market portfolio we can regard the regression as a high frequency estimate of a beta.

(iii) We derive the asymptotic distribution of the *realized correlation* between two assets. Realized correlations have been previously studied empirically by Andersen, Bollerslev, Diebold, and Labys (2001) but no distributional theory has been available to allow us to assess the precision of the realized correlations.

The structure of the paper is as follows. In Section 2 we will give two theorems concerning the asymptotic distribution of realized covariation. In Section 3 we transform the asymptotic theory to give us a theory for measuring correlation and performing regression using high frequency data. Section 4 performs some Monte Carlo experiments to assess the accuracy of the theory

[3]Since this paper was accepted for publication, we have been able to lift this restriction; see Barndorff-Nielsen and Shephard (2004c). The substantial feasible results considered in the present paper continue to hold in this more general setting.

for finite values of M. Section 5 illustrates the use of this theory on some financial data. Section 6 concludes, while two Appendices contain the proofs of the theorems given in the paper.

2. ASYMPTOTIC THEORY FOR REALIZED COVARIATION

2.1. *Main Results*

This section will present the three main results of the paper. Our first theorem derives the asymptotic distribution of $[y_M^*]_i$, regarding the paths of α^* and Σ as fixed. Corollary 1 generalizes this to the case where these processes are random. Theorem 2 shows how to feasibly estimate the asymptotic covariance matrix that appears in the distribution theory.

The results are dependent on the log-prices obeying the $\mathcal{SVSM}^c$ structure[4]

$$y^*(t) = \alpha^*(t) + \int_0^t \Theta(u)\, dw(u) \quad \text{and} \quad \Sigma(t) = \Theta(t)\Theta(t)'.$$

Often we will use the index $k = 1, 2, \ldots, q$, to denote the kth asset, e.g., $\alpha_{(k)}^*$ denotes the drift process of the kth asset, while its log-price will be written as $y_{(k)}^*$. We bracket k in the subscript in order to make it distinct from the other subscripts i and j. A similar style of bracketing notation is used by Foster and Nelson (1996). Then the high frequency returns, on the ith day for the kth asset, will be written as

$$y_{(k)i,j}^* \quad \text{for} \quad j = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, q.$$

THEOREM 1: *Let* $y^* \in \mathcal{SVSM}^c$, $\delta = \hbar/M$, *and suppose the following conditions are satisfied*:

(a) *For every* $k = 1, \ldots, q$ *and* $i = 1, 2, \ldots, I$, *the quantities*

$$\delta^{-1} \int_{\hbar(i-1)+\delta(j-1)}^{\hbar(i-1)+\delta j} \Sigma_{kk}(u)\, du \tag{15}$$

are bounded away from 0 *and infinity*, *uniformly in* j *and* δ.

(b) *For every* $k = 1, \ldots, q$, *the mean process* $\alpha_{(k)}^*$ *satisfies* (*pathwise*), *as* $\delta \to 0$,

$$\delta^{-3/4} \max_{1 \le j \le M} \left| \alpha_{(k)}^*(\hbar(i-1) + j\delta) - \alpha_{(k)}^*(\hbar(i-1) + (j-1)\delta) \right| = o(1). \tag{16}$$

[4]From the viewpoint of the general semimartingale theory it would be technically correct to start with the martingale component of y^* in Definition 1 being $m^*(t) = \int_0^t \Theta(u-)\, dw(u)$, where we have written $\Theta(u-)$ as Θ is only assumed to be càdlàg, which means $\Theta(u-)$ is càglàd and so predictable. Predictability is usually needed in the context of stochastic integrals with respect to semimartingales to ensure that m^* is a local martingale. It turns out that this is of no consequence here, for we are integrating with respect to the continuous Brownian motion w. Hence m^* becomes $m^*(t) = \int_0^t \Theta(u)\, dw(u)$, which still means that m^* is a local martingale.

(c) *The processes α^* and Σ are jointly independent of the Brownian motion w. Then, conditionally on (α^*, Σ), the realized covariation matrix,*

$$[y_M^*]_i = \sum_{j=1}^{M} y_{j,i} y'_{j,i} = \left\{ \sum_{j=1}^{M} y_{(k)j,i} y_{(l)j,i} \right\}_{k,l=1,2,\ldots,q}, \tag{17}$$

follows asymptotically, as $M \to \infty$, the normal law with $q \times q$ matrix of means

$$\Sigma_i = \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\, du.$$

The asymptotic covariance of

$$\sqrt{\frac{M}{\hbar}} \left\{ [y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\, du \right\}$$

is Ω_i, a $q^2 \times q^2$ array with elements

$$\Omega_i = \left\{ \int_{\hbar(i-1)}^{\hbar i} \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u)\}\, du \right\}_{k,k',l,l'=1,\ldots,q}. \tag{18}$$

COROLLARY 1: *Suppose $y^* \in \mathcal{SVSM}^c$ and conditions* (a)–(c) *of Theorem* 1 *hold; then the asymptotic law of*

$$\sqrt{\frac{M}{\hbar}} \left\{ [y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\, du \right\} \tag{19}$$

is mixed normal with mean 0 *and random covariance matrix Ω_i.*

Taking these two results together a number of points can be made.

REMARK 3: (i) Condition (a) in Theorem 1 essentially means that, on any bounded interval, $\Sigma_{kk}(t)$ itself is bounded away from 0 and infinity. This is the case, for instance, for the CIR process (due to it having a reflecting barrier at zero) and the OU volatility process considered in Barndorff-Nielsen and Shephard (2001). Condition (b) implies that α^* is continuous but is less strong than it being differentiable.[5]

[5]In Remark 1(ii) to the definition of the $\mathcal{SMSV}^c$ class we noted that predictability of α_k^*, assumption (b) of that definition, plus a lack of arbitrage, imply α_k^* is absolutely continuous. Recall this means that we can then write $\alpha_k^*(t) = \int_0^t \alpha_k(u)\, du$. A sufficient condition for (16) to hold is that α_k is bounded. However, the condition of absolute continuity is not on its own enough to ensure that (16) holds. An example outside the scope of Assumption (b) is where $\alpha_k^*(t) = t^\eta/(1+t^\eta)$, where $\eta \in (0, 3/4)$ for $t \geq 0$.

(ii) The rate of convergence is $\sqrt{M}$ for all components of the realized covariation.

(iii) The limit theorem is mixed Gaussian, that is Ω_i is a stochastic matrix. Recall that in general mixed Gaussian variables are heavier tailed than Gaussian variables.

(iv) The k, k', l, l'element of Ω_i corresponds to the asymptotic covariance between the k, lth and the k', l'th elements of

$$\sqrt{\frac{M}{\hbar}}\left\{[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du\right\}.$$

Inevitably the matrix Ω_i is singular, due to the symmetric nature of $[y_M^*]_i$ and Σ_i. Later we will write the theory in terms of the identifying elements of these matrices, by employing vech transformations. However for the moment we prefer to maintain the general structure as this makes the result more transparent and eases the proof since it can be carried out using standard tensor notation.

(v) In the univariate case Theorem 1 represents a more general result than that proved by Barndorff-Nielsen and Shephard (2002b). They assumed that $\Sigma(t)$ was of finite variation and that $\alpha^*(t) = \mu t + \beta \int_0^t \Sigma(u)\,du$. Relaxing both of these assumptions represents an important widening of the applicability of the theory. Of course this previous paper also did not deal with the multivariate case, which is the main contribution of the present paper. Condition (b) is related to, but not the same as, that called $A(i)$ in Foster and Nelson (1996).

(vi) The results of Theorem 1 and Corollary 1 are proved under the assumption that (α^*, Σ) are jointly independent from w. This no leverage assumption is a very important limitation of the result. Lengthy simulations reported in Barndorff-Nielsen and Shephard (2002a) suggested that the feasible limit theory is rather robust to relaxing this assumption; however formally developing the relevant theory for this case is demanding. A discussion of what is known from a theoretical viewpoint in the univariate case is given in Barndorff-Nielsen, Graversen, and Shephard (2004).

(vii) In many cases

$$\Omega_i = \left\{\int_{\hbar(i-1)}^{\hbar i} \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u)\}\,du\right\}_{k,k',l,l'=1,\ldots,q}$$

will have an unconditional mean

$$E(\Omega_i) = \left\{\int_{\hbar(i-1)}^{\hbar i} \big[E\{\Sigma_{kk'}(u)\Sigma_{ll'}(u)\} + E\{\Sigma_{kl'}(u)\Sigma_{lk'}(u)\}\big]\,du\right\}_{k,k',l,l'=1,\ldots,q}.$$

This expectation is the asymptotic unconditional covariance matrix of

$$\sqrt{\frac{M}{\hbar}}\left\{[y_M^*]_i - \int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\,du\right\}.$$

One would expect this to exist if the fourth moments of the returns exist. Previously Meddahi (2001) has studied this covariance quantity, but using very different methods. Importantly he proved that this result is independent of the no leverage assumption in the case where volatility is a stationary diffusion.

Unfortunately Ω_i is not known and so the result in Corollary 1 is statistically infeasible. However, the following theorem means that Ω_i can be replaced by a consistent, positive semi-definite estimator, thus providing a feasible theory.

THEOREM 2: *Suppose* $y^* \in \mathcal{SVSM}^c$ *and conditions* (a)–(c) *of Theorem* 1 *hold*; *then defining*

$$x_{j,i} = \mathrm{vec}(y_{j,i}y_{j,i}'),$$

where the vec *notation stacks the columns of a matrix into a vector*, *there exists a random* $q^2 \times q^2$ *positive semi-definite matrix*

$$H_i = \sum_{j=1}^{M} x_{j,i}x_{j,i}' - \frac{1}{2}\sum_{j=1}^{M-1}(x_{j,i}x_{j+1,i}' + x_{j+1,i}x_{j,i}'), \tag{20}$$

such that $(M/\hbar)H_i \to^p \Omega_i$ *as* $M \to \infty$.

REMARK 4: (i) The feasible theory does not require knowledge of the drift or volatility processes.

(ii) In the univariate case

$$H_i = \sum_{j=1}^{M} y_{j,i}^4 - \sum_{j=1}^{M-1} y_{j,i}^2 y_{j+1,i}^2$$

has the property that

$$\frac{M}{\hbar}H_i \xrightarrow{p} \Omega_i = 2\int_{\hbar(i-1)}^{\hbar i} \Sigma^2(u)\,du.$$

Thus this feasible limit theory is asymptotically equivalent to, but not equal to, that developed in Barndorff-Nielsen and Shephard (2002b) whose result is stated in (11). In particular the new feasible theory gives, in the univariate case,

$$\frac{[y_M^*]_i - \Sigma_i}{\sqrt{\sum_{j=1}^{M} y_{j,i}^4 - \sum_{j=1}^{M-1} y_{j,i}^2 y_{j+1,i}^2}} \xrightarrow{L} N(0,1). \tag{21}$$

(iii) The obvious multivariate generalization of Barndorff-Nielsen and Shephard (2002b) is to use $M\hbar^{-1}\sum_{j=1}^{M} x_{j,i}x_{j,i}'$ to estimate Ω_i but this converges to a $q^2 \times q^2$ array with elements

$$\left\{\int_{\hbar(i-1)}^{\hbar i} \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u) + \Sigma_{kl}(u)\Sigma_{k'l'}(u)\}\, du\right\}_{k,k',l,l'=1,\ldots,q}.$$

This cannot be scaled to deliver Ω_i except in special cases (e.g., the univariate case). Hence we need to use the more sophisticated estimator H_i.

(iv) Limit theories for sums of arbitrary powers of absolute returns have been recently studied by Barndorff-Nielsen and Shephard (2003b) and Barndorff-Nielsen and Shephard (2003a). Examples of such sums are, in the univariate case, $\sum_{j=1}^{M} |y_{j,i}|^r$ for $r > 0$.

(v) It is clear from the proof of Theorem 2 that we can replace the subscript $j+1$ with $j+q$ where q is any positive but finite integer. Of course, in practice, if q is large, then the generalized version of (20) is likely to have a large finite sample bias.

2.2. *Avoiding Singular Covariance Matrices*

It is sometimes convenient to avoid the symmetric replication in the realized covariation matrix by employing a vech transformation. Then the limit theory can be written as follows.

COROLLARY 2: *As* $M \to \infty$, *conditionally on* (α^*, Σ),

$$\sqrt{\frac{M}{\hbar}}\left\{\text{vech}([y_M^*]_i) - \text{vech}\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma(u)\, du\right)\right\} \xrightarrow{L} N(0, \Pi_i), \tag{22}$$

while, defining $x_{j,i} = \text{vech}(y_{j,i}y_{j,i}')$ *and*

$$G_i = \sum_{j=1}^{M} x_{j,i}x_{j,i}' - \frac{1}{2}\sum_{j=1}^{M-1}(x_{j,i}x_{j+1,i}' + x_{j+1,i}x_{j,i}'), \tag{23}$$

we have that

$$\frac{M}{\hbar}G_i \xrightarrow{p} \Pi_i. \tag{24}$$

REMARK 5: (i) The matrix G_i is still only guaranteed to be positive semi-definite. The positive semi-definiteness follows from the property of the first-

order serial correlation coefficient that its square is less than or equal to one. More specifically, for any conformable vector c,

$$c'G_ic = \sum_{j=1}^{M}(c'x_{j,i})^2 - \sum_{j=1}^{M-1}(c'x_{j,i})(c'x_{j+1,i}) \geq 0.$$

Of course, G_i is likely to be positive definite in practice.

(ii) We can write, using the elimination matrix L (e.g., Magnus (1988)), $\text{vech}(y_{j,i}y'_{j,i}) = L\,\text{vec}(y_{j,i}y'_{j,i})$, and so $G_i = LH_iL'$. Likewise $\Pi_i = L\Omega_iL'$. Similarly we can express $H_i = DG_iD'$ and $\Omega_i = D\Pi_iD'$, where D is the duplication matrix.

2.3. *The Bivariate Case*

The general results are quite compact. It is helpful to focus on the bivariate case in order to gain further understanding. We will look at results for assets k and l, whose log-prices will be written as $y^*_{(k)}$ and $y^*_{(l)}$ respectively. Then the high frequency returns, on the ith day, will be

$$y^*_{(k)i,j} \quad \text{and} \quad y^*_{(l)i,j} \quad \text{for} \quad j = 1, 2, \ldots, M.$$

In that case Theorem 1 tells us that the joint asymptotic distribution for identifying elements of realized covariation becomes

$$\sqrt{\frac{M}{\hbar}}\begin{pmatrix} \sum_{j=1}^{M} y^2_{(k)j,i} - \int_{\hbar(i-1)}^{\hbar i}\Sigma_{kk}(u)\,du \\ \sum_{j=1}^{M} y_{(k)j,i}y_{(l)j,i} - \int_{\hbar(i-1)}^{\hbar i}\Sigma_{kl}(u)\,du \\ \sum_{j=1}^{M} y^2_{(l)j,i} - \int_{\hbar(i-1)}^{\hbar i}\Sigma_{ll}(u)\,du \end{pmatrix} \tag{25}$$

$$\xrightarrow{L} N\left[0, \int_{\hbar(i-1)}^{\hbar i}\begin{Bmatrix} 2\Sigma^2_{kk}(u) & 2\Sigma_{kk}(u)\Sigma_{kl}(u) & 2\Sigma^2_{kl}(u) \\ 2\Sigma_{kk}(u)\Sigma_{kl}(u) & \Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma^2_{kl}(u) & 2\Sigma_{ll}(u)\Sigma_{kl}(u) \\ 2\Sigma^2_{kl}(u) & 2\Sigma_{ll}(u)\Sigma_{kl}(u) & 2\Sigma^2_{ll}(u) \end{Bmatrix} du\right]. \tag{26}$$

The result on realized variances $\sum_{j=1}^{M} y^2_{(k)j,i}$ and $\sum_{j=1}^{M} y^2_{(l)j,i}$ was first derived in Barndorff-Nielsen and Shephard (2002b). The result on the marginal distribution of realized covariance as $M \to \infty$ is that

$$\frac{\sqrt{\frac{M}{\hbar}}\left\{\sum_{j=1}^{M} y_{(k)j,i}y_{(l)j,i} - \int_{\hbar(i-1)}^{\hbar i}\Sigma_{kl}(u)\,du\right\}}{\sqrt{\int_{\hbar(i-1)}^{\hbar i}\{\Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma^2_{kl}(u)\}\,du}} \xrightarrow{L} N(0,1), \tag{27}$$

which seems new, as does the joint distribution. The corresponding feasible limit theory for the realized covariance is

$$\frac{\sum_{j=1}^{M} y_{(k)j,i}y_{(l)j,i} - \int_{\hbar(i-1)}^{\hbar i}\Sigma_{kl}(u)\,du}{\sqrt{\sum_{j=1}^{M} y^2_{(k)j,i}y^2_{(l)j,i} - \sum_{j=1}^{M-1} y_{(k)j,i}y_{(l)j,i}y_{(k)j+1,i}y_{(l)j+1,i}}} \xrightarrow{L} N(0,1). \tag{28}$$

Notice that when the spot correlation is zero, then Σ is diagonal and so the asymptotic covariance in (25) becomes

$$\int_{\hbar(i-1)}^{\hbar i} \begin{Bmatrix} 2\Sigma_{kk}^2(u) & 0 & 0 \\ 0 & \Sigma_{kk}(u)\Sigma_{ll}(u) & 0 \\ 0 & 0 & 2\Sigma_{ll}^2(u) \end{Bmatrix} du.$$

When $\Sigma_{kk}(t) = \Sigma_{ll}(t)$, then the asymptotic covariance becomes

$$\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}^2(s) \begin{Bmatrix} 2 & 2\rho_{(kl)}(u) & 2\rho_{(kl)}^2(u) \\ 2\rho_{(kl)}(u) & (1+\rho_{(kl)}^2(u)r) & 2\rho_{(kl)}(u) \\ 2\rho_{(kl)}^2(u) & 2\rho_{(kl)}(u) & 2 \end{Bmatrix} du,$$

where

$$\rho_{kl}(u) = \frac{\Sigma_{kl}(u)}{\sqrt{\Sigma_{kk}(u)\Sigma_{ll}(u)}}.$$

This last result is a generalization of that given in Anderson (1984, p. 121) on the asymptotic joint distribution in the case of independent and identically distributed multivariate Gaussian random variables.

3. ASYMPTOTIC THEORY FOR REGRESSION AND CORRELATION

3.1. *Realized Regression*

In this section we will study the asymptotic properties of some statistics that are transformations of the realized covariation matrix. The focus will be on realized regression and realized correlation. We start with the regression case.

Regression plays a central role both in theoretical and empirical financial economics (e.g., see Cochrane (2001, Chapter 12) and Campbell, Lo, and MacKinlay (1997, Chapter 5)). In this subsection we use our distribution theory for realized covariation to derive a theory for univariate regression. Again this will be based on fixed intervals of time allowing the number of high frequency observations to go to infinity within that interval. Although these measures are informative, it is important to understand that much of the financial theory based on time varying covariances directly connects to conditional spot regression and correlation quantities rather than the integrated quantities that are our focus. An interesting paper that attempts to directly estimate such spot quantities is Foster and Nelson (1996).

We regress variable l on variable k, to obtain the realized regression

$$\widehat{\beta}_{(lk),i} = \frac{\sum_{j=1}^{M} y_{(k)j,i} y_{(l)j,i}}{\sum_{j=1}^{M} y_{(k)j,i}^2}. \tag{29}$$

Notice this realized regression does not employ an intercept for the same reasons as the realized covariation matrix does not subtract sample means. This realized regression involves just elements of the realized covariation and so we can use the asymptotic theory of the previous section to derive its asymptotic distribution. The probability limit for the regression case follows from the theory of QV. In particular if $y^* \in \mathcal{SM}$, then

$$\widehat{\beta}_{(lk),i} \xrightarrow{p} \frac{[y^*_{(k)}, y^*_{(l)}]_i}{[y^*_{(k)}]_i} = \beta_{(lk),i}, \tag{30}$$

where $[y^*_{(k)}, y^*_{(l)}]_i$ is the (k,l)th element of $[y^*]_i$ while $[y^*_{(k)}]_i$ denotes the corresponding (k,k)th element. The above result for regression is discussed at some length in, for example, Back (1991) and Andersen, Bollerslev, Diebold, and Labys (2001). Empirical estimates of regression parameters using high frequency data have been computed by, for example, Andersen, Bollerslev, Diebold, and Ebens (2001). Here we extend the theoretical results to derive the asymptotic distribution. When $y^* \in \mathcal{SVSM}^c$, then $\beta_{(lk),i}$ has the simpler form

$$\beta_{(lk),i} = \frac{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kl}(u)\,du}{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\,du}. \tag{31}$$

The asymptotic distribution can be derived using standard linearization methods. It yields the following result.

PROPOSITION 1: *Under the conditions given in Theorem* 1, *as* $M \to \infty$, *so*

$$\frac{\sqrt{\frac{M}{\hbar}}(\widehat{\beta}_{(lk),i} - \beta_{(lk),i})}{\sqrt{\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\,du\right)^{-2} g_{(lk),i}}} \xrightarrow{L} N(0,1). \tag{32}$$

Here

$$g_{(lk),i} = d'_{(lk),i} \Psi_{(lk),i} d_{(lk),i},$$

where

$$\Psi_{(lk),i} = \int_{\hbar(i-1)}^{\hbar i} \begin{Bmatrix} \Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma^2_{kl}(u) & 2\Sigma_{kk}(u)\Sigma_{kl}(u) \\ 2\Sigma_{kk}(u)\Sigma_{kl}(u) & 2\Sigma^2_{kk}(u) \end{Bmatrix} du$$

and

$$d_{(lk),i} = \begin{pmatrix} 1 \\ -\beta_{(lk),i} \end{pmatrix}.$$

EXAMPLE 1: When $\Sigma_{kl}(u) = 0$ and so $\beta_{(lk),i} = 0$, we have that

$$\frac{\sqrt{\frac{M}{\hbar}}\widehat{\beta}_{(lk),i}}{\sqrt{\left(\int_{\hbar(i-1)}^{\hbar i}\Sigma_{kk}(u)\,du\right)^{-2}\int_{\hbar(i-1)}^{\hbar i}\Sigma_{kk}(u)\Sigma_{ll}(u)\,du}} \xrightarrow{L} N(0,1). \tag{33}$$

In practice we have to replace $\Psi_{(lk)}$ and $d_{(lk)}$ by estimators to make the above regression theory feasible. Theorem 2 from the previous section implies that this is straightforward. In particular, we can state the following proposition.

PROPOSITION 2: *Under the conditions given in Theorem* 1, *as* $M \to \infty$, *so*

$$\frac{\widehat{\beta}_{(lk),i} - \beta_{(lk),i}}{\sqrt{\left(\sum_{j=1}^{M} y_{(k)j,i}^2\right)^{-2}\widehat{g}_{(lk),i}}} \xrightarrow{L} N(0,1). \tag{34}$$

Here

$$\widehat{g}_{(lk),i} = \sum_{j=1}^{M} x_{j,i}^2 - \sum_{j=1}^{M-1} x_{j,i}x_{j+1,i}$$

and

$$x_{j,i} = y_{(k)j,i}y_{(l)j,i} - \widehat{\beta}_{(lk)}y_{(k)j,i}^2.$$

An attractive feature of this theory is that all of the required terms are straightforward to compute.

3.2. *Realized Correlation*

The same strategy can be used to derive the asymptotic distribution of the realized correlation coefficient. It is defined as

$$\widehat{\rho}_{(lk),i} = \frac{\sum_{j=1}^{M} y_{(k)j,i}y_{(l)j,i}}{\sqrt{\sum_{j=1}^{M} y_{(k)j,i}^2 \sum_{j=1}^{M} y_{(l)j,i}^2}}. \tag{35}$$

The probability limit of the realized correlation is known by the theory of QV. If $y^* \in \mathcal{SM}$, then

$$\widehat{\rho}_{(lk),i} \xrightarrow{p} \frac{[y_{(k)}^*, y_{(l)}^*]_i}{\sqrt{[y_{(k)}^*]_i[y_{(l)}^*]_i}} = \rho_{(lk),i}, \tag{36}$$

a result that is discussed at some length in Andersen, Bollerslev, Diebold, and Labys (2001). Here we extend this result to derive the asymptotic distribution,

under our additional assumptions given above. In the case where $y^* \in \mathcal{SVSM}^c$, then

$$\rho_{(lk),i} = \frac{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kl}(u)\,du}{\sqrt{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\,du \int_{\hbar(i-1)}^{\hbar i} \Sigma_{ll}(u)\,du}}. \tag{37}$$

The asymptotic distribution can be derived using standard linearization methods. It yields the following infeasible result.

PROPOSITION 3: *Under the conditions given in Theorem* 1, *as* $M \to \infty$, *so*

$$\frac{\sqrt{\frac{M}{\hbar}}(\widehat{\rho}_{(lk),i} - \rho_{(lk),i})}{\sqrt{\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\,du \int_{\hbar(i-1)}^{\hbar i} \Sigma_{ll}(u)\,du\right)^{-1} g_{(l,k),i}}} \xrightarrow{L} N(0,1). \tag{38}$$

Here

$$g_{(lk),i} = d'_{(lk),i} \Pi_{(lk),i} d_{(lk),i},$$

where

$$\Pi_{(lk),i} = \int_{\hbar(i-1)}^{\hbar i} \begin{Bmatrix} 2\Sigma_{kk}^2(u) & 2\Sigma_{kk}(u)\Sigma_{kl}(u) & 2\Sigma_{kl}^2(u) \\ 2\Sigma_{kk}(u)\Sigma_{kl}(u) & \Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma_{kl}^2(u) & 2\Sigma_{ll}(u)\Sigma_{kl}(u) \\ 2\Sigma_{k2}^2(u) & 2\Sigma_{ll}(u)\Sigma_{kl}(u) & 2\Sigma_{ll}^2(u) \end{Bmatrix} du$$

and

$$d_{(lk),i} = \begin{pmatrix} -\frac{1}{2}\beta_{(lk),i} \\ 1 \\ -\frac{1}{2}\beta_{(kl),i} \end{pmatrix}.$$

Further, $\beta_{(lk),i}$ *is the population regression of the* lth *variable on the* kth, *defined in* (31), *while* $\beta_{(kl),i}$ *is the regression of the* kth *on the* lth.

The following example illustrates the theory.

EXAMPLE 2: When $\Sigma_{kl}(s) = 0$, we have that $\beta_{(lk),i} = \beta_{(kl),i} = 0$ and $\rho_{(lk),i} = 0$, so

$$\frac{\sqrt{\frac{M}{\hbar}}\widehat{\rho}_{(lk),i}}{\sqrt{\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\,du \int_{\hbar(i-1)}^{\hbar i} \Sigma_{ll}(u)\,du\right)^{-1}\left(\int_{\hbar(i-1)}^{\hbar i} \Sigma_{kk}(u)\Sigma_{ll}(u)\,du\right)}} \xrightarrow{L} N(0,1). \tag{39}$$

Thus, in this case, we have a very explicit expression for $g_{(lk),i}$. Notice the asymptotic variance of this quantity is unaffected by the scaling of any of the

assets. This is potentially a useful property given the degree of time varying volatility observed in some financial markets.

The infeasible results can be used in practice by employing the following proposition.

PROPOSITION 4: *Under the conditions given in Theorem* 1, *as* $M \to \infty$, *so*

$$\frac{\widehat{\rho}_{(lk),i} - \rho_{(lk),i}}{\sqrt{\left(\sum_{j=1}^{M} y_{(k)j,i}^2 \sum_{j=1}^{M} y_{(l)j,i}^2\right)^{-1} \widehat{g}_{(lk),i}}} \xrightarrow{L} N(0,1). \tag{40}$$

Here

$$\widehat{g}_{(lk),i} = \sum_{j=1}^{M} x_{j,i}^2 - \sum_{j=1}^{M-1} x_{j,i} x_{j+1,i}, \quad \textit{where}$$

$$x_j = y_{(k)j,i} y_{(l)j,i} - \frac{1}{2}\widehat{\beta}_{(lk),i} y_{(k)j,i}^2 - \frac{1}{2}\widehat{\beta}_{(kl),i} y_{(l)j,i}^2$$

$$= \frac{1}{2} y_{(k)j,i}\left(y_{(l)j,i} - \widehat{\beta}_{(lk),i} y_{(k)j,i}\right) + \frac{1}{2} y_{(l)j,i}\left(y_{(k)j,i} - \widehat{\beta}_{(kl),i} y_{(l)j,i}\right).$$

Here we recall $\widehat{\beta}_{(lk),i}$ *is defined in* (29).

4. SMALL SAMPLE EFFECTIVENESS OF THE LIMIT THEORY

4.1. *Simulation Design*

In this section we document some Monte Carlo experiments that assess the finite sample performance of our asymptotic theory for realized covariation. Throughout we work with a bivariate SV model and $\hbar = 1$, which we think of as a day. In particular we assume that

$$dy^*(t) = \Theta(t)\, dw(t), \quad \Sigma(t) = \Theta(t)\Theta(t)',$$

where

$$\Sigma(t) = \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{12}(t) & \Sigma_{22}(t) \end{pmatrix} = \begin{pmatrix} \sigma_1^2(t) & \sigma_{1,2}(t) \\ \sigma_{1,2}(t) & \sigma_2^2(t) \end{pmatrix}$$

and $\sigma_{1,2}(t)$ is $\sigma_1(t)\sigma_2(t)\rho(t)$.

Our model for σ_1^2 is derived from some empirical work reported in Barndorff-Nielsen and Shephard (2002b) who used realized variances to fit the variance of the DM/Dollar rate from 1986 to 1996 by the sum of two uncorrelated processes

$$\sigma_1^2(t) = \sigma_1^{2(1)}(t) + \sigma_1^{2(2)}(t).$$

Their results are compatible with using CIR processes for the $\sigma_1^{2(1)}$ and $\sigma_1^{2(2)}$ processes. In particular we will write these, for $s = 1, 2$, as

$$d\sigma_1^{2(s)}(t) = -\lambda_s\left\{\sigma_1^{2(s)}(t) - \xi_s\right\} dt + \omega_s \sigma^{(s)}(t)\, db_s(\lambda_s t), \quad \xi_s \geq \omega_s^2/2, \tag{41}$$

where b is a vector of standard Brownian motions, independent from w. The process (41) has a gamma marginal distribution

$$\sigma_1^{2(s)}(t) \sim Ga(2\omega_s^{-2}\xi_s, 2\omega_s^{-2}) = Ga(\nu_s, a_s), \quad \nu_s \geq 1,$$

with a mean of ν_s/a_s and a variance of ν_s/a_s^2. The parameters ω_s, λ_s, and ξ_s were calibrated by Barndorff-Nielsen and Shephard (2002b) as follows. Setting $p_1 + p_2 = 1$, they estimated

$$\mathrm{E}(\sigma_1^{2(s)}) = p_s .509, \quad \mathrm{Var}(\sigma_1^{2(s)}) = p_s .461, \quad s = 1, 2,$$

with

$$p_1 = .218, \quad p_2 = .782, \quad \lambda_1 = .0429, \quad \text{and} \quad \lambda_2 = 3.74,$$

which means the first, smaller component of the variance process is slowly reverting with a half-life of around 16 days while the second has a half-life of around 4 hours. Bollerslev and Zhou (2002) have found similar results using a shorter span of this type of exchange rate data.

Our model for σ_2^2 is taken from Andersen and Bollerslev (1998). They calibrated the parameters of a GARCH diffusion from the fit of a GARCH(1, 1) model to daily returns for the DM/Dollar from 1987 until 1992 using the temporal aggregation results in Drost and Nijman (1993). These parameters have been used in Monte Carlo studies by Andreou and Ghysels (2002) and Andersen, Bollerslev, and Meddahi (2004). The model takes on the form

$$d\sigma_2^2(t) = -.035\{\sigma_2^2(t) - .636\}\, dt + .236\sigma_2^2(t)\, db_3(t).$$

Although we are calibrating both σ_1^2 and σ_2^2 to DM/Dollar data, it turns out these two variance processes have very different dynamic properties.

Finally, we specify an ad hoc model for $\rho(t)$, roughly matching the patterns in correlations we have observed in empirical data, as

$$\rho(t) = \frac{e^{2x(t)} - 1}{e^{2x(t)} + 1},$$

where x follows the GARCH diffusion

$$dx(t) = -.03\{x(t) - .64\}\, dt + .118x(t)\, db_4(t).$$

To produce an impression of this bivariate process we have drawn Figure 1. It reports results based on $M = 48$, simulating up to 200 complete days. Figure 1(a) shows the first 10 days of the sample, plotting the bivariate 30 minute

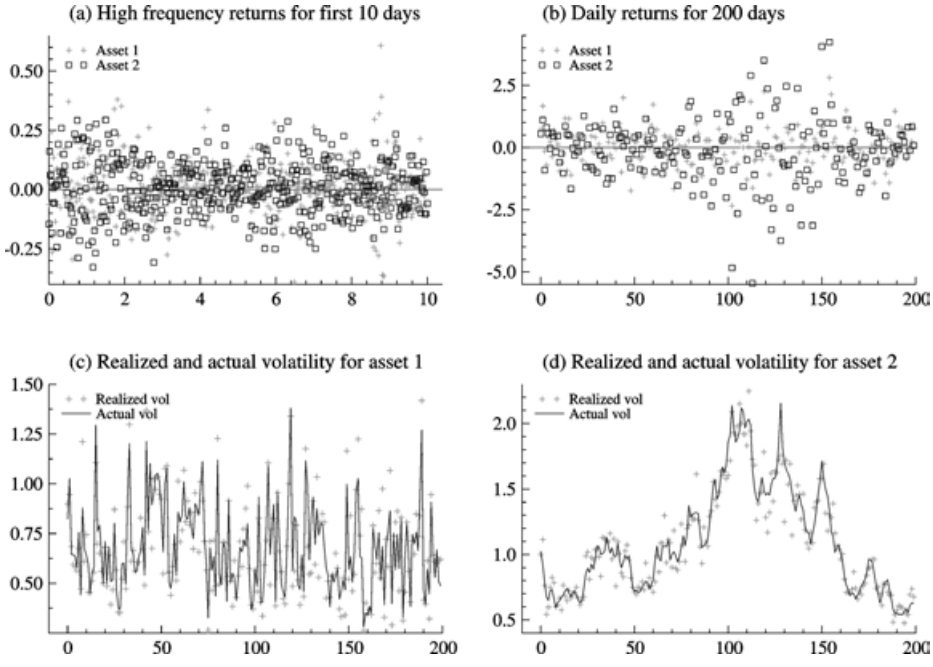


FIGURE 1.—Simulated bivariate SV model using $M = 48$: (a) 30 minute returns $y_{(1)j,i}$ and $y_{(2)j,i}$ for $i = 1, \ldots, 10$; (b) daily returns $\sum_{j=1}^{M} y_{(1)j,i}$ and $\sum_{j=1}^{M} y_{(2)j,i}$; (c) realized volatility $\sqrt{\sum y_{(1)j,i}^2}$ and actual volatility for asset 1; (d) realized and actual volatility for asset 2.

high frequency return data $y_{(1)j,i}$ and $y_{(2)j,i}$. The x-axis represents days in this picture. Figure 1(b) shows the daily returns $\sum_{j=1}^{M} y_{(1)j,i}$ and $\sum_{j=1}^{M} y_{(2)j,i}$ drawn against i. This indicates that the variability of the second asset increased in the middle of the sample, before falling back. Figure 1(c) shows the realized volatility $\sqrt{\sum_{j=1}^{M} y_{(1)j,i}^2}$ for asset 1 together with the corresponding actual volatility $\sqrt{\int_{\hbar(i-1)}^{\hbar i} \sigma_1^2(u)\,du}$. These are very jagged time series, reflecting the fast mean-reverting component in this process. The corresponding results for asset 2, given in Figure 1(d), show pronounced and quite persistent movements in the level of volatility through time. Of course, there are often quite large differences between the realized and actual quantities in these pictures.

Figure 2 gives the corresponding results for the measures of dependence between the two sets of asset returns. Figure 2(a) depicts

$$\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i} \quad \text{and} \quad \int_{\hbar(i-1)}^{\hbar i} \Sigma_{12}(u)\,du,$$

the realized and actual covariances, drawn against i. It shows that realized co-

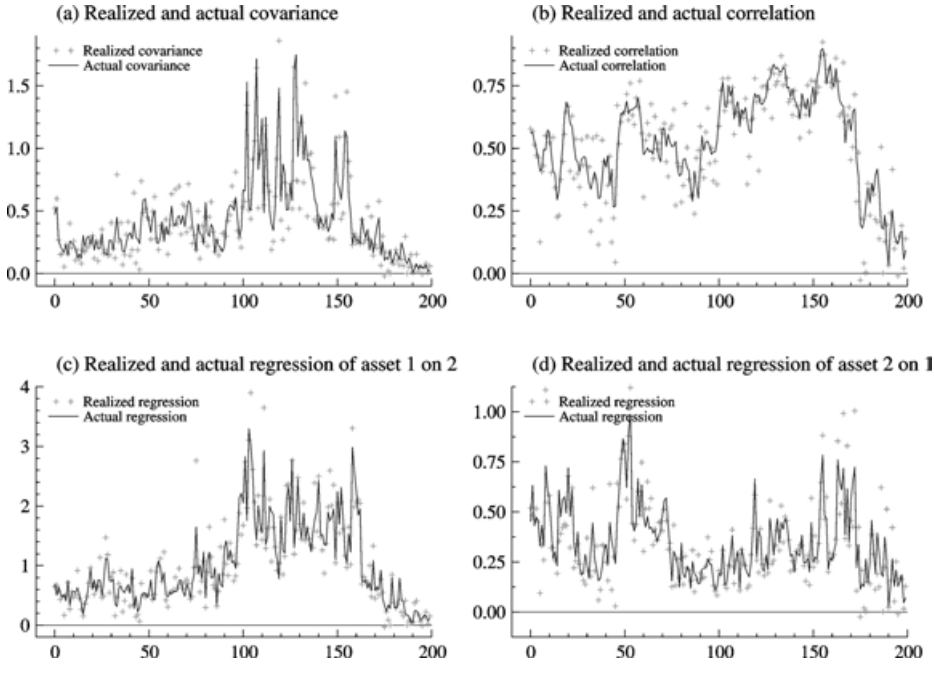


FIGURE 2.—Simulation of measures of dependence, using $M = 48$: (a) realized covariance and actual covariance; (b) realized and actual correlations; (c) realized and actual regressions of returns on asset 1 on asset 2; (d) same as (c) but asset 2 on asset 1.

variance is quite a noisy estimator, with the size of the errors being large when the level of covariance is high. The dependence structure of the data is much clearer in Figure 2(b), which draws

$$\frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sqrt{\sum_{j=1}^{M} y_{(1)j,i}^2 \sum_{j=1}^{M} y_{(2)j,i}^2}} \quad \text{and} \quad \frac{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{12}(u)\,du}{\sqrt{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{11}(u)\,du \int_{\hbar(i-1)}^{\hbar i} \Sigma_{22}(u)\,du}},$$

the realized and actual correlations amongst the two series. The estimation errors seem to have a long left-hand tail. A less stable picture appears in Figures 2(c) and (d), which report

$$\frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sum_{j=1}^{M} y_{(2)j,i}^2}, \qquad \frac{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{12}(u)\,du}{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{22}(u)\,du},$$

and

$$\frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sum_{j=1}^{M} y_{(1)j,i}^2}, \qquad \frac{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{12}(u)\,du}{\int_{\hbar(i-1)}^{\hbar i} \Sigma_{11}(u)\,du},$$

the realized and actual regression of asset 1 on asset 2 and the corresponding regressions for asset 2 on asset 1. Here the magnitude of the difference between these two quantities seems to vary quite significantly through time.

4.2. *Assessing the Performance of the Feasible Asymptotic Theory*

4.2.1. *Realized Covariance*

Our asymptotic theory for the realized covariance tells us that the normalized estimator error

$$\frac{\sum_{j=1}^{M} y_{(1)j,i}y_{(2)j,i} - \int_{\hbar(i-1)}^{\hbar i} \Sigma_{12}(u)\,du}{\sqrt{\sum_{j=1}^{M} y_{(1)j,i}^2 y_{(2)j,i}^2 - \sum_{j=1}^{M-1} y_{(1)j,i}y_{(2)j,i}y_{(1)j+1,i}y_{(2)j+1,i}}} \xrightarrow{L} N(0,1). \tag{42}$$

How close to normality is this ratio for small and moderate values of M? Figure 3 plots the realized covariance errors, $\sum_{j=1}^{M} y_{(1)j,i}y_{(2)j,i}$ minus $\int_{\hbar(i-1)}^{\hbar i} \Sigma_{21}(u)\,du$, against i for the Monte Carlo design discussed in the pre-

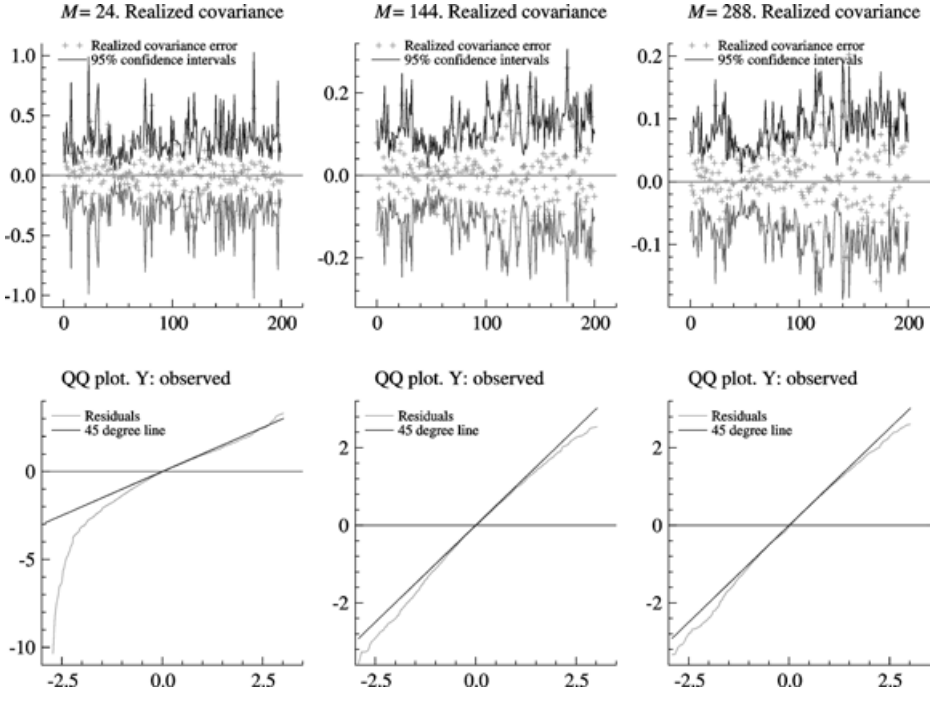


FIGURE 3.—Realized covariance. Top line: the realized covariance errors and their 95% confidence intervals. Bottom line: normal QQ plots for the standardized errors. Recall QQ plots: y-axis, the ranked observed standardized errors; x-axis, corresponding expected quantities under $N(0,1)$, left to right M increases through 24, 144, 288.

vious section. As we move from the left-hand side across the page we increase the value of M and we can see the decrease in the spread of these errors. The figure also gives 95% confidence intervals for the errors generated using the feasible limit theory (42). These also quickly contract with M. An important feature of the confidence intervals is that they vary dramatically, sometimes being quite small, other times being large. This reflects the changing volatility in the series.

The coverage of the limit theory is assessed by the normal QQ plots given in the lower three plots in Figure 3. These are based on 2,000 simulated daily observations. They are quite poor for small values of M, but improve as M increases. The degree of improvement is modest when M is high, although this may be because the sample size has only doubled.

4.2.2. *Realized Regression*

The limit theory for the normalized estimation error for realized regression of the returns of asset one on asset two has

$$\frac{\widehat{\beta}_{(12)i} - \beta_{(12)i}}{\sqrt{\left(\sum_{j=1}^{M} y_{(2)j,i}^2\right)^{-2}\left\{\sum_{j=1}^{M} x_{j,i}^2 - \sum_{j=1}^{M-1} x_{j,i}x_{j+1,i}\right\}}} \xrightarrow{L} N(0,1), \tag{43}$$

where $x_{j,i}$ is $y_{(1)j,i}y_{(2)j,i} - \widehat{\beta}_{(12)i}y_{(2)j,i}^2$. The graphs in the top half of Figure 4 show the regression errors, $\widehat{\beta}_{(12)i}$ minus $\beta_{(12)i}$, plotted against time together with asymptotically valid 95% confidence intervals for these errors based on (43). They show that the confidence intervals are more stable through time than was the case for the realized covariance. This is caused by the stablizing $(\sum_{j=1}^{M} y_{(2)j,i}^2)^{-2}$, which appears in the denominator of (43). This means realized regression is less sensitive to changes in the volatility of asset 2.

Figure 4 also shows the corresponding QQ plots for the standardized regression errors using 2,000 simulated days. They indicate that the theory provides a reasonable basis for inference when M is beyond 100.

4.2.3. *Realized Correlation*

The feasible limit theory for correlation of the returns of asset one and asset two is

$$\frac{\widehat{\rho}_{(12)i} - \rho_{(12)i}}{\sqrt{\left(\sum_{j=1}^{M} y_{(1)j,i}^2 \sum_{j=1}^{M} y_{(2)j,i}^2\right)^{-1}\left(\sum_{j=1}^{M} x_{j,i}^2 - \sum_{j=1}^{M-1} x_{j,i}x_{j+1,i}\right)}} \xrightarrow{L} N(0,1), \tag{44}$$

where $x_{j,i}$ is $y_{(1)j,i}y_{(2)j,i} - \frac{1}{2}\widehat{\beta}_{(12)i}y_{(2)j,i}^2 - \frac{1}{2}\widehat{\beta}_{(21)i}y_{(1)j,i}^2$. Here

$$\widehat{\beta}_{(12)i} = \frac{\sum_{j=1}^{M} y_{(1)j,i}y_{(2)j,i}}{\sum_{j=1}^{M} y_{(2)j,i}^2} \quad \text{and} \quad \widehat{\beta}_{(21)i} = \frac{\sum_{j=1}^{M} y_{(1)j,i}y_{(2)j,i}}{\sum_{j=1}^{M} y_{(1)j,i}^2}.$$

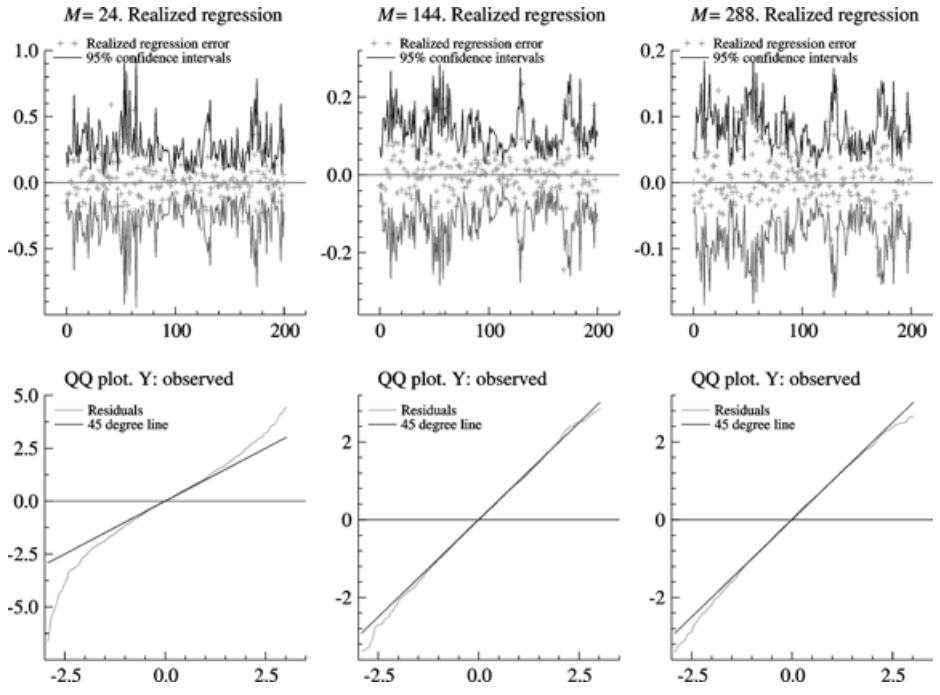


FIGURE 4.—Realized regression. Top line: the realized regression of asset 1 on asset 2. Bottom line: the corresponding QQ plots.

Importantly the scaling $(\sum_{j=1}^{M} y_{(1)j,i}^2 \sum_{j=1}^{M} y_{(2)j,i}^2)^{-1}$ adjusts the denominator in (44) to make it invariant as we scale either of the asset returns within each time period. This suggests it should be less sensitive to changes in the level of volatility in either of the assets.

Figure 5 shows the realized correlation error, $\widehat{\rho}_{(12)i}$ minus $\rho_{(12)i}$, plotted against i, together with 95% asymptotic confidence intervals. The precision of the estimators now does not vary as much with i. This conclusion is reinforced as M increases. However, for small values of M there are some very large negative errors, suggesting the estimator has a small sample downward bias.

The bottom plots in Figure 5 give QQ plots for the simulations of the realized correlation based on the asymptotic theory. They suggest the asymptotic theory is a poor guide for small values of M, but for moderate to large values of M it is reasonably accurate.

One possible way of improving the finite sample behavior of the asymptotic distribution of $\widehat{\rho}_{i(12)}$ is by using the Fisher (1921) z transformation (e.g., see the exposition in Anderson (1984, pp. 122–124) and Gourieroux, Renault, and

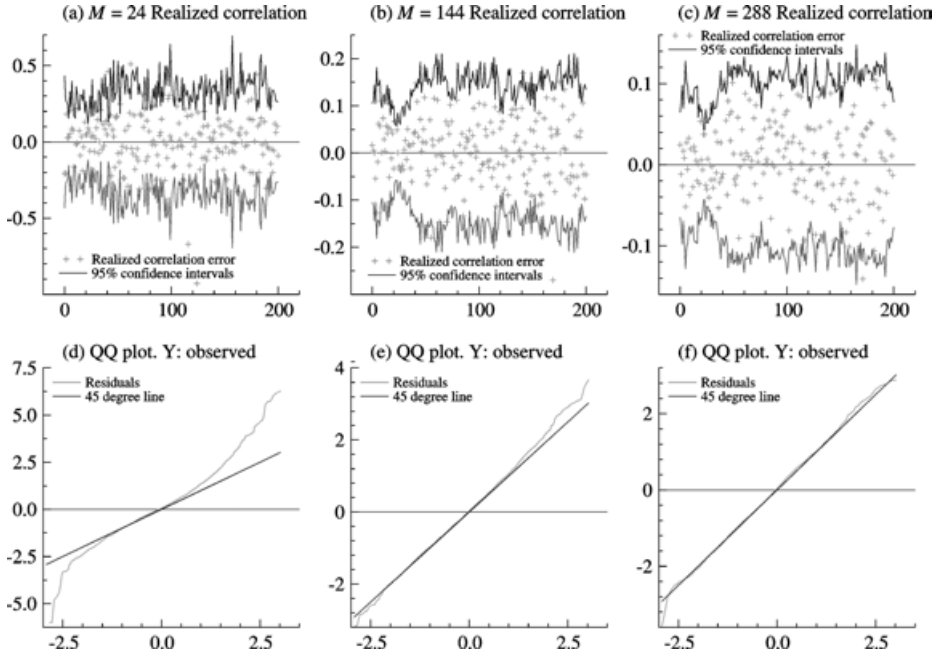


FIGURE 5.—Realized correlations. Top line: the realized correlation errors of asset 1 and asset 2 together with the associated confidence intervals. Bottom line: the corresponding QQ plots.

Touzi (2000) for an alternative approach):

$$z_{(12)i} = \frac{1}{2}\log\frac{1+\widehat{\rho}_{(12)i}}{1-\widehat{\rho}_{(12)i}} \quad \text{and} \quad \zeta_{(12)i} = \frac{1}{2}\log\frac{1+\rho_{(12)i}}{1-\rho_{(12)i}}.$$

Recall Fisher's analysis is based on M multivariate, independent and identically distributed Gaussian random variables, in which case his transformation has the important feature that $\sqrt{M}(z_{i(12)} - \zeta_{i(12)})$ has a standard normal limit distribution and it is well known its asymptotic distribution provides an excellent approximation to the exact distribution (e.g., David (1938)). In our more general case we could use

$$(45) \qquad \frac{z_{(12)i} - \zeta_{(12)i}}{\sqrt{\{1-\widehat{\rho}^2_{(12)i}\}^{-2}\left(\sum_{j=1}^{M} y^2_{(1)j,i}\sum_{j=1}^{M} y^2_{(2)j,i}\right)^{-1}\left(\sum_{j=1}^{M} x^2_{j,i} - \sum_{j=1}^{M-1} x_{j,i}x_{j+1,i}\right)}} \xrightarrow{L} N(0,1).$$

Figure 6 repeats the previous experiment but now using the Fisher transformation. These pictures demonstrate that the Fisher transformation improves

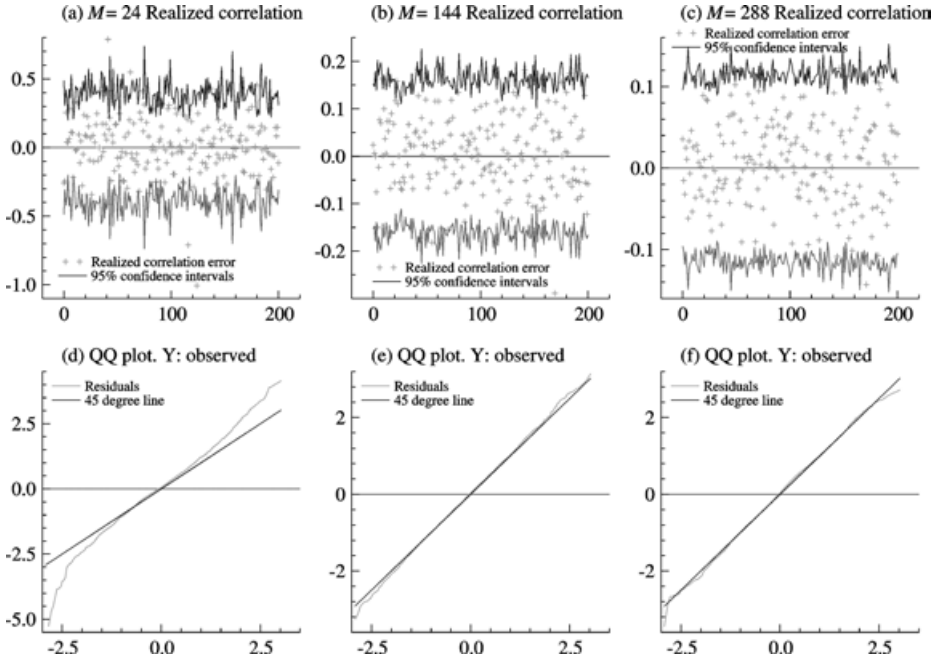


FIGURE 6.—Fisher transformation of the realized correlation. Top line: the Fisher transformed realized correlation error of assets 1 and 2 together with the associated asymptotic standard errors. Bottom line: corresponding QQ plots.

the performance of the asymptotic approximation and that the confidence intervals are made even more stable through time. This is particularly apparent when we look at the cases where M is moderately high. This means the errors for the transformed realized correlations are approximately unconditionally normal.

5. EMPIRICAL ILLUSTRATION

To illustrate some of the empirical features of realized covariation, and particularly their precision as estimators of actual covariation, we have used a subset of the return data employed by Andersen, Bollerslev, Diebold, and Labys (2001), although we have made slightly different adjustments to deal with some missing data. These adjustments are described in detail in Barndorff-Nielsen and Shephard (2002b). The bivariate series in question records the United States Dollar/German Deutsche Mark and Dollar/Japanese Yen series. It covers the ten year period from 1st December 1986 until 30th November 1996. The subset we have selected to illustrate our theory starts on February 4th, 1991 and covers the next 50 trading days. The original dataset records every 5 minutes the most recent mid-quote to appear on the Reuters screen. It has been kindly

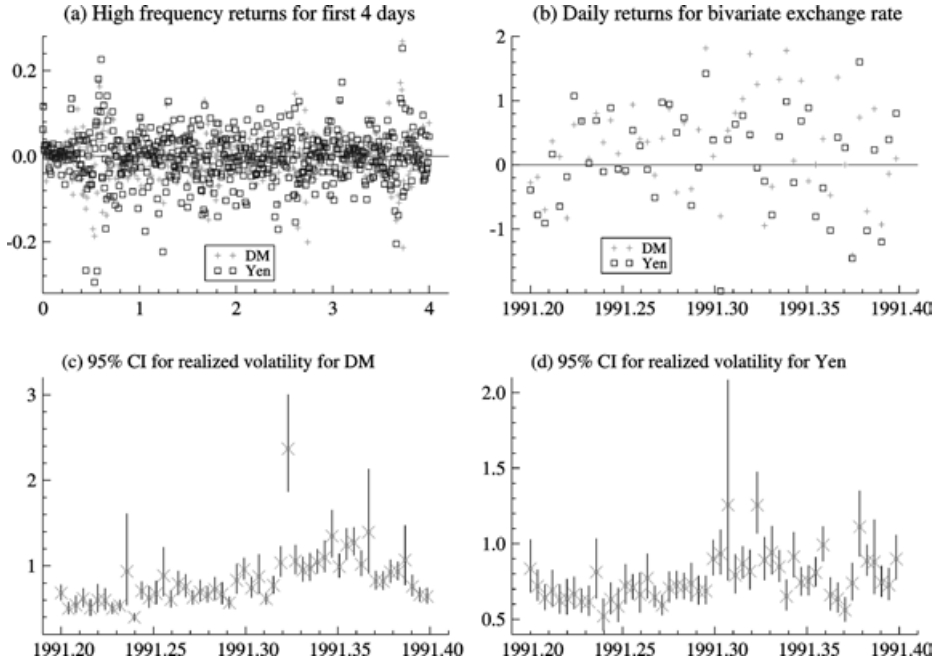


FIGURE 7.—DM and Yen against the Dollar, based on the Olsen dataset. Data are 4th February 1991 onwards for 50 active trading days. (a) 10 minute returns on the two exchange rates for the first 4 days of the dataset. (b) Daily returns for the first 50 days of the dataset. (c) Realized volatility $\sqrt{\sum y_{(1)j,i}^2}$ for the DM series. This is marked with a cross, while the bars denote 95% confidence intervals. (d) Realized volatility $\sqrt{\sum y_{(2)j,i}^2}$ for the Yen series.

supplied to us by Olsen and Associates in Zurich, who document their path-breaking work in this area in Dacorogna, Gencay, Müller, Olsen, and Pictet (2001). Throughout the analysis presented in this section we will set $M = 144$, thus basing our analysis on 10 minute return data with $\hbar$ representing one day.

Figure 7 provides some descriptive statistics for the empirical observations. Figure 7(a) shows the first four active days of the dataset, displaying the bivariate 10 minute returns. Figure 7(b) shows the daily returns for each of the 50 active days drawn against i. Figures 7(c) and (d) detail the realized volatilities for the two exchange rates, together with 95% confidence intervals. These confidence intervals are based on the log-version of the limit theory for the realized variance. Recall, for the first asset,

$$\frac{\sum_{j=1}^{M} y_{(1)j,i}^2 - \int_{\hbar(i-1)}^{\hbar i} \Sigma_{11}(u)\,du}{\sqrt{\sum_{j=1}^{M} y_{(1)j,i}^4 - \sum_{j=1}^{M-1} y_{(1)j,i}^2 y_{(1)j+1,i}^2}} \xrightarrow{L} N(0,1), \tag{46}$$

which implies, using the standard linearization method, that

$$\frac{\log \sum_{j=1}^{M} y_{(1)j,i}^2 - \log \int_{\hbar(i-1)}^{\hbar i} \Sigma_{11}(u)\,du}{\sqrt{\left(\sum_{j=1}^{M} y_{(1)j,i}^2\right)^{-2}\left\{\sum_{j=1}^{M} y_{(1)j,i}^4 - \sum_{j=1}^{M-1} y_{(1)j,i}^2 y_{(1)j+1,i}^2\right\}}} \xrightarrow{L} N(0,1).$$

This type of log-based distribution theory for realized variance is studied in Barndorff-Nielsen and Shephard (2004a), where it was shown to have better finite sample behavior than the raw theory given in (46). Figure 7(c) shows a steady increase in the level of volatility during this period, with occasional large values of volatility. When the volatility is high, the confidence intervals tend to be very large as well. Figure 7(d) shows a similar type of result with the first half of the sample having quite a low level of volatility, which has risen and become more variable in the second half of the sample.

We now turn our attention to the measures of dependence between the assets. In Figure 8(a) we have drawn the realized covariance $\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}$ against i, together with the associated 95% confidence intervals constructed using our asymptotic theory. These terms move rather violently through this period, although the confidence intervals show that when the realized covariance is particularly high, then so is the width of the confidence intervals. The corresponding realized correlations

$$\frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sqrt{\sum_{j=1}^{M} y_{(1)j,i}^2 \sum_{j=1}^{M} y_{(2)j,i}^2}} \tag{47}$$

are given in Figure 8(b). These are quite stable through time with only a single realized correlation standing out from the others in the sample. An important feature is that the confidence intervals, which are based on the Fisher type theory given in (45), are nonsymmetric with longer left-hand tails than right-hand tails. The correlations are not particularly precisely estimated, with the confidence intervals typically being around .2 wide. Interestingly the largest shifts in the correlation structure during this time period do not appear at times of particularly large volatility spikes.

Figures 8(c) and (d) also pick up the large movements in the volatility, for they show the regressions of the DM on the Yen and the Yen on the DM,

$$\frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sum_{j=1}^{M} y_{(2)j,i}^2} \quad \text{and} \quad \frac{\sum_{j=1}^{M} y_{(1)j,i} y_{(2)j,i}}{\sum_{j=1}^{M} y_{(1)j,i}^2},$$

respectively. These move very significantly over time, but again this is mostly due to volatility shifts, not changes in the pattern of correlation.

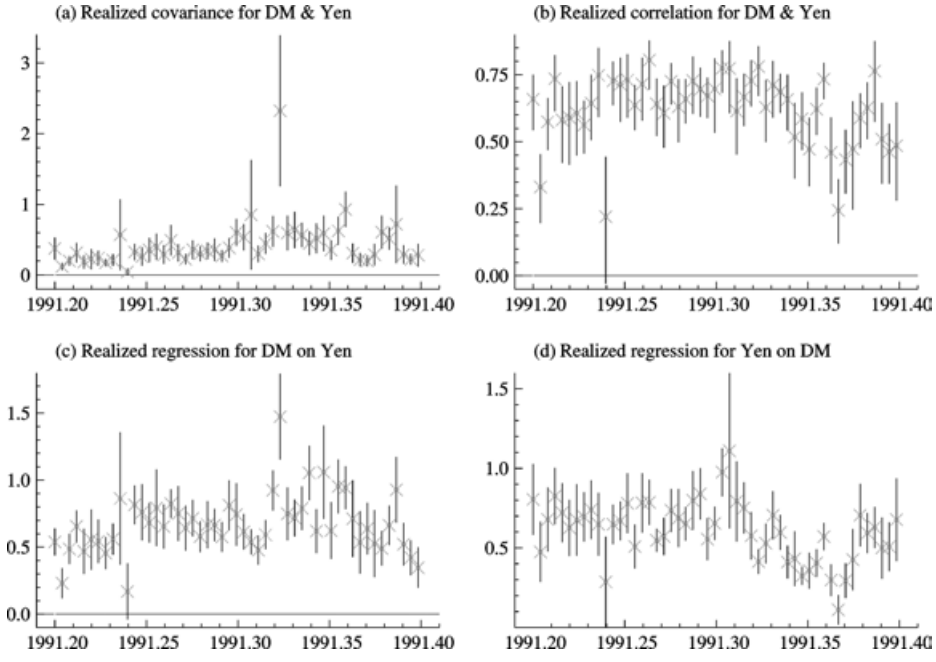


FIGURE 8.—Dependence measures between the DM and Yen returns based on the Olsen dataset. Data are 4th February 1991 onwards for 50 active trading days, using 10 minute returns. Vertical lines always denote 95% confidence intervals. (a) Realized covariance. (b) Realized correlation and confidence intervals using the Fisher transformation theory. (c) Realized regression of asset 1 on asset 2. (d) Realized regression of asset 2 on asset 1.

6. CONCLUSIONS

In this paper we have developed a distribution theory for realized covariation, a quantity that appears both in finance theory and in a great deal of empirical financial econometrics. Based on some rather weak assumptions on a multivariate stochastic volatility process, our new theory can be used to derive a feasible limit theory for realized regression and realized correlation. The limit theory is robust as it does not require the empirical researcher to specify a model for the spot covolatility or the drift process. In this sense it is semiparametric. Our approach has the virtues that there are no tuning parameters to choose and that the theory is both easy to code and self-contained. Monte Carlo results suggest the theory may well be useful in practice for it seems a good guide to the finite sample behavior.

An important theme in theoretical econometrics and statistics is that covariances are not very robust objects, as they are highly sensitive to large movements in asset prices. It may be desirable to construct economic theory and econometrics on more robust quantities such as mean absolute errors. In some recent work Barndorff-Nielsen and Shephard (2003b, 2004b) have studied the

univariate version of this problem, in particular establishing limiting distribution theories for $M^{-1/2}\sum_{j=1}^{M}|y_{j,i}|$ and $\sum_{j=2}^{M}|y_{j-1,i}||y_{j,i}|$, which are somewhat robust to jumps in the price process. We are currently working on developing multivariate versions of these statistics.

The Network for Mathematical Physics and Stochastics (MaPhySto), University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark; oebn@imf.au.dk

and

Nuffield College, University of Oxford, Oxford OX1 1NF, UK; www.nuff.ox.ac.uk/users/shephard/.

Manuscript received December, 2002; final revision received November, 2003.

APPENDIX A: PROOF OF THEOREM 1

A.1. *Prelude*

A.1.1. *Multivariate SV Model in Tensor Notation*

We start by recalling that $y^* \in \mathcal{SVSM}^c$, which means that

$$y^*(t) = \alpha^*(t) + \int_0^t \Theta(u)\,dw(u) \quad \text{and} \quad \Sigma(t) = \Theta(t)\Theta(t)', \tag{48}$$

where w is a q-dimensional vector of independent Brownian motions. In our proofs it will be more convenient to employ tensor notation (cf., for instance, McCullagh (1987, Chapter 1) or Barndorff-Nielsen and Cox (1994, p. 3)) than vector and matrix notation. Thus we will write the q stochastic processes $y^*_{(k)}$ $(k = 1, \ldots, q)$ as

$$y^*_{(k)}(t) = \alpha^*_{(k)}(t) + \int_0^t \gamma^a_{(k)}(u)\,dw_a(u) \tag{49}$$

with initial condition $y^*_{(k)}(0) = 0$ and

$$\Theta(t) = \{\gamma^a_{(k)}(t)\}_{k,a=1,2,\ldots,q}.$$

Recall that in tensor notation we use the Einstein summation convention, which means that if an index is repeated in a single expression then summation over that index is understood. Thus, in particular (49) is understood to mean

$$y^*_{(k)}(t) = \alpha^*_{(k)}(t) + \sum_{a=1}^{q}\int_0^t \gamma^a_{(k)}(u)\,dw_a(u).$$

However, unless otherwise mentioned, we apply the summation convention only to indices a, b, c, d and not to the indices k, l, k', l'.

Further, in what is to follow we will write

$$\gamma^{ab}_{(kl)} = \gamma^a_{(k)}\gamma^b_{(l)},$$

with similar notation for other index combinations. This is quite a standard formulation in tensor notation. Notice that in this object no superscripts or subscripts are repeated and so no summation operator is generated. Combining the Einstein summation convention and the notational rule for $\gamma^{ab}_{(kl)}$, the (k, l)th element of the spot covolatility matrix of the SV model is then

$$\Sigma_{kl}(t) = \gamma^{aa}_{(kl)}(t) = \sum_{a=1}^{q}\gamma^a_{(k)}(t)\gamma^a_{(l)}(t). \tag{50}$$

A.1.2. *Notation*

Throughout this Appendix we will set $i = 1$, for the more general result holds by analogy if we can establish the $i = 1$ case. By focusing solely on the $i = 1$ case we are able to drop i from our notation and so reduce the clutter in our subscripts. Further, in the main text we have written $y_{(k)j,i}$ as the high frequency return for the kth asset. As the i notation has been removed we take this opportunity to also drop the bracket around the k notation in the sequel. Hence, for example,

$$y^*_{(k)} \quad \text{becomes} \quad y^*_k.$$

As a result in what follows we have set, as before,

$$\delta = \hbar/M$$

as the time interval for our high frequency returns and we let our high frequency returns be written as

$$y_{kj} = y^*_k(j\delta) - y^*_k((j-1)\delta) \qquad (j = 1, \ldots, M). \tag{51}$$

Note that y_{kj} suppresses the dependence on δ and so on M. Below we apply the same kj convention in similar cases.

Then using this notation we will prove the following result under Assumptions (a)–(c) of Theorem 1. Conditionally on (α^*, Σ) the realized covariation matrix

$$[y^*_M] = \sum_{j=1}^{M} y_j y_j' = \left\{ \sum_{j=1}^{M} y_{kj} y_{lj} \right\}_{k,l=1,2,\ldots,q} \tag{52}$$

follows asymptotically, as $M \to \infty$, the normal law with $q \times q$ matrix of means $\int_0^\hbar \Sigma(u)\,du$. The asymptotic covariance of $\delta^{-1/2}\{[y^*_M] - \int_0^\hbar \Sigma(u)\,du\}$ is Ω, a $q^2 \times q^2$ array with elements

$$\Omega = \left\{ \int_0^\hbar \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u)\}\,du \right\}_{k,k',l,l'=1,\ldots,q}. \tag{53}$$

It will be convenient for us to define the following

$$\alpha_{kj} = \alpha^*_k(j\delta) - \alpha^*_k((j-1)\delta), \tag{54}$$

$$\Gamma_{kl}(t) = \int_0^t \gamma^{aa}_{kl}(s)\,ds, \tag{55}$$

and

$$\Gamma_{klj} = \int_{(j-1)\delta}^{j\delta} \gamma^{aa}_{kl}(s)\,ds. \tag{56}$$

Then, in fact,

$$\Gamma_{kl}(t) = \int_0^t \Sigma_{kl}(u)\,du$$

and

$$\Gamma_{klj} = \int_{(j-1)\delta}^{j\delta} \Sigma_{kl}(u)\,du.$$

When $k = l$ it will be convenient to use the shorthand

$$\Gamma_{kj} = \Gamma_{kkj}.$$

We should note that in the main text we wrote $\Sigma^*(t) = \int_0^t \Sigma(u)\,du$. We have introduced the additional notation $\Gamma_{kl}(t)$ and Γ_{klj} in order to reduce notational clutter in superscripts.

A.2. *Structure of the Proof*

We give the proof of Theorem 1 in several steps, each constituting a section. First we derive the means, variances, and covariances of the variates

$$[y^*_{kM}, y^*_{lM}] = \sum_{j=1}^{M} y_{kj} y_{lj} = \sum_{j=1}^{M} \{y^*_k(\delta j) - y^*_k((j-1)\delta)\}\{y^*_l(j\delta) - y^*_l((j-1)\delta)\}. \tag{57}$$

The second step is to prove Theorem 1 for the case where the mean processes α^*_k are identically 0. In the final section the latter restriction is lifted.

Throughout the rest of this proof we reason conditionally on α^* and Σ and so can regard the processes as if they were deterministic.

A.3. *Mean and Variance*

We are interested in the limiting behavior of $[y^*_{kM}, y^*_{lM}]$ when the processes α^*_k, γ^a_k are considered given due to conditioning.

To the asymptotic order considered, the limit behavior is dominated by the infinitesimal variation of the Brownian motion w, so that, as we shall show, the variation of the (vector) process α^* does not influence the limit laws. Accordingly we suppose in this section that α^* is identically 0, in which case

$$y_{kj} = \int_{(j-1)\delta}^{j\delta} \gamma^a_k(s)\, dw_a(s) \tag{58}$$

and we proceed to determine the means, variances, and covariances of the quantities $[y^*_{kM}, y^*_{lM}]$.

For this we first recall the fact, which follows from the multidimensional version of Ito's formula (cf., for instance, Protter (1990, p. 74)), that for any continuous semimartingales $Y_1(t), \ldots, Y_m(t)$ (with starting value 0) we have that

$$Y_1(t)\cdots Y_m(t) = \sum_{j=1}^{m} \int_0^t \prod_{k\neq j} Y_k(s)\, dY_j(s) + \sum_{1\le j<k\le m} \int_0^t \prod_{l\neq j,k} Y_l(s)\, d[Y_j, Y_k](s). \tag{59}$$

For $m = 2$ this reduces to

$$Y_1(t)Y_2(t) = \int_0^t Y_1(s)\, dY_2(s) + \int_0^t Y_2(s)\, dY_1(s) + \int_0^t d[Y_1, Y_2](s), \tag{60}$$

while for $m = 4$

$$\begin{aligned} Y_1(t)Y_2(t)Y_3(t)Y_4(t) = {} & \int_0^t Y_1(t)Y_2(t)Y_3(t)\, dY_4(s) \qquad [4] \\ & + \int_0^t Y_1(s)Y_2(s)\, d[Y_3, Y_4](s) \quad [6], \end{aligned} \tag{61}$$

where the symbol [4] indicates the sum of the term given plus 3 similar terms obtained via permutation of the indices, etc. Specifically,

$$\begin{aligned} & \int_0^t Y_1(t)Y_2(t)Y_3(t)\, dY_4(s) \quad [4] \\ & = \int_0^t Y_1(t)Y_2(t)Y_3(t)\, dY_4(s) + \int_0^t Y_1(t)Y_2(t)Y_4(t)\, dY_3(s) \\ & \quad + \int_0^t Y_1(t)Y_3(t)Y_4(t)\, dY_2(s) + \int_0^t Y_2(t)Y_3(t)Y_4(t)\, dY_1(s), \end{aligned}$$

which, we note in passing, is a local martingale, and

$$\begin{aligned}
&\int_0^t Y_1(s)Y_2(s)\,d[Y_3,Y_4](s) \quad [6]\\
&\quad=\int_0^t Y_1(s)Y_2(s)\,d[Y_3,Y_4](s)+\int_0^t Y_2(s)Y_3(s)\,d[Y_1,Y_4](s)\\
&\qquad+\int_0^t Y_3(s)Y_4(s)\,d[Y_1,Y_2](s)+\int_0^t Y_1(s)Y_4(s)\,d[Y_2,Y_3](s)\\
&\qquad+\int_0^t Y_1(s)Y_3(s)\,d[Y_2,Y_4](s)+\int_0^t Y_2(s)Y_4(s)\,d[Y_1,Y_3](s).
\end{aligned}$$

By (58) and (59) we find

$$\mathrm{E}\{y_{kj}y_{lj}\}=\Gamma_{klj}$$

and hence we have

$$\mathrm{E}\{[y^*_{kM},y^*_{lM}]\}=\Gamma_{kl}(\hbar).$$

Furthermore, for any indices k,l,k',l' in $\{1,\dots,q\}$,

$$\begin{aligned}
\mathrm{Cov}\{[y^*_{kM},y^*_{lM}],[y^*_{k'M},y^*_{l'M}]\}&=\mathrm{E}\left\{\sum_{j'=1}^{M}(y_{kj'}y_{lj'}-\Gamma_{klj'})\sum_{j=1}^{M}(y_{k'j}y_{l'j}-\Gamma_{k'l'j})\right\}\\
&=\sum_{j=1}^{M}\mathrm{E}\{y_{kj}y_{lj}y_{k'j}y_{l'j}\}-\sum_{j=1}^{M}\Gamma_{klj}\Gamma_{k'l'j}.
\end{aligned}\tag{62}$$

Consider now the case $j=1$. Using (61) and similarly for other index combinations, we find

$$\begin{aligned}
\mathrm{E}\{y_{k1}y_{l1}y_{k'1}y_{l'1}\}&=\int_0^\delta \mathrm{E}\left\{\int_0^u\int_0^u \gamma^a_k(s)\gamma^b_l(s)\,db_a(s)\,db_b(s)\right\}\gamma^c_{k'}(u)\gamma^c_{l'}(u)\,du \quad [6]\\
&=\int_0^\delta \gamma^{cc}_{k'l'}(u)\int_0^u \gamma^{aa}_{kl}(s)\,ds\,du \quad [6] \qquad (63)\\
&=\int_0^\delta \gamma^{cc}_{k'l'}(u)\Gamma_{kl}(u)\,du \quad [6]. \qquad (64)
\end{aligned}$$

Next we note that

$$\frac{d}{ds}\{\Gamma_{kl}(s)\Gamma_{k'l'}(s)\}=\gamma^{cc}_{kl}(s)\Gamma_{k'l'}(s)+\gamma^{cc}_{k'l'}(s)\Gamma_{kl}(s),$$

or, in other words,

$$\int_0^\delta \gamma^{cc}_{k'l'}(u)\Gamma_{kl}(u)\,du+\int_0^\delta \gamma^{cc}_{kl}(u)\Gamma_{k'l'}(u)\,du=\Gamma_{kl}(\delta)\Gamma_{k'l'}(\delta).$$

The terms for other values of the indices behave similarly and so, all in all, we obtain

$$\mathrm{Cov}\{[y^*_{kM},y^*_{lM}],[y^*_{k'M},y^*_{l'M}]\}=\sum_{j=1}^{M}(\Gamma_{kk'j}\Gamma_{ll'j}+\Gamma_{kl'j}\Gamma_{lk'j}).\tag{65}$$

Now, when $\delta\to 0$ the sum in (65) turns into an integral and behaves as $\delta\Omega_{kl,k'l'}(\hbar)$, i.e.,

$$\delta^{-1}\,\mathrm{Cov}\{[y^*_{kM},y^*_{lM}],[y^*_{k'M},y^*_{l'M}]\}\to\Omega_{kl,k'l'}(\hbar),\tag{66}$$

where (recalling that $\Gamma_{klj} = \int_{(j-1)\delta}^{j\delta} \gamma_{kl}^{aa}(u)\,du$ and invoking Riemann integration)

$$\Omega_{kl,k'l'}(t) = \int_0^t \{\gamma_{kk'}^{aa}(s)\gamma_{ll'}^{cc}(s) + \gamma_{kl'}^{aa}(s)\gamma_{lk'}^{cc}(s)\}\,ds, \tag{67}$$

as stated in Theorem 1.

A.4. *Proof of Asymptotic Normality*

As above, indices k, l, k', and l' will run from 1 to q. To prove the result of Theorem 1 in the case where the mean processes α_k^* are identically 0, it suffices,[6] in view of (66), to show that for any real constants c^{kl} we have, as $\delta \downarrow 0$,

$$\delta^{-1/2}\sum_{j=1}^{M} c^{kl}(y_{kj}y_{lj} - \Gamma_{klj}) \xrightarrow{L} N\big(0, c^{kl}c^{k'l'}\Omega_{kl,k'l'}(\hbar)\big), \tag{68}$$

where we are now applying the Einstein summation convention also to the indices $k, l, \ldots$. By the above calculations,

$$\operatorname{Var}\left\{\delta^{-1/2}\sum_{j=1}^{M} c^{kl}(y_{kj}y_{lj} - \Gamma_{klj})\right\} \to c^{kl}c^{k'l'}\Omega_{kl,k'l'}(\hbar).$$

We now invoke the following theorem from Gnedenko and Kolmogorov (1954, pp. 102–103). Let $x_{n1}, \ldots, x_{nk_n}$ ($n = 1, 2, \ldots$, $i = 1, 2, \ldots, k_n$; with $k_n \to \infty$ as $n \to \infty$) be a triangular array of independent random variables and let $x_n = x_{n1} + \cdots + x_{nk_n}$.

THEOREM 3: *Suppose that* $\mathrm{E}\{x_{nj}\} = 0$ *for all* n *and* j *and that* $\operatorname{Var}\{x_n\} = 1$ *for all* n. *Then* $x_n \to^L N(0,1)$ *if and only if, for arbitrary* $\psi > 0$,

$$\sum_{j=1}^{k_n} \mathrm{E}\{x_{nj}^2 \mathbf{1}_{(\psi,\infty)}(|x_{nj}|)\} \to 0, \tag{69}$$

as $n \to \infty$.

This result yields the following Corollary.

COROLLARY 3: *Suppose that* $\mathrm{E}\{x_{nj}\} = 0$ *for all* n *and* j *and that there exists a nonnegative number* v *such that* $\operatorname{Var}\{x_n\} \to v$ *for* $n \to \infty$. *Then*

$$x_n \xrightarrow{L} N(0, v)$$

if and only if (69) *is satisfied*.

[6]Recall that if $z_n = (z_{n1}, \ldots, z_{nq})$ is a sequence of random vectors having mean 0, then to prove that $z_n \to^L N_q(0, \Psi)$ for some nonnegative definite matrix Ψ it suffices to show that for arbitrary real constants $c_1, \ldots, c_q$ we have

$$c'z_n \xrightarrow{L} N_q\big(0, c'\Psi c\big),$$

where $c = (c_1, \ldots, c_q)'$. (This follows directly from the characterization of convergence in law in terms of convergence of the characteristic functions.)

PROOF: When $v = 0$ we have $\mathrm{Var}\{x_n\} \to 0$, which implies

$$x_n \xrightarrow{p} 0 = N(0, 0).$$

When $v > 0$, writing $v_n = \mathrm{Var}\{x_n\}$ and $\bar{x}_n = x_n/\sqrt{v_n}$ we find, by the above theorem, that $\bar{x}_n \to^L N(0, 1)$ and hence

$$x_n = \sqrt{v} \cdot \sqrt{v_n/v} \cdot \bar{x}_n \xrightarrow{L} N(0, v). \qquad \textit{Q.E.D.}$$

By a standard type of argument (cf., for instance, Billingsley (1995, Theorem 27.3)), a sufficient condition for (69) is that

$$\sum_{j=1}^{k_n} \mathrm{E}\{|x_{nj}|^{2+\varepsilon}\} \to 0$$

for some $\varepsilon > 0$.

We have

$$y_{kj} \overset{L}{=} \sqrt{\Gamma_{kj}}\, u_{kj},$$

where u_{kj} is a standard normal variate. Hence, letting

$$x_{Mj} = \delta^{-1/2} c^{kl} (y_{kj} y_{lj} - \Gamma_{klj}),$$

we find

$$x_{Mj} \overset{L}{=} \delta^{-1/2} c^{kl} \big(\sqrt{\Gamma_{kj}\Gamma_{lj}}\, u_{kj} u_{lj} - \Gamma_{klj}\big)$$

or, equivalently,

$$x_{Mj} \overset{L}{=} \delta^{1/2} c^{kl} \sqrt{\bar{\Gamma}_{kj}\bar{\Gamma}_{lj}}\, (u_{kj} u_{lj} - \rho_{klj}), \tag{70}$$

where

$$\bar{\Gamma}_{kj} = \delta^{-1} \Gamma_{kj}$$

and

$$\rho_{klj} = \frac{\Gamma_{klj}}{\sqrt{\Gamma_{kj}\Gamma_{lj}}}$$

is the correlation coefficient between u_{kj} and u_{lj}. Note that, by our Assumption (a) on the Σ process, as δ varies the quantities $\bar{\Gamma}_{kj}$ are bounded away from 0 and infinity, uniformly in k and j. This implies that

$$\mathrm{E}\big\{\big|c^{kl}\sqrt{\bar{\Gamma}_{kj}\bar{\Gamma}_{lj}}\,(u_{kj} v_{lj} - \rho_{klj})\big|^{2+\varepsilon}\big\}$$

is uniformly bounded above, and hence, by (70), that

$$\sum_{j=1}^{M} \mathrm{E}\{|x_{Mj}|^{2+\varepsilon}\} \to 0,$$

as was to be shown.

A.5. *Negligibility of Mean Process* α^*

To prove that the same limiting laws hold when the mean processes α_ν^* are not 0, we first note that

$$y_{kj}y_{lj} = \alpha_{kj}\alpha_{lj} + \alpha_{kj}y_{0lj} + \alpha_{lj}y_{0kj} + y_{0kj}y_{0lj},$$

where

$$y_{0kj} = \int_{(j-1)\delta}^{j\delta} \gamma_k^a(s)\, dw_a(s).$$

Under condition (16), α_{kj} is $o(\sqrt{\delta})$ uniformly in k and j. Furthermore we have

$$\alpha_{kj}y_{0lj} + \alpha_{lj}y_{0kj} \sim N(0, \alpha_{kj}^2\Gamma_{llj} + 2\alpha_{kj}\alpha_{lj}\Gamma_{klj} + \alpha_{lj}^2\Gamma_{kkj}).$$

Consequently,

$$\begin{aligned}[y_{kM}^*, y_{lM}^*] &= \sum_{j=1}^{M} \alpha_{kj}\alpha_{lj} + \sum_{j=1}^{M}(\alpha_{kj}y_{0lj} + \alpha_{lj}y_{0kj}) + \sum_{j=1}^{M} y_{0kj}y_{0lj} \\ &= o(\sqrt{\delta}) + o_p(\sqrt{\delta}) + [y_{0kM}^*, y_{0lM}^*],\end{aligned}$$

where

$$[y_{0kM}^*, y_{0lM}^*] = \sum_{j=1}^{M} y_{0kj}y_{0lj}.$$

It follows that, conditionally,

$$\delta^{-1/2}\{[y_M^*] - \Gamma\}$$

has the same limit law as

$$\delta^{-1/2}\{[y_{0M}^*] - \Gamma\}$$

and the latter is as given in Theorem 1.

APPENDIX B: STATISTICALLY FEASIBLE RESULTS

B.1. *Preamble*

The matrix Ω of Theorem 1 is not known and so the results of the theorem and its corollary are infeasible. In this Appendix we continue with the same notation and assumptions used in the previous section, but now we focus on feasible limit theorems. The following theorem and corollary, which then immediately deliver the proof of Theorem 2 in the main text, mean that Ω can be replaced by a consistent estimator. We have thus delivered a feasible theory.

THEOREM 4: *Let the assumptions be as in Theorem* 1, *and let*

$$\widehat{\psi}_{klk'l'} = \frac{M}{\hbar}\sum_{j=1}^{M} y_{kj}y_{lj}y_{k'j}y_{l'j}$$

and

$$\widetilde{\psi}_{klk'l'} = \frac{M}{\hbar}\sum_{j=1}^{M-1} y_{kj}y_{lj}y_{k'(j+1)}y_{l'(j+1)}.$$

Then, as $M \to \infty$,

$$\widehat{\psi}_{klk'l'} \xrightarrow{p} \int_0^{\hbar} \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u) + \Sigma_{kl}(u)\Sigma_{k'l'}(u)\}\, du,$$

while

$$\widetilde{\psi}_{klk'l'} \xrightarrow{p} \int_0^{\hbar} \Sigma_{kl}(u)\Sigma_{k'l'}(u)\, du.$$

Note that, importantly,

$$\widetilde{\psi}_{k'l'kl} \xrightarrow{p} \int_0^{\hbar} \Sigma_{kl}(u)\Sigma_{k'l'}(u)\, du.$$

COROLLARY 4: *Defining*

$$\overline{\psi}_{klk'l'} = \widehat{\psi}_{klk'l'} - \frac{1}{2}\big(\widetilde{\psi}_{klk'l'} + \widetilde{\psi}_{k'l'kl}\big),$$

we see that

$$\overline{\psi}_{klk'l'} \xrightarrow{p} \int_0^{\hbar} \{\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{lk'}(u)\}\, du,$$

and so there exists a random $q^2 \times q^2$ *matrix*

$$H = \sum_{j=1}^{M} x_j x_j' - \frac{1}{2}\sum_{j=1}^{M-1}(x_j x_{j+1}' + x_{j+1}x_j'), \tag{71}$$

where $x_j = \mathrm{vec}(y_j y_j')$, *explicitly calculable in terms of* y_M^*, *such that*

$$\frac{M}{\hbar} H \xrightarrow{p} \Omega$$

as $M \to \infty$.

The above corollary follows immediately from Theorem 4; hence the only remaining issue is the Proof of Theorem 4.

B.1.1. *Proof of Theorem 4*

Recall from (62) that

$$\mathrm{Cov}\{[y_{kM}^*, y_{lM}^*], [y_{k'M}^*, y_{l'M}^*]\} = \sum_{j=1}^{M} E\{y_{kj}y_{lj}y_{k'j}y_{l'j}\} - \sum_{j=1}^{M} \Gamma_{klj}\Gamma_{k'l'j}.$$

From the previous discussion we have

$$\delta^{-1}\,\mathrm{Cov}\{[y_{kM}^*, y_{lM}^*], [y_{k'M}^*, y_{l'M}^*]\} \to \Omega_{kl,k'l'}(\hbar).$$

On the other hand, arguing as above it is seen, using formula (63) and Riemann integration, that

$$\delta^{-1}\sum_{j=1}^{M} E\{y_{kj}y_{lj}y_{k'j}y_{l'j}\} = \delta^{-1}\sum_{j=1}^{M}\int_{(j-1)\delta}^{j\delta} \gamma_{k'l'}^{cc}(u)\int_{(j-1)\delta}^{j\delta}\gamma_{kl}^{aa}(u)\, ds\, du \quad [6]$$

$$\to \int_0^{\hbar} \{\gamma_{kl}^{aa}(s)\gamma_{k'l'}^{cc}(s) + \gamma_{kk'}^{aa}(s)\gamma_{ll'}^{cc}(s) + \gamma_{kl'}^{aa}(s)\gamma_{lk'}^{cc}(s)\}\, ds$$

and, by (56),

$$\delta^{-1}\sum_{j=1}^{M}\Gamma_{klj}\Gamma_{k'l'j}\to\int_0^{\hbar}\gamma_{kl}^{aa}(s)\gamma_{k'l'}^{cc}(s)\,ds,$$

and, moreover, that

$$\delta^{-1}\sum_{j=1}^{M}y_{kj}y_{lj}y_{k'j}y_{l'j}$$

must converge in probability to the same limit as

$$\delta^{-1}\sum_{j=1}^{M}\mathrm{E}\{y_{kj}y_{lj}y_{k'j}y_{l'j}\}.$$

Thus to obtain a consistent estimator of $\Omega_{kl,k'l'}(\hbar)$ it suffices to find a consistent estimator of

$$\int_0^{\hbar}\gamma_{kl}^{aa}(s)\gamma_{k'l'}^{cc}(s)\,ds.$$

The quantity

$$\sum_{j=1}^{M-1}y_{kj}y_{lj}y_{k'(j+1)}y_{l'(j+1)}$$

solves this problem, as is seen by again applying formula (63).

REFERENCES

ANDERSEN, T. G., AND T. BOLLERSLEV (1998): "Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts," *International Economic Review*, 39, 885–905.

ANDERSEN, T. G., T. BOLLERSLEV, AND F. X. DIEBOLD (2004): "Parametric and Nonparametric Measurement of Volatility," in *Handbook of Financial Econometrics*, ed. by Y. Aït-Sahalia and L. P. Hansen. Amsterdam: North-Holland, forthcoming.

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND H. EBENS (2001): "The Distribution of Realized Stock Return Volatility," *Journal of Financial Economics*, 61, 43–76.

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND P. LABYS (2001): "The Distribution of Exchange Rate Volatility," *Journal of the American Statistical Association*, 96, 42–55. Correction (2003), 98, 501.

——— (2003): "Modeling and Forecasting Realized Volatility," *Econometrica*, 71, 579–625.

ANDERSEN, T. G., T. BOLLERSLEV, AND N. MEDDAHI (2004): "Analytic Evaluation of Volatility Forecasts," *International Economic Review*, 45, forthcoming.

ANDERSON, T. W. (1984): *An Introduction to Multivariate Statistical Analysis* (Second Edition). New York: John Wiley & Sons.

ANDREOU, E., AND E. GHYSELS (2002): "Rolling-Sampling Volatility Estimators: Some New Theoretical, Simulation and Empirical Results," *Journal of Business and Economic Statistics*, 20, 363–376.

BACK, K. (1991): "Asset Pricing for General Processes," *Journal of Mathematical Economics*, 20, 371–395.

BAI, X., J. R. RUSSELL, AND G. C. TIAO (2000): "Beyond Merton's Utopia: Effects of Non-normality and Dependence on the Precision of Variance Estimates Using High-Frequency Financial Data," Unpublished paper, Graduate School of Business, University of Chicago.

BARNDORFF-NIELSEN, O. E., AND D. R. COX (1994): *Inference and Asymptotics*. London: Chapman & Hall.

BARNDORFF-NIELSEN, O. E., AND N. SHEPHARD (2001): "Non-Gaussian Ornstein–Uhlenbeck-Based Models and Some of Their Uses in Financial Economics (with Discussion)," *Journal of the Royal Statistical Society*, Series B, 63, 167–241.

——— (2002a): "Econometric Analysis of Realised Covariation: High Frequency Covariance, Regression and Correlation in Financial Economics," Unpublished paper, Nuffield College, Oxford, Economics Working Paper 2002-W13.

——— (2002b): "Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models," *Journal of the Royal Statistical Society*, Series B, 64, 253–280.

——— (2003a): Power Variation and Time Change," Unpublished paper, Nuffield College, Oxford.

——— (2003b): "Realized Power Variation and Stochastic Volatility," *Bernoulli*, 9, 243–265 and 1109–1111.

——— (2004a): "How Accurate Is the Asymptotic Approximation to the Distribution of Realized Volatility?" in *Identification and Inference for Econometric Models. A Festschrift in Honour of T. J. Rothenberg*, ed. by D. W. K. Andrews, J. Powell, P. A. Ruud, and J. H. Stock. Cambridge: Cambridge University Press, forthcoming.

——— (2004b): "Power and Bipower Variation with Stochastic Volatility and Jumps (with Discussion)," *Journal of Financial Econometrics*, 2, 1–48.

——— (2004c): "A Feasible Central Limit Theory for Realised Volatility Under Leverage," Unpublished paper, Nuffield College, Oxford.

BARNDORFF-NIELSEN, O. E., S. E. GRAVERSEN, AND N. SHEPHARD (2004): "Power Variation and Stochastic Volatility: A Review and Some New Results," *Journal of Applied Probability*, 41, forthcoming.

BILLINGSLEY, P. (1995): *Probability and Measure* (Third Ed.). New York: John Wiley & Sons.

BOLLERSLEV, T., AND H. ZHOU (2002): "Estimating Stochastic Volatility Diffusion Using Conditional Moments of Integrated Volatility," *Journal of Econometrics*, 109, 33–65.

BOLLERSLEV, T., R. F. ENGLE, AND J. M. WOOLDRIDGE (1988): "A Capital Asset Pricing Model with Time Varying Covariances," *Journal of Political Economy*, 96, 116–131.

CALVET, L., AND A. FISHER (2002): "Multifractality in Asset Returns: Theory and Evidence," *Review of Economics and Statistics*, 84, 381–406.

CAMPBELL, J. Y., A. W. LO, AND A. C. MACKINLAY (1997): *The Econometrics of Financial Markets*. Princeton: Princeton University Press.

CHAMBERLAIN, G. (1988): "Asset Pricing in Multiperiod Securities Markets," *Econometrica*, 56, 1283–1300.

COCHRANE, J. H. (2001): *Asset Pricing*. Princeton: Princeton University Press.

COMTE, F., AND E. RENAULT (1998): "Long Memory in Continuous-time Stochastic Volatility Models," *Mathematical Finance*, 8, 291–323.

DACOROGNA, M. M., R. GENCAY, U. A. MÜLLER, R. B. OLSEN, AND O. V. PICTET (2001): *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

DAVID, F. N. (1938): *Tables of the Ordinates and Probability Integral of the Distribution of the Correlation Coefficient in Small Samples*. Cambridge: Cambridge University Press.

DOOB, J. L. (1953): *Stochastic Processes*. New York: John Wiley & Sons.

DOORNIK, J. A. (2001): *Ox: Object Oriented Matrix Programming 3.0.* London: Timberlake Consultants Press.

DROST, F. C., AND T. E. NIJMAN (1993): "Temporal Aggregation of GARCH Processes," *Econometrica*, 61, 909–927.

FISHER, R. A. (1921): "On the Probable Error of a Coefficient of Correlation Deduced from a Small Sample," *Metron*, 1, 3–32.

FOSTER, D. P., AND D. B. NELSON (1996): "Continuous Record Asymptotics for Rolling Sample Variance Estimators," *Econometrica*, 64, 139–174.

GENON-CATALOT, V., C. LAREDO, AND D. PICARD (1992): "Non-Parametric Estimation of the Diffusion Coefficient by Wavelet Methods," *Scandinavian Journal of Statistics*, 19, 317–335.

GHYSELS, E., A. C. HARVEY, AND E. RENAULT (1996): "Stochastic Volatility," in *Statistical Methods in Finance*, ed. by C. R. Rao and G. S. Maddala. Amsterdam: North-Holland, 119–191.

GNEDENKO, B. W., AND A. N. KOLMOGOROV (1954): *Limits Distributions for Sums of Independent Random Variables*. Cambridge: Addison-Wesley.

GOURIEROUX, C., E. RENAULT, AND N. TOUZI (2000): "Calibration by Simulation for Small Sample Bias Correction," in *Simulation-Based Inference in Econometrics: Methods and Applications*, ed. by R. Mariano, T. Schuermann, and M. J. Weeks. Cambridge: Cambridge University Press, 328–358.

HANSEN, B. E. (1995): "Regression with Non-Stationary Volatility," *Econometrica*, 63, 1113–1132.

JACOD, J., AND A. N. SHIRYAEV (1987): *Limit Theorems for Stochastic Processes*. Berlin: Springer-Verlag.

KARATZAS, I., AND S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus* (Second Edition), Vol. 113 of *Graduate Texts in Mathematics*. Berlin: Springer-Verlag.

——— (1998): *Methods of Mathematical Finance*. New York: Springer-Verlag.

LUTKEPOHL, H. (1996): *Handbook of Matrices*. Chichester: John Wiley & Sons.

MAGNUS, J. R. (1988): *Linear Structures*. London: Griffin.

MCCULLAGH, P. (1987): *Tensor Methods in Statistics*. London: Chapman & Hall.

MEDDAHI, N. (2001): "A Theoretical Comparison Between Integrated and Realized Volatilities," Unpublished Paper, CIRANO DP 2001s-71.

——— (2002): "A Theoretical Comparison Between Integrated and Realized Volatilities," *Journal of Applied Econometrics*, 17, 479–508.

——— (2003): "ARMA Representation of Integrated and Realized Variances," *Econometrics Journal*, 6, 334–355.

MUNROE, M. E. (1953): *Introduction to Measure and Integration*. Cambridge: Addison-Wesley.

PROTTER, P. (1990): *Stochastic Integration and Differential Equations: A New Approach*. New York: Springer-Verlag.